J. B. Cooper

STRENGTH OF MATERIALS

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BY

JAMES E. BOYD, M. S.
Professor of Mechanics, The Ohio State University

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PREFACE TO THE FOURTH EDITION

Few changes in arrangement have been made in the fourth edition. The fundamentals of shear have been advanced to the second chapter while stress beyond the elastic limit, now given in the third chapter, includes yield point and failure in shear as well as in tension and compression.

Most of the problems are new, and many more problems are based upon experimental tables. The curves and photographs which accompany these data help to bridge the gap between theory and its applications, especially when laboratory and class work are not concurrent.

Since some teachers prefer to present deflection of beams by the classical method of successive integrations with arbitrary constants, while others prefer area moments or a combination of the two methods, these methods are given in separate chapters in such a way that choice may conveniently be made. A short chapter is allotted to successive integration between limits—a superior method for a beam with several concentrated loads.

Considerable addition has been made to the discussion of indeterminate beams. The material is arranged to present the usual short course or a rather complete course in preparation for the study of rigid frames. (A note to the teacher at the beginning of Chapter XI explains the use of the text for these purposes.) The theorem of three moments includes beams which carry uniformly distributed loads or uniformly increasing loads over parts of the span, and beams for which the supports are not in one plane. Table XXI gives expressions by which the theorem of three moments or the theorem of two moments may be written for a number of types of loading.

The tables and curves for the secant formula and Euler's formula of the previous editions have been recalculated from the better values of the modulus of elasticity and yield point of structural steel adopted by the Column Committee of the American Society of Civil Engineers. The chapter on working formulas for columns has been extended to include the parabolic formulas recommended by this committee and equations for columns of aluminum alloys.

As further preparation for the study of rigid frames, the chapter on resilience of beams has been enlarged by adding the method of "elastic energy" with applications to indeterminate beams and closed circular rings.

The theory of curved beams has been simplified by the derivation of a general equation for the resisting moment, which greatly reduces the labor of computation.

The more advanced portions of the text are illustrated by the solution of many numerical examples, which will make it easier for the engineer to study these subjects without a teacher.

The book is designed for use with the 1934 edition of the American Institute of Steel Construction Handbook or the 1934 edition of the Carnegie Pocket Companion.

The author is indebted to Profs. Rolland W. Chase, Frederick J. Converse, Philip G. Laurson, Elmer K. Timby, George E. Tomlinson, and his colleagues: P. W. Ott, S. B. Folk, R. W. Powell, E. C. Clark, and Leroy Tucker for suggestions, corrections, and criticisms. Professor Folk assisted in the correction of the manuscript and proof, and Mr. H. H. Brittingham, of the Department of Engineering Drawing, prepared the illustrations.

J. E. B.

COLUMBUS, OHIO,
January, 1935.

PREFACE TO THE FIRST EDITION

This book is intended to give the student a grasp of the physical and mathematical ideas underlying the Mechanics of Materials, together with enough of the experimental facts and simple applications to sustain his interest, fix his theory, and prepare him for the technical subjects as given in works on Machine Design, Reinforced Concrete, or Stresses in Structures.

It is assumed that the reader has completed the Integral Calculus, and has taken a course in Theoretical Mechanics which includes statics and the moment of inertia of plane areas. Chapters XVI and XVII* give a brief discussion of center of gravity and moment of inertia. Students who have not mastered these subjects should study these chapters before taking up Chapter V (preferably before beginning Chapter I).

The problems, which are given with nearly every article, form an essential part of the development of the subject. They were prepared with the twofold object of fixing the theory and enabling the student to discover for himself important facts and applications. The first problems of each set usually require the use of but one new principle—the one given in the text which immediately precedes; the later problems aim to combine this principle with others previously studied and with the fundamental operations of Mathematics and Mechanics. The constants given in the data or derived from the results of the problems fall within the range of the figures obtained from actual tests of materials. Many of the problems are taken directly from such measurements. Some of them are from tests made by the author or his colleagues at the Ohio State University; others are from bulletins of the University of Illinois Engineering Experiment Station, from "Test of Metals" at the Watertown Arsenal, and from the Transactions of the American Society of Civil Engineers.

This book is designed for use with "Cambria Steel," to which references are made by title instead of by page, so that they are adapted to any edition of the handbook.

* Chapter XX of the Fourth Edition.

The author acknowledges his indebtedness for suggestions and criticisms to Professors C. T. Morris, E. F. Coddington, Robert Meiklejohn, K. D. Swartzel, and many others of the Faculty of the College of Engineering; and to Professor Horace Judd of the Department of Mechanical Engineering for the material for several of the half-tones. He also expresses his obligations to the books which have helped to mold his ideas of the subject,—Johnson's "Materials of Construction," Ewing's "Strength of Materials," and especially the textbooks which he has used with his classes,—Merriman's "Mechanics of Materials," Heller's "Stresses in Structures," and Goodman's "Mechanics Applied to Engineering."

The symbols used in the mathematical expressions are much the same as in Heller's "Stresses in Structures."

J. E. B.

COLUMBUS, OHIO,
November 6, 1911.

CONTENTS

NOTATION

The symbols which are frequently used in this book are:

a = radius of circle; distance of concentrated load from left end of span.
A = area; area of cross section.
A' = some special area; part of the area of a cross section.
b = breadth; breadth of rectangular section; base of triangle; distance of concentrated load from right end of span.
B = some special value of b, usually a maximum.
c = distance from neutral axis to extreme outer fiber; distance of center of a circular curve beam from center of curvature; distance in figure.
C = distance from center of curvature of trapezoidal curved beam to intersection of sides.
C_1, C_2, C_3 = integration constants.
d = depth; depth of rectangular section; diameter; distance between parallel axes.
D = some special depth; diameter of boiler.
e = eccentricity of a load on a column; distance in figure.
E = modulus of elasticity.
E_c = modulus of elasticity in compression; modulus of elasticity of concrete.
E_s = modulus of elasticity in shear; tension modulus of elasticity of steel reinforcement.
E_t = modulus of elasticity in tension.
E_v = modulus of volume elasticity.
E_w = working modulus of elasticity.
h = height; distance from vertex to base of triangle.
hp = horsepower.
H = product of inertia; distance of larger base of trapezoidal curved beam from intersection of nonparallel sides.
H_c = product of inertia with respect to axes which intersect at center of gravity.
H' = product of inertia for inclined axes.
I = moment of inertia.
I_c = moment of inertia with respect to axis through center of gravity.
I_m = maximum moment of inertia of beam of variable section.
I_x = moment of inertia with respect to X axis.
I_y = moment of inertia with respect to Y axis.
j = ratio of moment arm to total depth of a reinforced concrete beam.
J = polar moment of inertia.
k = a constant coefficient; radius of gyration (in Chapter XX); a ratio less than unity.
l = length; length of beam between supports; length of column between points of inflection.

L = length; total length of column.
m = mass of particle; slope of tangent at support; a ratio.
M = moment; mass.
M_o = moment at origin of coördinates.
M_a, M_b, M_c = moment over three consecutive supports.
M_1, M_2, M_3, etc. = moment over first, second, third, etc., supports.
M_p = moment caused by a dummy load P.
M_q = moment caused by actual loads.
M_t = moment caused by a dummy couple.
n = ratio; number of turns in a helical spring.
N = normal force at surface; number of revolutions per minute.
p = pitch of rivets; slope of tangent; ratio of steel area to concrete area.
P = concentrated load or force.
q = coefficient in Rankine's formula.
Q = concentrated load or force.
r = distance from origin; radius of gyration (in column formulas); radius.
R = reaction at support; resultant force; radius; radius of coil.
R_1 = reaction at left support; radius of inside surface of curved beam or hook.
R_2 = reaction at second support; radius of outside surface of curved beam or hook.
R_0 = radius of neutral surface of curved beam or hook.
$\bar{R}$ = radius of curved beam to center of gravity of section.
s = unit stress.
s_t, s_s, s_c = unit tensile, shearing, and compressive stress.
s_u = ultimate unit stress.
s_w = allowable unit stress.
s' = unit stress resulting from combined shear and tension or compression.
S = unit stress in extreme fibers.
S_1 = unit stress at concave surface of curved beam.
S_2 = unit stress at convex surface of curved beam.
S_s = unit shearing stress at surface of shaft.
t = thickness.
T = torque; tension.
U = work.
U_p = modulus of resilience.
v = distance from neutral axis.
$\bar{v}$ = distance from neutral axis to center of gravity of A'.
V = total vertical shear.
V_{ab} = total shear near support A in span joining A to B.
w = distributed load per unit of length.
W = total load uniformly distributed.
$\bar{x}$, $\bar{y}$, $\bar{z}$ = coördinates of center of gravity.
y = deflection in a beam or column.
y_a = deflection at A caused by unit load at A.
y_{ba} = deflection at A caused by unit load at B.
y_p = deflection under load P caused by this load.
y_{max} = maximum deflection of a beam or column.
Z = section modulus.

α, β, θ, ϕ = angles in figure.
δ; δ_s = unit deformation; unit shearing deformation.
μ = Poisson's ratio; coefficient of friction.
ρ = density; radius of curvature.
θ_1, θ_2 = slope at 1 and 2 from left to right.
θ_{21} = slope at 2 from right to left.

STRENGTH OF MATERIALS

CHAPTER I

STRESSES

1. Strength of Materials.—That branch of mechanics which treats of the changes in form and dimensions of elastic solids and the forces which cause these changes is called *the mechanics of materials*. When the physical constants and the results of experimental tests upon the materials of construction are included with the theoretical discussion of the ideal elastic solid, the entire subject is called *the strength of materials* or *the resistance of materials*.

2. Tension.—Figure 1 shows a rubber band which is suspended from a horizontal bar and carries a hook at the lower end. When a small weight is hung on the hook, the rubber band is stretched; its length is increased by an amount a, while its cross section is reduced. When a second weight is added, there is an additional elongation b. If the weights are equal, the elongation a caused by the first weight is equal to the elongation b caused by the second weight. When the weights are removed, the rubber band returns to its original length and cross section.

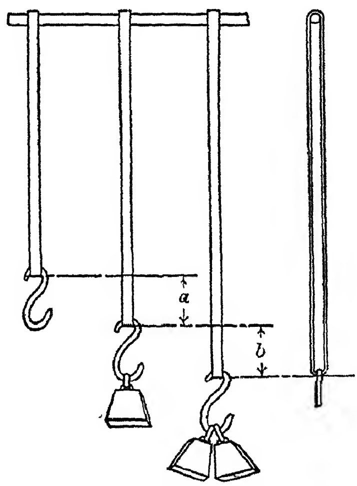


FIG. 1.—Rubber bands in tension.

If steel, iron, wood, concrete, stone, or other solid material is used instead of rubber, the results are similar. There is this apparent difference: while the rubber may be stretched to twice or three times its original length and still return to its original size and shape after the load is removed, one of the other materials

may be stretched only a very small amount (usually less than 0.002 of its length), without receiving a permanent change in its dimensions. Again, the force required to produce a relatively small increase in the length of a rod of wood or steel, for instance, is many times greater than that necessary to *double* the length of a soft rubber band of equal cross section. These differences between the behavior of soft rubber and other solid materials are differences of degree and not of kind. Essentially they are alike.

The rubber bands shown in Fig. 1 are subjected to the action of two forces: the force of the weights pulling downward, and the reaction of the support pulling upward. The bands are in *tension*. A body is said to be in tension when it is subjected to two sets of forces whose resultants are in the same straight line, opposite in direction, and directed *away* from each other.

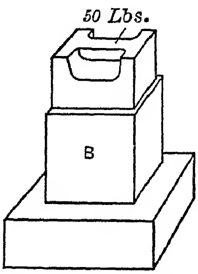


FIG. 2.—Compression.

3. Compression.—When a body is subjected to two sets of forces whose resultants are in the same straight line, opposite in direction, and directed *toward* each other, it is said to be in *compression*. In Fig. 2, the block B is in compression under the action of the 50 pounds pushing down and the reaction of the support pushing up. The effect of compression upon a body is to shorten it in the line of the forces and increase its dimensions in the plane perpendicular to this line.

Tension and compression may be represented as in Fig. 3, in which the arrows represent the forces, and the small rectangles represent the bodies, or portions of a body, upon which the forces act. The rectangles are often omitted; a pair of arrows with their heads together indicates compression, and a pair with their heads in the opposite sense indicates tension.

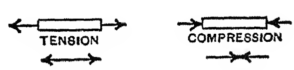


FIG. 3.

4. Stress: Total Stress.—The force exerted by one body on another at their surface of contact is called the *stress* between the bodies or the *total stress* between the bodies. In Fig. 2, the total stress between the 50-pound weight and the block B upon which it rests is 50 pounds. This stress is compressive. The weight pushes down upon the block with a *force* of 50 pounds and the block pushes up against the weight with an equal force. Action and reaction are equal.

Figure 4 represents a bar subjected to a horizontal pull of **P** pounds. If the bar is supposed to be cut by an *imaginary plane* at C, the portion A to the left of this *plane section* is in equilibrium under the action of the external pull $\mathbf{P}_1$ toward the left and an equal opposite pull $\mathbf{P}_3$ at the section **C**. This force $\mathbf{P}_3$ across the section is the pull exerted by the right portion B upon the left portion A. This pull is the *internal stress* at the section. In like manner, the right portion B is in equilibrium under the external pull $\mathbf{P}_2$ at the right end of the bar and the internal stress $\mathbf{P}_4$, equal and opposite to $\mathbf{P}_3$, exerted by the left portion A upon the right portion B across the section, as shown separately in Fig. 4, II.

Stress at any section is the force with which *the material at the section* resists deformation or rupture.

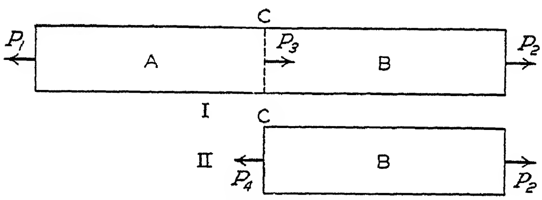


FIG. 4.—Stress at section.

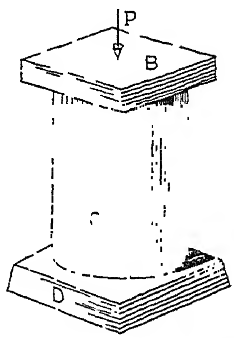


FIG. 5.—Area under stress.

5. Unit Stress; Intensity of Stress.—The *unit compressive* or *tensile stress* at any section of a body is calculated by dividing the total external force by the area of the cross section at right angles to the force. If a vertical force **P** is applied to the cylinder C of Fig. 5 by means of the plate B and the reaction of the support D, the unit stress at any section is given by the equation

$$s = \frac{P}{A}, \qquad \text{Formula I*}$$

in which s is the *unit stress*, P is the *external* force equal to the *total internal stress*, and A is the area of cross section perpendicular to the direction of the stress. *Unit stress* frequently is called *intensity* of *stress*. In American engineering practice, unit stresses

* Important formulas, which should be understood and memorized, are designated by Roman numerals in this book.

generally are given in pounds per square inch or kips per square inch. (One *kip* or kilo pound is 1,000 pounds.) Frequently compressive stresses in large masonry structures are expressed in tons per square foot. It is the common practice to give bearing pressure of masonry on soils in this way. British engineers employ long tons per square inch as well as pounds per square inch to express the intensity of stress in steel and similar materials. Continental* engineers, of course, use kilograms per square centimeter. Physicists prefer dynes per square centimeter or dynes per square millimeter. Stress in pounds per square inch may be written lb./in.^{2}

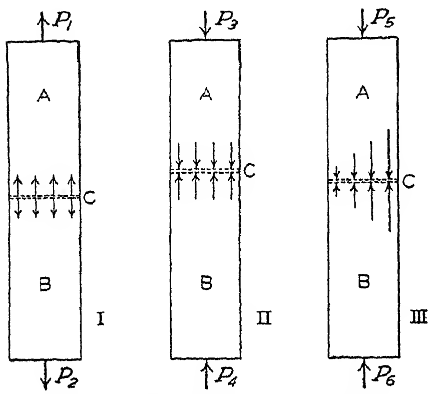


FIG. 6.—Representation of stress.

In elementary mechanics the tensile or compressive stress exerted by a bar is assumed to lie in the axis of the member. In reality each longitudinal element exerts its portion of the stress. The force assumed to act along the axis is the resultant of the forces exerted by all the elements. The unit stress obtained by dividing the total applied force by the area of the cross section is the *average unit stress* in the member.

Figure 6, I, shows a bar under tensile stress which is uniform in all parts of the section. The arrows which represent the stress of different elements are all of equal length. Figure 6, II, shows a bar under uniform compressive stress. Figure 6, III, shows compressive stress which increases uniformly from left to right.

When the stress is uniform, the resultant stress passes through the center of gravity of each cross section which corresponds to the center of gravity of a short piece of uniform length cut from the bar. When the stress is not uniform, the location of the resultant may be found by calculating the sum of the moments with respect to some parallel plane of the force on each element of area and dividing this moment by the sum of the forces. In other words, the resultant passes through the center of gravity of a solid whose base is the section of the bar and whose altitude at any point is proportional to the unit stress at that point.

* They sometimes use atmospheres. One atmosphere equals 14.7 pounds per square inch, or 1.033 kilograms per square centimeter.

Problems

(*Find dimensions of steel shapes in steel handbook.*)

1. The cylinder of Fig. 5 is 2 in. in diameter and 5 in. long. Find the unit stress when a vertical load of 14,000 lb. is applied by means of the plate *B*. *Ans.* 4,456 lb./in.^{2}
2. A piece 6 in. long is cut from a 5-in. by 3-in. by ½-in. angle section by planes perpendicular to its length. The piece stands vertical and a load of 30,000 lb. is applied at the top by means of a 5-in. by 3-in. by 1-in. steel plate. Find the unit stress in the angle. *Ans.* $s = 8{,}000$ lb./in.^{2}
3. Two edges of the plate in Problem 2 lie in the planes of back of the legs of the angle section. The load is applied to the plate by means of a steel ball. Where must this ball be placed in order that the unit stress in the angle may be uniform?
Ans. 1.75 in. from one 3-in. edge, and 0.75 in. from one 5-in. edge.
4. A rectangular wood block, 2 in. by 3 in., is 10 in. long. Find the unit stress when a load of 7,200 lb. is applied parallel to the length.
5. A block in the form of a frustum of a pyramid is 2 in. square at the top, 3 in. square at the bottom, and 8 in. high. Find the unit stress 2 in. from the bottom and 4 in. from the bottom when a load of 7,200 lb. is placed on the top. *Ans.* 952.1 lb./in.^{2}; 1152 lb./in.^{2}
6. In a short block 2 in. square the unit stress increases uniformly from 100 lb. per sq. in. in the left face to 700 lb. per sq. in. in the right face. Find the total load. *Ans.* $P = 1{,}600$ lb.
7. In Problem 6, find the location of the resultant force. Represent the stress in the front face by a trapezoid 100 units high on the left and 700 units high on the right. Find the center of gravity of the trapezoidal wedge which represents the stress by combining the moment and area of two triangles, or the moment and area of a triangle and a rectangle.
Ans. 1.25 in. from the left face; 1 in. from the front face.
8. A short block of triangular section has two faces each 13 in. wide, and one face 10 in. wide. The block is subjected to compression parallel to its length which causes the unit stress to increase uniformly from 100 lb. per sq. in. at the intersection of the 13-in. faces to 700 lb. per sq. in. in the 10-in. face. Find the total load by integration. Show that this load equals the area of the section multiplied by the unit stress at the center of gravity of the cross section. *Ans.* $P = 30{,}000$ lb.
9. By integration of moments find the line of action of the resultant force of Problem 8. *Ans.* 8.8 in. from intersection of 13-in. faces.

6. Working Stress; Allowable Unit Stress.—Working stresses are the unit stresses to which the materials of a machine or structure are subjected. The *allowable unit stress* for a given material is the maximum unit stress which, in the judgment of some competent and official authority, should be applied to this material. For instance, the specifications of the American Institute of Steel Construction and the building laws of New

York City give 18,000 pounds per square inch as the unit tensile stress for structural steel. For the compressive stress in relatively short blocks of select-grade white oak in situations which are always dry, the American Society for Testing Materials specifies 1,000 pounds per square inch parallel to the grain. The Joint Committee of Concrete and Reinforced Concrete* gives 25 per cent of the compressive strength at 28 days as the allowable compressive stress of concrete.

Table I gives a few allowable stresses in tension and compression. The values for concrete which have been used in some cities are recommended where tests are not made. The other allowable stresses are from various official sources.

TABLE I.—ALLOWABLE UNIT STRESS
(This table should be memorized.)

Material	Tension, pounds per square inch	Compression, pounds per square inch	
Structural steel	18,000	18,000	
Cast steel	16,000	16,000	
Wrought iron	12,000	12,000	
Cast iron	3,000	15,000	
Portland-cement concrete 1:2:4	......	450	
Portland-cement concrete 1:3:6	......	300	
		With grain	Across grain
Common grade timber in dry location:			
Douglas fir, coast region	......	880	325
Southern yellow pine	......	880	325
White or red oak	......	800	500

A steel bar one foot long and one square inch in cross section weighs 3.4 *pounds.* For methods of calculating areas and weights of rolled sections see "Carnegie Pocket Companion,"† 1934, page 133.

* This committee is made up of representatives from the American Society of Civil Engineers, the American Society for Testing Materials, the American Railway Engineering Association, the American Concrete Institute, and the Portland Cement Association.

† Same book as the "Illinois Pocket Companion."

Problems

(Use the data of Table I unless otherwise specified.)

1. Find the total allowable load, in compression parallel to the grain, which may be applied to a 4-in. by 6-in. short block of southern yellow pine. *Ans.* 21,120 lb.
2. What must be the dimensions of a cubical block of white oak which supports a load of 50,000 lb? (Two solutions.)
3. An eyebar of structural steel, 1 in. thick, exerts a pull of 60,000 lb. What is its minimum width?
4. A piece of 6-in. wrought-iron water pipe is 2 ft. long and 6⅝ in. outside diameter. What is the allowable load on the pipe standing on end? *Ans.* 66,970 lb.
5. A yellow-pine beam 8 in. wide rests on the end of the pipe of Problem 4. A steel plate, 10 in. square, transmits the load from the beam to the pipe. Find the allowable load. *Ans.* 26,000 lb.
6. Solve Problem 5 if the beam is made of white oak.
7. In Problem 5, the steel plate is 8 in. wide. What must be its length in order that the pipe may carry its full allowable load?
8. A 1-in. steel bolt supports a load by means of a nut. What is the allowable load? (Use handbook.) *Ans.* 9,918 lb.
9. In Problem 8, what is the bearing stress on the nut when the bolt carries its allowable load? *Ans.* 6,772 lb./in.^{2} for square nut.
10. The bolt of Problem 8 runs vertically through an oak beam. Find the diameter of the washer required. *Ans.* 5⅛ in.
11. A short oak block, 12 in. square, stands on a square steel plate, which is supported by a pier of 1:3:6 concrete. What must be the dimensions of the plate and the size of the pier in order that the timber and concrete may reach their allowable stress at the same load? *Ans.* 384 sq. in.^{2} Use plate 20 in. square.
12. The pier of Problem 11 is a frustum of a pyramid, 20 in. square at the top and 5 ft. high. The concrete weighs 144 lb. per cu. ft. The soil which supports the pier has a bearing strength of 2.4 tons per sq. ft. What should be the dimensions of the base if the weight of the pier is neglected? *Ans.* Area = 24 sq. ft.; 4 ft. 11 in. square, nearly.
13. If the pier of Problem 12 were made 5 ft. square at the base, would the additional square foot of area be sufficient to support the weight of the concrete? Would a base 62 in. square be large enough?
14. *Assuming* that the bearing strength of the soil is accurately known, what is the minimum area of the base? *Ans.* 61 in. square.
15. What is the allowable load in tension on a steel rod which is 5 ft. 6 in. long and weighs 70 lb.?

7. Deformation; Unit Deformation.—The changes in dimensions which occur when forces are applied to a body are called *deformations.* In Fig. 1, the increase in length a, which takes place when the first load is applied, is the deformation caused by that load; the increase b is the deformation caused by the second

load; and $a + b$ is the deformation caused by the two loads. The deformation produced by a *tensile* force or *pull* is an *elongation.* The deformation produced by a *compressive* force or *push* is a *compression.* Compression is negative elongation. A deformation which remains after the force is removed is called a *set.*

Unit deformation in a body is the deformation per unit length. In a bar of uniform cross section, the unit deformation is calculated by dividing the total deformation of a given portion of the bar by the original length of the portion. In Fig. 1, the length a divided by the original length of the band is the unit deformation caused by the first load. Unit deformation is frequently called *relative deformation.*

In algebraic equations many authors represent unit deformation by the letter δ (delta).

Deformation is frequently called *strain.* The word *strain* was formerly used as a synonym for *stress* and is still sometimes heard in that sense. The general practice of technical literature, however, is now to use *strain* to mean *deformation.* When employed in this book, it will always have that meaning. Total deformation in a length l sometimes is represented by e. Unit deformation is then

$$\delta = \frac{e}{l}. \qquad \text{Formula II}$$

Problems

1. When a steel bar is subjected to a tensile stress, a portion, originally 8 in. long, is stretched 0.0052 in. Find the unit elongation. *Ans.* 0.00065.
2. An oak post under compression is shortened 0.1476 in. in a length of 15 ft. Find the unit deformation. *Ans.* 0.00082.
3. A ⅞-in. steel rod, 20 in. long, is subjected to a pull of 15,176 lb. A portion of the rod, originally 8 in. long, is stretched 0.0054 in. when the force is applied. Find the unit stress and the unit deformation.
4. The coefficient of expansion of steel is 0.000012 for 1°C. Find the unit deformation and the total deformation in a steel rod 15 ft. long when the temperature changes from 50°C. to 20°C. Solve when the temperature changes from 14°F. to 14°C. *Ans.* 0.000288; 0.05184 in.

8. Elastic Limit.—When a force is applied to a solid body and then removed, the body returns to its original size and shape, provided the unit stress developed by the force has not exceeded a certain limit. If the stress has gone beyond this limit, the body does not return entirely to its original dimensions but retains

some permanent deformation or *set*. The *unit stress* at this limit is called the *elastic limit* of the material. A soft-steel rod may be stretched 0.0054 inch in a gage length of 8 inches by a pull of 20,000 pounds per square inch. When this load is removed, the rod shortens to its original length. A pull of 30,000 pounds per square inch may stretch this rod 0.0081 inch, and the rod may return to its original length when the load is removed. A load of 32,000 pounds per square inch may stretch the rod 0.0200 inch. When this load is removed, the rod may have an elongation of 0.0110 inch. The rod shortens about 0.0090 inch, while the remaining elongation of 0.0110 inch persists as a permanent set. Evidently, the elastic limit is between 30,000 pounds per square inch and 32,000 pounds per square inch.

It is difficult to determine the elastic limit with exactness. A test piece may appear to have no residual deformation when measured with the usual apparatus and still show some set when more delicate instruments are employed. Time is a factor. If a load is applied for a considerable period, it causes somewhat greater deformation and considerably greater set than it would cause if the time of application were shorter. Some materials, such as steel, after having been subjected to comparatively large unit stress, frequently show a set of more or less temporary character. When the load is first removed, there is a residual deformation, which may partly or wholly vanish after some little interval.

9. Modulus of Elasticity.—For all stresses below the elastic limit, the ratio of the unit stress to the unit deformation is *nearly* constant. The quotient obtained by dividing any given change of unit stress by the accompanying change in unit deformation is called the *modulus of elasticity* or *Young's* modulus. Modulus of elasticity is represented in physical equations by the letter E. In algebraic language, the definition of the modulus of elasticity is

$$E = \frac{s}{\delta} \qquad \text{Formula III}$$

in which E is the modulus of elasticity, s represents a change in the unit stress, and δ is the change in unit deformation which accompanies this change of unit stress.

Problems

1. A 2-in. by 1.5-in. bar is tested in tension. When the load changed from 3,000 lb. to 48,000 lb., the dial reading for a gage length of 8 in. changed

from 0.00080 in. to 0.00492 in. Find the change in unit stress, the change in unit deformation, and the modulus of elasticity.

Ans. $E = 29{,}130{,}000$ lb./in.^{2}

2. A steel rod, 0.600 in. in diameter, is stretched 0.00536 in. in a gage length of 8 in. when the load changed from 1,415 lb. to 7,075 lb. Using the area to three significant figures, find the modulus of elasticity.

Ans. $E = 29{,}850{,}000$ lb./in.^{2}

3. A timber piece 2 in. square is shortened 0.014 in. in a length of 20 in. Find the force required, if the modulus of elasticity is 2,000,000 lb. per sq. in. Solve without writing. How does the unit stress compare with the allowable compressive stress for southern pine?
4. A spruce stick, 1.745 in. by 1.756 in., tested at the Bureau of Standards, was 25.25 in. in length. Deformations were measured on a 20-in. gage length. Some readings were

Total Load, Pounds	Average of Two Gages, Inches
1,224	0.00223
1,836	0.00424
4,284	0.01365
4,896	0.01620

Calculate the area to two decimal places. Find E from first and third readings and also from second and fourth readings.

Ans. 1,751,000; 1,672,000.

5. A 10-in. 30-lb. standard channel, 10 ft. long, is subjected to a compressive load of 88,000 lb. parallel to its length. How much is the channel shortened if $E = 29{,}300{,}000$? *Ans.* 0.0410 in.
6. A wrought-iron column, tested at Watertown Arsenal, was 11.31 sq. in. in cross section. When the load was changed from 5,000 lb. to 100,000 lb., the column was shortened 0.0610 in. in a length of 200 in. Find E for this wrought iron.
7. In a tension test of cast iron at the Watertown Arsenal, an increase of unit stress from 1,000 lb. per sq. in. to 6,000 lb. per sq. in. produced an increase in length of 0.0034 in. in a gage length of 10 in. Find E for this cast iron.

Table II gives values of the modulus of elasticity of a few common materials. This table should be memorized.

TABLE II.—MODULUS OF ELASTICITY

Material	Modulus, Pounds per Square Inch
Structural steel	29,000,000
Hard steel	30,000,000
Wrought iron	27,000,000
Cast iron	15,000,000
Timber (parallel to the grain)	1,000,000 to 2,800,000
Portland-cement concrete	2,000,000 to 4,000,000

8. A bar of cast iron, 3 in. by 1 in., is shortened 0.0056 in. in a length of 8 in. Find the load applied. *Ans.* 31,500 lb.
9. A structural steel bar is 5 ft. long and weighs 68 lb. How much will a pull of 60,000 lb. elongate it? *Ans.* 0.03103 in.
10. A 12-in. 35-lb. standard I-beam, 6 ft. long, is subjected to a load which shortens it $\frac{1}{16}$ in. Find the load.
11. The temperature coefficient of steel is 0.0000067 per degree Fahrenheit. What is the unit tensile stress developed in a structural steel rod when the temperature falls from 90°F. to 20°F., if the rod is not allowed to shorten?

10. Physical Meaning of E.—Formula III of Art. 9 may be written

$$\delta = \frac{s}{E}.$$

When s is made equal to unity, δ becomes equal to $\frac{1}{E}$. With the common engineering units, the reciprocal of E is numerically equal to the unit deformation produced by a unit load of 1 pound per square inch.

The modulus of elasticity of steel in tension is about 30,000,000 pounds per square inch. This means that a pull of 1 pound on a bar 1 inch square will stretch every inch of this bar one thirty-millionth of an inch. A pull of 1,000 pounds applied to a bar 1 inch square will stretch every inch of its length one thirty-thousandth of an inch. If the elastic limit is not exceeded, a pull of 30,000 pounds per square inch of cross section will stretch each inch of the bar one-thousandth of an inch.

Examples

(*Solve without writing.*)

1. A rod of machine steel, for which E is 30,000,000, is 1 in. square and 40 in. long. What is the unit deformation caused by a pull of 15,000 lb.? How much is 30 in. of the rod stretched?
Ans. One two-thousandth of an inch; 0.015 in.
2. If wood having a modulus of 1,200,000 lb. per sq. in. is subjected to a load of 1,200 lb. per sq. in., what is the compression per inch of length? What is the total compression in a length of 10 ft. 5 in.? *Ans.* $\frac{1}{8}$ in.
3. A 3-in. by 4-in. wooden block is subjected to a compressive load of 9,600 lb. If the modulus of elasticity is 1,600,000 lb. per sq. in., what is the deformation in a length of 5 ft.?
4. What is the unit tensile deformation in hard steel when the unit tensile stress is 18,000 lb. per sq. in.?

Also, Formula III of Art. 9 may be written

$$s = E\delta,$$

which defines E as the coefficient to be multiplied into the unit deformation to obtain the unit stress. A deformation of 0.001 inch per inch of length is generally not far from the elastic limit. Since many micrometers measure in thousandths of an inch, this length has a definite meaning to all persons who do exact mechanical work or make precise measurements. It is desirable, therefore, to fix the attention on the unit stress which accompanies a unit deformation of one one-thousandth of the original length or, expressed in a slightly different way, on the unit stress which accompanies a relative deformation of one-tenth of 1 per cent.

Examples

(*Solve without writing.*)

5. What is the unit stress in structural steel when the unit deformation is 0.001 in.?
6. What is the unit stress in cast iron when the unit deformation is 0.0008 in.?
7. A structural-steel rod is stretched 0.004 in. in a length of 8 in. What is the unit stress? What is the total pull if the rod is 2 in. square?
8. A hard-steel rod, 1 in. in diameter, is stretched 0.004 in. in a length of 12 in. Find the total force.
9. A bar of hard steel, which is 4 ft. long and weighs 27.2 lb., is shortened 0.032 in. What is the total compressive load?
10. A 6-in. by 10-in. timber post, 10 ft. long, is shortened 0.0600 in. If the modulus of elasticity is 1,400,000 lb. per sq. in., what is the load?

If δ is made equal to unity in Formula III (Art. 9), s becomes equal to E. From this relation the modulus of elasticity is sometimes defined as the unit stress which would double the length of a rod of uniform cross section, if such doubling were possible *without breaking the rod* or *exceeding the elastic limit.*

11. Work and Resilience.—When force acts on a body and the body or the point of the body at which the force is applied moves in the direction of the force, the force is said to do *work.* The distance which the point of application of the force moves is called the *displacement.* The *component* of the *displacement* in the direction of the force is the *effective displacement.* If the magnitude of the force is represented by P and the effective displacement is represented by x, work $= P \times x$. If the force is in pounds and the displacement is in feet, the work is expressed

in *foot-pounds*. If the force is not constant, the work is the product of the *average force* multiplied by the displacement. When an elastic body is deformed, the force varies directly as the displacement (provided the elastic limit is not exceeded) and the average force is half the sum of the initial and final forces.

Example I

A helical spring is stretched 1 in. by a load of 12 lb. What force will stretch the spring 3 in.? What is the average force for the elongation of 3 in.? What is the work done in stretching the spring 3 in.?

The force required to stretch the spring 3 in. is $P = 12 \times 3 = 36$ lb. The average force for first interval of 3 in. is $\frac{0 + 36}{2} = 18$ lb.

The average force is the force at the middle of the interval, which is an elongation of 1.5 in. Average force equal $1.5 \times 12 = 18$ lb. Work of displacement $= 18 \times 3 = 54$ in.-lb. $= 4.5$ ft.-lb.

Example II

After the spring of Example I has been stretched 3 in., an additional force is applied, which produces an additional elongation of 4 in. What is the additional force? What is the average force while the spring is stretched the last 4 in.? What is the work done in stretching the spring the last 4 in.?

Ans. Additional force $= 48$ lb.; average force $= 60$ lb.; work $= 20$ ft.-lb.

Problems

1. Find the work done in stretching the spring of Examples I and II from zero elongation to 7 in. elongation. Compare with the sum of the answers of the examples.
2. Find the work done in stretching the foregoing spring from 0 in. to 4 in. and then from 4 in. to 7 in. Check.
3. A load of 36,000 lb. is applied gradually to a steel rod which has no initial load. The elongation is 0.03 in. Find the work in foot-pounds.
Ans. 45 ft.-lb.
4. An additional load of 24,000 lb. is applied to the rod of Problem 3. If the stress does not exceed the elastic limit, find the additional work in foot-pounds. *Ans.* 80 ft.-lb.
5. Find the total work on the rod of Problem 3 when a load of 60,000 lb. is gradually applied. Check.
6. The rod of Problem 3 is 2 in. square and the modulus of elasticity is 30,000,000 lb. per sq. in. What is the length? What is the elastic energy per cubic inch when the total load is 60,000 lb.?
Ans. 3.75 in.-lb./in.^{3}

12. Modulus of Resilience.—The work expended in deforming *unit volume* of any solid to the *elastic limit* is called the *modulus of resilience* of the material. It is the *elastic potential energy*

of unit volume when stressed to the elastic limit. The modulus of resilience is a measure of the amount of elastic energy which may be stored in unit volume of a given material and recovered as mechanical work without loss.

If a unit cube of a solid is subjected to unit stress s, the deformation is $\frac{s}{E}$ and the total work is

$$U = \frac{s}{2} \times \frac{s}{E} = \frac{s^2}{2\,E}, \qquad \text{Formula IV}$$

in which U is the work done in deforming the volume or the stored potential energy.

Formula IV gives the elastic energy at any stress *below* the elastic limit. For the particular value of s at the elastic limit, the expression represents the *modulus* of *resilience*.

When s and E are given in pounds per square inch, Formula IV gives energy in inch-pounds per cubic inch.

If all parts of a solid body are subjected to the same unit stress s, the total elastic energy is obtained by multiplying the total volume of the body by the energy per unit volume.

The increase in energy per unit volume when the unit stress changes from s_1 to s_2 is expressed by the equation

$$U_2 - U_1 = \frac{s_2^2 - s_1^2}{2\,E}, \tag{1}$$

in which $\frac{s_2^2}{2\,E}$ is the total energy when the unit stress is s_2 and $\frac{s_1^2}{2\,E}$ is the total energy when the unit stress is s_1. The difference as given by Equation (1) represents the work done in changing the stress of unit volume from s_1 to s_2.

Equation (1) may be derived directly from the definition of work without reference to Formula IV. The average force is $\frac{s_2 + s_1}{2}$, the change in unit deformation is $\frac{s_2 - s_1}{E}$,

$$\frac{s_2 + s_1}{2} \times \frac{s_2 - s_1}{E} = \frac{s_2^2 - s_1^2}{2\,E}. \tag{2}$$

When the stress is uniform throughout the body, the total change of elastic energy when the unit stress changes from

s_1 to s_2 is

$$\frac{s_2^2 - s_1^2}{2E} \times \text{volume.} \tag{3}$$

Problems

1. Find the modulus of resilience for steel having a modulus of elasticity of 30,000,000 lb. per sq. in. and an elastic limit of 60,000 lb. per sq. in. *Ans.* 60 in.-lb./in.^{3}.
2. How high will the elastic energy of a block of steel having the constants of Problem 1 lift its own weight?
3. What is the modulus of resilience of timber for which the modulus of elasticity is 1,200,000 lb. per sq. in. and the elastic limit is 3,600 lb. per sq. in.? How high will the elastic energy lift its own weight if the density of this timber is 36 lb. per cu. ft.?
4. Timber having an elastic limit of 4,800 lb. per sq. in. has a modulus of resilience of 1.6 in.-lb. per cu. in. Find E.
5. A piece of timber, 2 in. square and 5 ft. long, is shortened 0.096 in. in a length of 4 ft. when the load is changed from 4,800 lb. to 12,800 lb. Find the total work in the gage length. *Ans.* U = 844.8 in.-lb.
6. From the answer of Problem 5, calculate the work per cubic inch. Check by Eq. (1).
7. In Problem 4 of Art. 9, find the work done by the external load on the 20-in. gage length when the load changed from 1,224 lb. to 4,896 lb. Divide the result by the volume to get the work per unit volume. Check by Eq. (1).

The energy of resilience has been calculated by multiplying the average force by the deformation. With the stress constant throughout the body at any given load, and directly proportional to the unit deformation, the average stress is one-half the sum of the initial and final stresses. Under these conditions, the derivation of the energy equations is accomplished by elementary algebra. On the other hand, when the stress varies with the

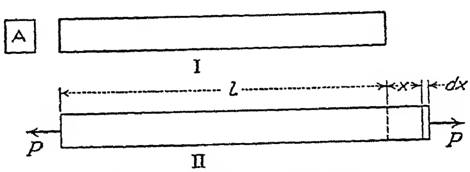


FIG. 7.—Stress at element.

position in the body, an integration with respect to space is necessary in order to determine the total energy change.

For the purpose of securing some practice in the integration of an elastic-energy problem which may be checked conveniently,

it is advisable to solve by calculus the elementary problem of uniform cross section under uniform tension or compression. Figure 7, I, shows a bar (or portion of a bar) of constant cross section A and initial length l. Figure 7, II, shows the same bar elongated a distance x by a pull P applied along the axis of the bar to make the stress constant at all parts of the cross section.

$$\delta = \frac{x}{l}; \quad s = \frac{E\,x}{l}; \quad P = \frac{A\,E\,x}{l}.$$

When the deformation changes by a small increment dx, the increment of work is

$$dU = P\,dx = \frac{A\,E}{l} \times dx. \tag{4}$$

Integrating:

$$U = \frac{A\,E}{l}\left[\frac{x^2}{2}\right]. \tag{5}$$

When x changes from x_1 to x_2, the limits in Equation (5) are x_1 and x_2;

$$U_2 - U_1 = \frac{A\,E}{l} \times \frac{x_2^2 - x_1^2}{2}. \tag{6}$$

Since

$$s = \frac{E\,x}{l}, \quad x = \frac{l\,s}{E} \quad \text{and} \quad x^2 = \frac{l^2 s^2}{E^2}.$$

From the corresponding values of x_2^2 and x_1^2, Equation (6) becomes

$$U_2 - U_1 = \frac{A\,l}{E} \times \frac{s_2^2 - s_1^2}{2} = \frac{s_2^2 - s_1^2}{2\,E} \times \text{volume}. \tag{7}$$

13. Poisson's Ratio.—A body subjected to tensile stress is elongated, and the magnitude of the elongation, provided the elastic limit is not exceeded, is proportional to the unit stress. At the same time the dimensions at right angles to the direction of the tensile stress become smaller. A body under compressive load is shortened in the direction of the load, while its transverse dimensions are increased. The ratio of the unit deformation at right angles to the direction of the load to the unit deformation in the direction of the load is called *Poisson's ratio*. Since this ratio is approximately 0.25 for some common materials, and

since very exact measurements are required for its determination, Poisson assumed that the value is always $\frac{1}{4}$. In reality, Poisson's ratio varies from below 0.15 for concrete to over 0.40 for hard rubber. For steel and steel alloys with high elastic limit, Poisson's ratio is about 0.26. If a bar of steel is elongated 0.001 of its original length by a tensile force, its transverse dimensions are reduced by about 0.00026 of their original value. For low-carbon steel, Poisson's ratio is less than 0.25 at ordinary temperatures.

Poisson's ratio is represented in this book by the Greek letter μ (mu).*

Problems

1. A steel rod 2 in. in diameter is stretched 0.0160 in. in a gage length of 20 in. Its diameter is reduced 0.00043 in. Find Poisson's ratio. *Ans.* 0.27.
2. If Poisson's ratio is 0.26 and the modulus of elasticity of structural steel is 29,000,000 lb. per sq. in., how much is the width of a 6-in. by 1-in. steel bar decreased by a pull of 121,800 lb.? *Ans.* 0.00109 in.
3. Poisson's ratio for copper is about ⅓ and the modulus of elasticity is 16,000,000. How much is the width of a plate, originally 8 in. wide and 0.162 inch thick, decreased by a pull of 12,960 lb.?
4. A rod of 0.49 per cent carbon steel tested by J. McLean Jasper† was 0.749 in. in diameter. When the load changed from 1,135 lb. to 6,784 lb., the unit longitudinal deformation changed 0.000421 and the unit transverse dimension changed 0.000099. Find E and Poisson's ratio from this test. *Ans.* $\mu = 0.235$.
5. A cylindrical core of concrete, tested by Dean A. N. Johnson,‡ was about 9 in. long, 4.5 in. in diameter, and twelve months old. When the load changed from 100 lb. per sq. in. to 900 lb. per sq. in., the unit longitudinal deformation changed from -0.000026 to -0.000286 and the unit transverse deformation changed from 0.000005 to 0.000040. Find E and Poisson's ratio. *Ans.* $E = 3{,}077{,}000$; $\mu = 0.134$.
6. A steel plate is subjected to a tensile stress of 12,000 lb. per sq. in. parallel to the X axis and a tensile stress of 8,400 lb. per sq. in. parallel to the

* There is no unanimity as to the symbol of Poisson's ratio. The fraction $\frac{1}{m}$ is the most common. A single letter for Poisson's ratio is better than a fraction. The equations are simpler and the numerical calculations are less laborious, especially since m is not an integer. The Greek letters ρ, σ, λ, and μ are used. The letter σ was employed in the earlier editions of this book. The change is made to μ because that letter seems to have the preference of writers who are deserting the common fraction $\frac{1}{m}$.

† *Trans. A.S.T.M.*, 1924, vol. 24, Part II, p. 1015.

‡ *Ibid.*, p. 1030.

Y axis. If $E = 30{,}000{,}000$ and Poisson's ratio is $\frac{1}{4}$, what is the unit deformation parallel to each axis?

	Unit Deformation
Ans. X	0.00033
Y	0.00018
Z	−0.00017

7. Solve Problem 6 if the unit stress parallel to the Y axis is compressive.

Ans. X	0.00047
Y	−0.00038
Z	−0.00003

14. Volume Change and Modulus of Elasticity.—When a solid is subjected to a load in one direction, there is a slight change in volume. The relative change in area of cross section at right angles to the load is smaller than the unit deformation in the direction of the load. Consequently, when the load is compressive, the volume is reduced; and when the load is tensile, the volume is increased.

Problems

1. A steel bar, 2 in. square and 10 in. long, is subjected to a compressive load of 96,000 lb. in the direction of its length. If $E = 30{,}000{,}000$ and Poisson's ratio is 0.27, what are the length, area of cross section, and volume when the load is on?
Ans. 9.992 in.; 4.001728 sq. in.; 39.98527 cu. in.
2. Find the work done by the load of Problem 1. Find the work per unit volume two ways. *Ans.* 32 ft.-lb.; 0.8 ft.-lb./in.^{3}.
3. A round rod, 2 in. in diameter and 20 in. long, has its diameter reduced 0.0005 in. and its volume increased 0.0228 cu. in. by a load of 78,540 lb. Find E and Poisson's ratio.

If a unit cube is elongated an amount δ by an external pull, its length becomes $1 + \delta$ and its transverse dimensions become $1 - \mu\delta$ in which μ is Poisson's ratio.

$$\text{Area of cross section} = (1 - \mu\delta)^2 = 1 - 2\mu\delta + (\mu\delta)^2. \quad (1)$$

Since $\mu\delta$ is small, being never greater than 0.001, its square, which is relatively much smaller, may be neglected; hence the approximate cross section is

$$A = 1 - 2\mu\delta. \quad (2)$$

Multiplication of area by length gives

$$\text{Volume} = (1 - 2\mu\delta)(1 + \delta) = 1 + (1 - 2\mu)\delta - 2\mu\delta^2, \quad (3)$$

of which the last term $2\mu\delta^2$ may be neglected.

$$\text{Approximate volume} = 1 + (1 - 2\mu)\delta. \tag{4}$$

After the original volume of one cubic unit has been subtracted from the approximate volume under tension, the remainder gives

$$\text{Increment of volume} = (1 - 2\mu)\delta. \tag{5}$$

These formulas apply only to temporary deformations below the elastic limit. The permanent deformations which occur when the elastic limit is exceeded produce practically no change of volume.

Problems

4. If the external force is compressive, show that

$$\text{Approximate volume} = 1 - (1 - 2\mu)\delta,$$

and

$$\text{Increment of volume} = -(1 - 2\mu)\delta.$$

5. A block of hard steel, originally 2 in. square and 10 in. long, is subjected to a load of 144,000 lb. parallel to its length. If $E = 30{,}000{,}000$ and Poisson's ratio is 0.27, what is the increment of cross section and the total increment of volume? *Ans.* 0.002592 sq. in.; −0.02208 cu. in.
6. Find the area and volume of the block of Problem 5 to eight decimal places by direct multiplication without the use of foregoing equations. Find the increments of area and volume and compare with answers of Problem 5.

A solid submerged in a liquid is under pressure from all directions. The quotient obtained when the unit pressure is divided by the relative reduction of volume is called the *modulus of volume elasticity*. If, for instance, 1 cubic inch of a solid is reduced to 0.9995 cubic inch by a pressure of 10,000 pounds per square inch in all directions, the modulus of volume elasticity is

$$E_v = \frac{10{,}000}{0.0005} = 20{,}000{,}000 \text{ pounds per square inch.}$$

Problem

7. A block of steel has its volume changed from 40.320 cu. in. to 40.200 cu. in. by a liquid pressure of 60,000 lb. per sq. in. Find the modulus of volume elasticity. *Ans.* $E_v = 20{,}160{,}000$.

The modulus of volume elasticity may be computed from the modulus of linear elasticity (Young's modulus) and Poisson's ratio. If a cube of unit dimensions is subjected to unit pressure s in the direction of any axis, it is shortened $\frac{s}{E}$ in the direction

of the pressure and elongated $\frac{\mu s}{E}$ along each of the two axes at right angles to the direction of the pressure. When there is a compressive stress s in every direction, the compression along any axis is made up of the direct compression $\frac{s}{E}$, which is due to the pressure in that direction, and two elongations, each of magnitude $\frac{\mu s}{E}$, which are due to pressures along the two axes at right angles with the first. (It is assumed that the body is *isotropic*, having the same properties in every direction, so that E and μ are the same for all axes.)

In any direction,

$$\text{Total compression} = \frac{s}{E} - \frac{2\,\mu s}{E} = \frac{s}{E}(1 - 2\mu). \tag{6}$$

The length of each edge of the cube becomes $1 - \frac{s}{E}(1 - 2\mu)$.

$$\text{Volume} = \left(1 - \frac{s}{E}(1 - 2\mu)\right)^3 = 1 - \frac{3s}{E}(1 - 2\mu) + \frac{3\,s^2}{E^2}(1 - 2\mu)^2 + \cdots \tag{7}$$

Since $\frac{s}{E}$ is very small, the terms containing the higher powers may be dropped and Equation (7) becomes

$$\text{Final volume} = 1 - \frac{3\,s}{E}(1 - 2\mu). \tag{8}$$

Since the original volume was unity, the decrease in volume is $\frac{3\,s}{E}(1 - 2\mu)$ which is also the unit change of volume or the unit volume deformation. The modulus of volume elasticity is obtained by dividing the unit stress s by the unit volume deformation.

$$E_v = \frac{s}{\frac{3\,s}{E}(1 - 2\mu)} = \frac{E}{3(1 - 2\mu)}. \tag{9}$$

Problems

8. If Poisson's ratio is ¼, show that the modulus of volume elasticity is two-thirds the modulus of linear elasticity.

9. What would be the modulus of volume elasticity if Poisson's ratio were $\frac{1}{2}$?

10. If $E = 15{,}500{,}000$ and $E_v = 10{,}200{,}000$, what is Poisson's ratio?

Ans. 0.255.

(*Article* 15 *may be omitted.*)

15. Biaxial Stresses.—Problems 6 and 7 of Art. 13 are examples of *biaxial* stresses. The loads are applied parallel to the X and Y axes, and the deformation along any axis depends upon the stress along each axis. The calculation of the deformations when the stresses are given is a simple matter. On the other hand, it is sometimes desirable to compute the stresses from the unit deformations.

When the loads are applied parallel to the X and Y axes,

$$\delta_x = \frac{1}{E}(s_x - \mu s_y); \tag{1}$$

$$\delta_y = \frac{1}{E}(s_y - \mu s_x), \tag{2}$$

in which δ_x and δ_y are the unit deformations along the X and Y axes, respectively.

When the unit deformations are known and the unit stresses are to be calculated, elimination of s_y between Equations (1) and (2) gives

$$s_x(1 - \mu^2) = E(\delta_x + \mu\delta_y). \tag{3}$$

$$s_x = E\,\frac{\delta_x + \mu\delta_y}{1 - \mu^2}. \tag{4}$$

By symmetry,

$$s_y = E\,\frac{\delta_y + \mu\delta_x}{1 - \mu^2}. \tag{5}$$

When $\delta_x = \delta_y$, Equations (3) and (4) become

$$s_x = s_y = E\,\frac{\delta_x}{1 - \mu}. \tag{6}$$

Problems

1. A steel plate is subjected to tension along the X and the Y axes. The unit deformation along each of these is 0.00037. If $E = 30{,}000{,}000$ and Poisson's ratio is 0.26, what is the unit stress along the X and Y axes, and the unit deformation along the Z axis?

Ans. $s_x = s_y = 15{,}000$ lb./in.^{2}; $\delta_z = -0.00026$.

2. The unit elongation in a steel plate along the X axis is 0.00054 and along the Y axis is 0.00036 while Poisson's ratio is 0.25. Solve.

Ans. $s_x = 20{,}160$ lb./in.^{2}; $s_y = 15{,}840$ lb./in.^{2}; $\delta_z = -0.00030$.

3. Solve Problem 2 if the deformation along the Y axis is negative.

Ans. $s_x = 14{,}400$ lb./in.^{2}; $s_y = -7{,}200$ lb./in.^{2}; $\delta_z = -0.00006$.

4. A rectangular steel plate is held in two directions by walls which yield slightly. The temperature coefficient is 0.000012 per degree centigrade. When the temperature is raised 100°C., a 2-in. gage length in the direction of the X axis expands 0.00060 in. and a 2-in. gage length in the direction

of the Y axis expands 0.00096 in. Find the unit stress along each axis if $E = 30{,}000{,}000$ and Poisson's ratio $= \frac{1}{4}$.

The expansion per inch without restraint would be 0.0012 in. along each axis. Since the actual unit elongation along the X axis is 0.00030 in., the deformation caused by the restraint is $\delta_x = 0.00120 - 0.00030 = 0.00090$. $\delta_y = 0.00120 - 0.00048 = 0.00072$.

$$s_x = \frac{0.00090 + 0.00018}{\frac{15}{16}} \times 3 \times 10^7 = 34{,}560 \text{ lb./in.}^2;\ s_y = 30{,}240 \text{ lb./in.}^2$$

(*Article* 16 *may be omitted.*)

16. Triaxial Stresses.—The theory of volume elasticity in Art. 14 involves *triaxial* stresses. Triaxial stresses occur in the walls of a tank subjected to internal liquid pressure. An *axial* tensile stress parallel to the length resists the opposite pressures on the heads, which tend to rupture the tank around any circumference. A *circumferential* tensile stress resists the opposite pressures on any two halves of the side walls, which tend to split the tank longitudinally. Near the inner surface a *radial* compressive stress resists the normal pressure of the liquid on the inner surface and the radial components of the circumferential stress in the material outside the element under consideration. The approximate calculations of *thin-walled* tanks and boiler tubes are simple (see Arts. 55 and 56) but the calculations of *relatively thick-walled* tanks involve the principles of triaxial loading. Similar problems occur in the calculation of the stresses in boiler tubes and in cylinders of internal-combustion engines, in which large relative deformations are caused by the temperature difference between the inner and outer surfaces.

In the theory of volume elasticity, the stresses are given and the deformations are easily calculated. In order to find the stresses from the unit deformation, the equations must be transformed to express unit stress explicitly in terms of the unit deformations.

$$\delta_x = \frac{1}{E}(s_x - \mu s_y - \mu s_z). \tag{1}$$

$$\delta_y = \frac{1}{E}(-\mu s_x + s_y - \mu s_z). \tag{2}$$

$$\delta_z = \frac{1}{E}(-\mu s_x - \mu s_y + s_z). \tag{3}$$

From Equations (1) and (2), eliminating s_z:

$$\delta_x - \delta_y = \frac{1}{E}\Big((1 + \mu)s_x - (1 + \mu)s_y\Big). \tag{4}$$

From Equations (2) and (3), eliminating s_z:

$$\delta_y + \mu\delta_z = \frac{1}{E}\Big(-(\mu + \mu^2)s_x + (1 - \mu^2)s_y\Big). \tag{5}$$

Multiplying Equation (4) by μ and adding to Equation (5) eliminates s_x.

$$\mu\delta_x + (1 - \mu)\delta_y + \mu\delta_z = \frac{1}{E}(1 - \mu - 2\mu^2)s_y, \tag{6}$$

$$s_y = E\frac{\mu\delta_x + (1-\mu)\delta_y + \mu\delta_z}{1-\mu-2\mu^2}. \quad (7)$$

By symmetry,

$$s_x = E\frac{(1-\mu)\delta_x + \mu\delta_y + \mu\delta_z}{1-\mu-2\mu^2}; \quad (8)$$

$$s_z = E\frac{\mu\delta_x + \mu\delta_y + (1-\mu)\delta_z}{1-\mu-2\mu^2}.$$

Problems

1. Material for which $E = 30{,}000{,}000$ and Poisson's ratio $= 0.24$ is stressed in three directions. The deformation in 8 in. is 0.008192 in. along the X axis and 0.002240 in. along the Y axis. The deformation along the Z axis is -0.001672 in. in a gage length of 2 in. Find the unit stress in each direction. *Ans.* $s_z = -15{,}000$ lb./in.^{2}
2. If s_z is known to be zero, find the values of s_x and s_y in terms of δ_x and δ_y and compare with Eq. (4) of Art. 15.
3. If s_z is known to be zero and the unit deformations along the X and Y axes are the same, find s_x and compare with Eq. (6) of Art. 15.

Miscellaneous Problems

1. A longleaf yellow-pine post, tested at Watertown Arsenal ("Tests of Metals," 1897, p. 415), was 9.79 in. by 9.81 in. The gage length was 50 in. When the total load changed from 19,210 lb. to 211,290 lb., the compression in the gage length increased from 0.0035 in. to 0.0460 in. Calculate the area to two decimal places. Find E and the total work in the gage length. Divide the total work by the volume to get the work per cubic inch. Check by Eq. (3) of Art. 12. *Ans.* $E = 2{,}353{,}000$; $U = 1.019$ in.-lb./in.^{3}
2. A second post of longleaf yellow pine ("Tests of Metals," 1897, p. 417) was 9.76 in. by 9.79 in. When the load changed from the initial value of 1,911 lb. to 191,100 lb., the measured compression in the gage length of 50 in. changed from 0 in. to 0.0568 in. Find E and the work per unit volume. Check. *Ans.* $E = 1{,}743{,}000$.
3. The post of Problem 1 was 119.78 in. long and weighed 330 lb. The post of Problem 2 was 118.45 in. long and weighed 284 lb. Find the weight of each per cubic foot. *Ans.* 49.5 lb./ft.^{3}; 43.3 lb./ft.^{3}
4. A stick of Douglas fir, tested in tension ("Tests of Metals," 1896, p. 405), was 24 ft. 2⅝ in. long, 8.12 in. wide and 3.02 in. thick. The stick weighed 163 lb. When the load changed from 2.450 lb. to 51,450 lb., a gage length of 200 in. elongated 0.1493 in. Find the area to the first decimal place and the weight per cubic foot. Find E and the total work in the gage length. Find work per cubic inch and check by Eq. (3) of Art. 12. *Ans.* $E = 2{,}679{,}000$; U per cu. in. $= 0.8212$ in.-lb.
5. A compression piece 60 in. long cut from the tension piece of Problem 4 was shortened 0.0294 in. in a gage length of 50 in. when the load changed from 2,450 lb. to 49,000 lb., and was shortened 0.0455 in. when the load changed from 2,450 lb. to 73,500 lb. Find E and the work per unit volume for the two large intervals. Check by Eq. (3) of Art. 12.
   Ans. $E = 3{,}231{,}000$ and $3{,}187{,}000$; $U = 0.6174$ and ? ("Test of Metals," 1896, p. 412.)

6. In Problem 5, when the load changed from 2,450 lb. to 49,000 lb., a transverse gage length of 7 in. was elongated 0.0022 in., and when the load changed from 2,450 lb. to 73,500 lb., the elongation was 0.0034 in. Find Poisson's ratio. *Ans.* $\mu = 0.534$.

(Answer seems too large, but wood is not isotropic.)

7. A stick of Douglas fir ("Tests of Metals," 1896, p. 407), tested in tension, was stretched 0.2614 in. in a gage length of 200 in. when the unit stress changed from 100 lb. per sq. in. to 2,400 lb. per sq. in., while a transverse gage length of 12 in. shortened 0.0063 in. Find E and Poisson's ratio. *Ans.* $E = 1{,}760{,}000$; $\mu = 0.40$.

8. A compression piece, cut from the stick of Problem 7, shortened 0.0471 in. in a gage length of 50 in. when the load changed from 100 lb. per sq. in. to 2,000 lb. per sq. in. Find E. A transverse gage length of 12 in. elongated 0.0064 in. for the same change in load. Find Poisson's ratio. *Ans.* $E = 2{,}170{,}000$; $\mu = 0.566$.

9. A white-oak post ("Tests of Metals," 1896, p. 425) was 106.1 in. long, had a cross section of 9.95 in. by 11.98 in., and weighed 415.5 lb. A 50-in. gage length was shortened 0.0260 in. when the load changed from 11,920 lb. to 119,200 lb. Find E. *Ans.* 1,731,000.

10. A short block 11.97 in. by 8.10 in. cut from the post of Problem 9 was loaded transversely, perpendicular to the growth rings. When the load changed from 1,939 lb. to 38,784 lb., the compression in a 6-in. gage length was 0.0118 in. Find the modulus of elasticity of this oak across the grain. *Ans.* $E = 193{,}000$ lb./in.^{2}

11. A paving brick, tested in compression lengthwise ("Tests of Metals," 1896, p. 359), was 8.14 in. by 2.46 in. by 4.18 in. When the load changed from 1,028 lb. to 102,800 lb., the compression in a gage length of 5 in. was 0.0071 in. and the elongation in a transverse gage length of 3.5 in. was 0.0008 in. Find E and Poisson's ratio.

Ans. $E = 6{,}970{,}000$; $\mu = 0.161$.

12. A steel rod, 0.798 in. in diameter, elongates 0.01040 in. in a gage length of 20 in. when its temperature is raised from 60°F. to 140°F. Tested in tension, the rod is stretched 0.0064 in. in a gage length of 8 in. when the pull changed from 200 lb. to 12,000 lb. Find E and the coefficient of linear expansion. The ends of the rod are fastened to a rigid frame and the temperature is lowered from 120°F. to 60°F. What is the increase in the total tension and in the unit tensile stress if the resistance of the frame entirely prevents the rod from contracting? Solve also if a 20-in. gage length shortens 0.0028 in. with the fall of temperature.

Ans. $E = 29{,}500{,}000$; temperature coefficient = 0.0000065 per degree Fahrenheit; 11,505 lb./in.^{2}; 5,752 lb.; 7,375 lb./in.^{2}; 3,687 lb.

13. A steel bar in the form of a frustum of a pyramid is 1 in. square at one end, 2 in. square at the other end, and 10 in. long. A load of 30,000 lb. is applied in compression. If E is 30,000,000 lb., per sq. in., and if it is assumed that the stress in any transverse section is uniform throughout the section, calculate the decrease in length by means of integral calculus. Compare with a uniform bar 1.5 in. square.

Ans. Total compression is 0.005 in.

14. By integration find the total internal work of the bar of Problem 13.

Ans. 75 in.-lb.

15. The total external work of the bar of Problem 13 is the product of the total compression multiplied by the average load. Solve for the external work by means of the answer of Problem 13 and compare with the internal work.

16. A bar 1 in. thick and 20 in. long is 1 in. wide at one end and increases uniformly to a width of 3 in. at the other end. If a load of 30,000 lb. in compression is applied to the bar, find the unit stress at a distance x from the small end. If E is 30,000,000 lb. per sq. in., find the unit deformation, and find the total deformation in the entire length.

Ans. 0.010986 in.

17. By integration solve Problem 16 for the total internal work and then check Problem 16 by means of the external work.

18. A plate of uniform thickness t has a breadth b at one end of a given length l and a breadth c at the other end. Find the expression for the elongation of this length l due to a pull P. *Ans.* $\dfrac{Pl}{Et(c-b)} \log \dfrac{c}{b}$.

19. An oak block, 6 in. square and 30 in. long, is bolted between two steel plates, each 6 in. wide, ½ in. thick, and 30 in. long. A force applied lengthwise the combined block shortens it 0.008 in. in a length of 20 in. If E for the steel is 29,000,000 lb. per sq. in. and E for the oak is 1,600,000 lb. per sq. in., what is the total force? *Ans.* 92,640 lb.

20. In Problem 19, what is the unit stress in the steel when the unit stress in the oak is at its allowable value in compression?

21. A vertical pier, 26 in. square, is made of concrete in which four 4-in. by 3-in. by ½-in. structural-steel angles are imbedded. The modulus of elasticity of the steel is fifteen times the modulus of the concrete. When the pier carries a load of 360,360 lb., how much of this load is carried by the concrete and how much is carried by the steel? What is the unit stress in each?

22. A hollow steel cylinder, 1 in. inside diameter and 2 in. outside diameter, is supported in a vertical position by three small lugs which do not appreciably change the cross section. A 1-in. wrought-iron bolt passes through this cylinder. A nut near the bottom of the bolt is turned up against the lower end of the cylinder until the tensile stress in the bolt reaches 10,000 lb. per sq. in. If E is 30,000,000 lb. per sq. in. for the steel and 27,000,000 lb. per sq. in. for the wrought iron, what is the unit stress in each? A load of 6,000 lb. is then hung on the lower end of the bolt and supported by a second nut which does not touch the nut in contact with the cylinder. If the deformation of the upper nut and of the part of the bolt which passes through it be neglected, and if the deformation of the head of the bolt also be neglected, what is the total tension and what is the unit tensile stress in the bolt if the supporting lugs are at the top of the hollow cylinder?

Ans. Total, 9,239 lb. tension in bolt.

23. Solve Problem 22 if the supporting lugs are at the bottom of the cylinder.

CHAPTER II

SHEAR

17. Shear and Shearing Stress.—When a body is subjected to a pair of forces which are in the *same line* and directed *away* from each other, *tensile* stress is produced. When the pair of forces are in the *same line* and directed *toward* each other, *compressive* stress is produced. If the forces are in *parallel lines* or *planes*, *shearing* and bending stresses are produced in the portion of the body between them. In Fig. 8, the block A is securely held by the clamp B and a horizontal force P is applied by a block C. The force P is parallel to the upper surface of B. The clamp B exerts a horizontal force on the block A. This force is equal and opposite to the force P. The portion of the block A between the upper surface of the clamp and the lower surface $E\ F\ G$ of the block C is subjected to a pair of equal, opposite, parallel forces. The material in this portion of the block is subjected to shearing and bending stresses. The shearing stresses depend upon the magnitude of the forces and the area of the section of A. The bending stresses depend upon these and also upon the distance of the forces apart. If the body C is brought very close to B, so that the distance between the two forces P and P' becomes negligible, the unit bending stress becomes small, while the unit shearing stress is unchanged. The average unit shearing stress is calculated by dividing the force P by the area of the cross section $E\ F\ G$ or the area of any section parallel to it.

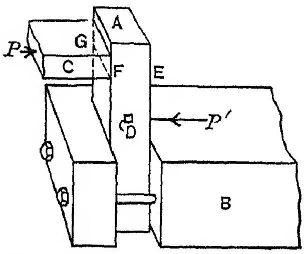


Fig. 8.—Shear and bending.

In tension and compression the unit stress is calculated by dividing the total force by the area of the cross section perpendicular to it. In shear, on the other hand, the unit stress is calculated by dividing the total force by the area of the cross section parallel to it.

In Fig. 8, as in all cases of application of force, the line P represents the resultant of a set of forces distributed over an

area. The resultant P' must fall some distance below the upper surface of B and the resultant P must lie above the lower surface of C. It is, therefore, not practicable to secure shearing stress entirely free from bending or compressive stress. It will be shown later that the distribution of shearing stress, when combined with bending, is not uniform over the section. At present, however, no account will be taken of this variation, and the average shearing stress will be calculated by dividing the total force by the area in shear.

Table III.—Allowable Unit Shearing Stress

Material	Pounds per square inch		
	A.R.E.A.	A.I.S.C.	A.S.M.E.
Steel web plates for girders	10,000	12,000	
Power-driven rivets	12,000	13,500	
Hand-driven rivets	9,000	10,000	
Turned bolts	9,000		
Turned bolts in reamed holes, clearance not over 0.02 inch	......	13,500	
Steel rivets in boilers	......	......	8,800
Timber parallel to grain:			
Douglas fir, coast region, select	......	90	
Oak, white or red, select	......	125	
Southern yellow pine, select	......	110	
Redwood, select	......	70	
Common grade—80 per cent of select			

Problems

(*Use A.R.E.A. Specifications for first three.*)

1. Two 3-in. by ½-in. plates are united by one ¾-in. power-driven rivet. What is the allowable load in shear? *Ans.* 5,300 lb.
2. One 3-in. by ½-in. plate is placed between two 3-in. by ⅜-in. plates and connected by one ⅞-in. hand-driven rivet which passes through all three plates. What is the allowable load in shear? *Ans.* 10,820 lb.
3. A 5-in. by 1-in. eyebar (see handbook) has one end between two plates. With 16,000 lb. per sq. in. as the allowable tensile unit stress in the eyebar, what is the minimum allowable diameter of the pin in double shear which connects it with the plates? *Ans.* $2\frac{3}{8}$ in.
4. A 2-in. by 4-in. yellow-pine block (Fig. 9) hung vertical and supported at the upper end, has a hole 1.2 in. square, which is perpendicular to the 4-in. faces. The lower edge of this hole is 4½ in. above the lower end of the block. If a load of 1,800 lb. is hung on a square bar passing through

this block, what is the unit shearing stress in the pine parallel to the grain? *Ans.* 100 lb./in.²

5. What is the maximum allowable load on the block of Fig. 9 when calculated for shear?

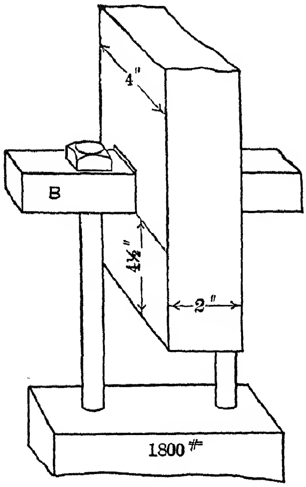


FIG. 9.—Shear in timber.

6. What is the unit tensile stress at the minimum section of Fig. 9 when the load is 1,800 lb.?

7. The head of a 1⅜-in. bolt is 1 in. high. A pull of 20,000 lb. applied to the bolt and resisted by the head tends to shear the head from the body of the bolt. Find the unit shearing stress. Find the unit tensile stress in the gross section of the bolt. If the force is applied at the opposite end of the bolt by means of a nut, find the unit tensile stress at the minimum section (see handbook). Which stress is above the allowable?
Ans. $s_s = 4{,}630$ lb./in.²; s_t 13,470 lb./in.²; $s_t = 18{,}972$ lb./in.²

8. The load of 20,000 lb. is applied to the bolt of Problem 7 by means of a nut. Find the unit shearing stress at the root of the threads. *Ans.* $s_s = 4{,}560$ lb./in.²

18. Shearing Deformation.—In Fig. 8, a small portion of section *D* extends through the block *A* with its long dimension perpendicular to the plane which contains the resultants *P* and *P'*. The cross section *D* is represented on a large scale by the rectangle *H I J K* of Fig. 10. When the shearing forces are applied as shown in Fig. 8, this rectangle is distorted to the form *H I'J'K*. If the lower line, *H K* be regarded as fixed, the total

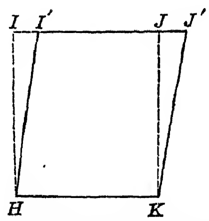


FIG. 10.—Shearing deformations.

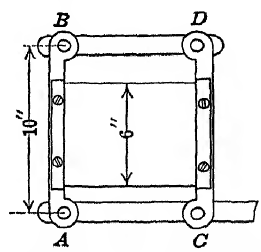


FIG. 11.—Device for illustrating shear.

displacement of any point in the upper line is *I I'* or *J J'*. The unit shearing deformation, which may be represented by δ_s, is the ratio of this horizontal displacement *I I'* to the vertical distance *H I*. In linear deformation, the unit deformation is obtained by dividing the total deformation by a length in the

same direction as the deformation; in shearing deformation, the displacement is divided by a distance at *right angles* to the *displacement.* The unit displacement is the tangent of the angle $I\ H\ I'$ or $J\ K\ J'$. The effect of the shearing forces is to lengthen the diagonal $H\ J$ and shorten the diagonal $I\ K$.

Problems

1. Two equal bars, $A\ B$ and $C\ D$ (Fig. 11) are hinged to a second pair of equal bars, $A\ C$ and $B\ D$, to form a parallelogram. A sheet of rubber, 6 in. wide, has one edge securely clamped to $A\ B$ and the other edge to $C\ D$. The length of $A\ B$, center to center of hinges, is 10 in. What is the unit shearing displacement when B is displaced 0.2 in. to the right of the vertical? *Ans.* Unit shear $\delta_s = 0.02$.
2. A shaft 4 in. in diameter is twisted 3° in a length of 10 ft. What is the total displacement of a point on the surface at one end if the other end is regarded as fixed? What is the unit displacement? *Ans.* 0.10472 in.; 0.000873.

19. Modulus of Elasticity in Shear.—The modulus of elasticity in shear is obtained by dividing the unit shearing stress by the unit shearing deformation, just as the modulus of elasticity in tension or compression is computed by dividing the unit tensile or compressive stress by the corresponding unit deformation.

$$E_s = \frac{s_s}{\delta_s}.$$

The modulus of shearing elasticity is frequently called the *modulus of rigidity.*

Forces applied as in Fig. 8 do not give pure shear. Even in Fig. 9, in which the plates which apply the parallel forces are as close together as possible, shear is combined with bending. Pure shear, free from bending or compression, may be secured by torsion, as in Problem 2 of Art. 18.

Problems

1. In Problem 2 of Art. 18, if E_s is 11,200,000 lb. per sq. in., what is the unit shearing stress at the surface of the shaft? *Ans.* $s_s = 9.778$ lb./in.^{2}
2. What is the maximum unit shearing deformation if the maximum allowable shearing stress is 10,000 lb. per sq. in. and the modulus of rigidity is 11,400,000 lb. per sq. in.? *Ans.* 0.000877.
3. A hollow shaft has an inside diameter of 4 in. and an outside diameter of 6 in. The shaft is twisted 0.02 radian in a length of 10 ft. What is the unit deformation at the inner surface? At the outer surface? What is the unit stress at each surface if the modulus of rigidity is 10,800,000 lb. per sq. in.? *Ans.* 3,600 lb./in.^{2} at inner surface.

20. Shear Caused by Compression or Tension.—Figure 12 shows a block subjected to a downward compressive force P in the direction of its length and an equal upward force at the bottom. This block may be supposed to be cut by a plane

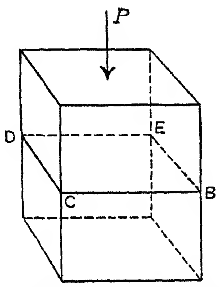


FIG. 12.—Section normal to force.

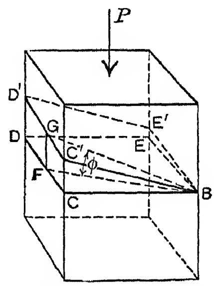


FIG. 13.—Section inclined to force.

$B\ C\ D\ E$, normal to its length, and then glued together. If the portion above this section be regarded as a free body in equilibrium, and if the weight of the portion be neglected, the downward force P must be equal to the upward reaction of the glued surface. If A is the area of the section, the unit compressive stress in the glue is given by

$$s_c = \frac{P}{A}. \tag{1}$$

Since the external force P has no horizontal component, the shearing force in the glue is zero. If the body were actually made of two portions, the upper portion would not slide on the lower portion, no matter how smooth the surface of contact.

Figure 13 represents a body similar to Fig. 12, loaded and supported in the same way. This body is assumed to be cut by a plane $B\ C'D'E'$, which makes an angle ϕ with the normal section. The portion above the inclined section may be taken as the free body, and the external force P may be resolved perpendicular and parallel to this plane. The component of P normal to the plane is $P \cos \phi$. The unit compressive stress is this component divided by the area of the section. If A is the area of the normal section, the area of the inclined section is $A \sec \phi$. The unit compressive stress is given by

$$s_c = \frac{P \cos \phi}{A \sec \phi} = \frac{P}{A} \cos^2 \phi = \frac{P}{2A}(1 + \cos 2\phi). \tag{2}$$

The component of the force P in the direction of the line $B\,G$, which makes the maximum angle with the normal plane, is $P \sin \phi$. This component is resisted by the shearing stress in the section $B\,C'D'E'$. The unit shearing stress is obtained by dividing the component parallel to the section by the area of the section.

$$s_s = \frac{P \sin \phi}{A \sec \phi} = \frac{P}{A} \sin \phi \cos \phi = \frac{P}{2A} \sin 2\phi. \tag{3}$$

If the body were in tension instead of compression, Equation (3) would still give the unit shearing stress in the section, and Equation (2) would give the unit tensile stress (instead of the unit compressive stress) normal to the section.

Problems

(*Use handbook for data not given in tables.*)

1. A 6-in. by 4-in. post is cut by a plane which makes an angle of 35° with the 6-in. faces and is normal to the 4-in. faces. What is the length of the intersection of this plane with the 4-in. faces? If a load of 10,800 lb. is placed on this post, what is the component parallel to this plane? What is the component perpendicular to this plane? What is the unit shearing stress along this plane? What is the unit compressive stress perpendicular to the plane?

Make sketch. Solve completely. Then check by Eqs. (2) and (3).

Ans. $s_c = 148.0$ lb./in.^{2}

2. Solve Problem 1 if the plane makes an angle of 55° with the 6-in. faces.

3. Show from Eqs. (2) and (3) that the shearing stress is zero and the compressive stress is a maximum when $\phi = 0$. Explain from your sketch.

4. A 6-in. by 6-in. post is subjected to a load of 10,800 lb. in the direction of its length. Find the unit shearing stress and the unit compressive stress with respect to a plane which makes an angle of 25° with the normal section.

5. Solve Problem 4 if the plane makes an angle of 65° with the normal section.

6. Solve Problem 4 without using Eqs. (2) and (3). Draw sketch with inclined plane normal to one 6-in. face.

7. What is the area of the section in Problem 4, and what is the area in Problem 5?

8. The grain of a 4-in. by 4-in. short post of common southern yellow pine makes an angle of 12° with the length. Find the total safe load. Find the total safe load if the grain makes an angle of 5° with the length.

Ans. 6,924 lb.; 14,080 lb.

9. Show that the unit shearing stress produced by a single tensile or compressive load is a maximum at 45° with the direction of the load, and that

this maximum shearing stress is one-half the unit tensile or compressive stress which produces it.

21. Shearing Forces in Pairs.—If a body is subjected to pure shearing stress (with no tension or compression except that which is due to shear), there must be two sets of shearing forces to secure equilibrium and the unit shearing stresses which these forces produce must be the same in both. Figure 14 represents a rectangular block A B with two other blocks C and D glued to

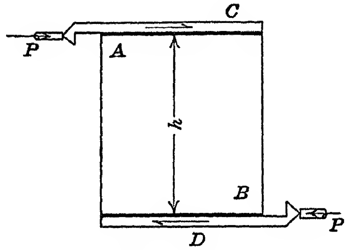


Fig. 14.—Pair of shearing forces.

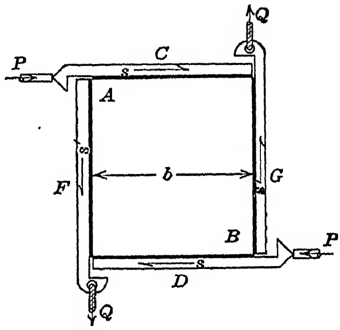


Fig. 15.—Two pairs of shearing forces.

the top and bottom, respectively. There is a horizontal force P, toward the right, acting on the block C and an equal and opposite force acting on the block D. These two forces form a couple tending to rotate the system in a clockwise direction. To produce equilibrium, a block F is glued to the left vertical face of AB (Fig. 15) and a block G is glued to the right vertical face. A downward force Q is applied to F and an equal upward force is applied to G. The breadth of A B is b and its height is h. Equilibrium will occur when the moments of the two couples are equal, *i.e.*, when

$$P\,h = Q\,b. \tag{1}$$

The force is transmitted from C and D to A B as a horizontal shear in the glue. Shearing stress is represented by an arrow with a single barb. The arrow in C, with barb above the shaft, represents the shearing stress from C to A B. If it were desired to represent the opposite shearing stress from A B to C, the arrow would be placed in A B, would point toward the left, and would have the barb below the shaft.

If l is the length of the block A B perpendicular to the plane of the paper, the top and bottom surfaces each have an area $b\,l$, and

$$P = s\,b\,l, \tag{2}$$

in which s is the unit horizontal shearing stress.

The area of each vertical face perpendicular to the plane of the paper is $h\,l$ and

$$Q = s'h\,l, \tag{3}$$

in which s' is the unit vertical shearing stress.

Since

$$P\,h = Q\,b,$$
$$s\,b\,l\,h = s'h\,l\,b, \tag{4}$$
$$s = s'. \qquad \text{Formula V}$$

Formula V applies to any portion of block $A\,B$ cut out by horizontal and vertical planes perpendicular to the plane of the paper. Figure 16 represents one such block.

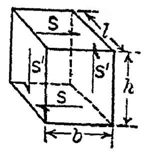


FIG. 16.—Equilibrium in shear.

Frequently, tensile or compressive stresses occur along with shearing stresses. Figure 17 represents a block which is supposed to be glued to the base and pushed toward the right by a force P applied near the top. To the left of the middle the glue is in tension; to the right of the middle it is in compression. All of the glue is in shear. A portion A of the block is in tension and shear, and a portion C is in compression and shear. The portion B at the middle is in shear only. The direction of the shear in A and C, for which the arrows are not shown, is the same as in B.

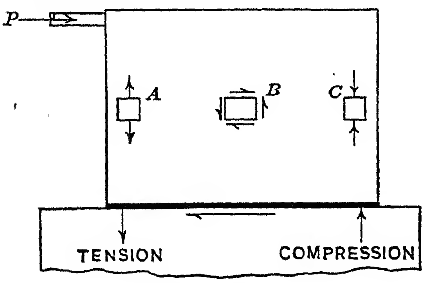


FIG. 17.—Shear with tension and compression.

If the tension in A is not the same at the top and bottom, the vertical shearing stress will not be exactly equal on the two sides. Ordinarily, if A is small, the difference is slight.

For a block of infinitesimal dimensions, the shearing stresses are practically equal on all sides, even if tensile or compressive forces exist in the body. A combination of shear with other stresses is considered at greater length in Chapter XVII.

22. Compressive and Tensile Stress Caused by Shear.—Figure 18, I, represents a rectangular parallelepiped of breadth b, height h, and length l, subjected to *pure* shearing stress. The shearing stress acts toward the right parallel to the breadth

at the top and toward the left at the bottom. As shown in Art. 21, there is also a shearing stress of the same intensity at the left surface acting downward and an equal shearing stress at the right surface acting upward. (If the direction of one of these shears is reversed, they must all be reversed to produce equilibrium.) Now consider the parallelepiped divided by the inclined plane containing the edges $C\,D$ and $G\,F$, and treat the triangular

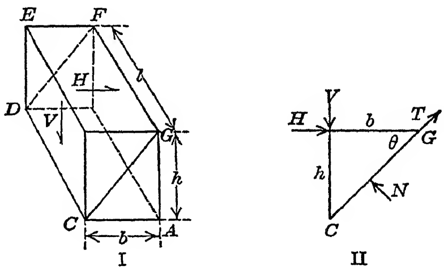


FIG. 18.—Shear causing compression.

prism to the left of this plane as a free body in equilibrium under the action of the forces at its surface. These forces are four in number: the shearing force H in the upper surface acting toward the right, the shearing force V in the left vertical surface acting downward, the compressive force N acting normal to the inclined surface (Fig. 18, II, which represents all the forces in the plane of the paper), and a shearing force T along this surface parallel to the diagonal line $C\,G$. If s_s is the intensity of the horizontal and vertical shear,

$$H = s_s b\,l, \quad V = s_s h\,l.$$

Resolving normal to the inclined plane,

$$N = H \sin\theta + V \cos\theta, \tag{1}$$

$$N = s_s b\,l \sin\theta + s_s h\,l \cos\theta, \tag{2}$$

in which θ is the angle which the inclined plane makes with the horizontal surface. The unit compressive stress on the inclined surface is obtained by dividing N by the area of this surface. If c is the length of the hypotenuse $C\,G$, the area of the inclined surface is $c\,l$. Dividing Equation (2) by $c\,l$,

$$\frac{N}{\text{Area}} = \frac{N}{c\,l} = \frac{s_s b \sin\theta}{c} + \frac{s_s h \cos\theta}{c}. \tag{3}$$

Since $\cos\theta = \frac{b}{c}$ and $\sin\theta = \frac{h}{c}$,

$$s_c = 2\,s_s \sin\theta\cos\theta = s_s \sin 2\theta. \tag{4}$$

When θ is 45 degrees, the compressive stress is a maximum and is equal to the shearing stress.

$$s_c = s_s \qquad \text{Formula VI}$$

If the plane which bisects the parallelepiped is taken parallel to $C\,D$ through the corners A and E, an expression similar to Equation (4) may be derived for the tensile stress. The maximum tensile stress is at 45 degrees with the shearing stress and at right angles to the maximum compressive stress. When a body is subjected to pure shear, there is a compressive stress of equal intensity at an angle of 45 degrees with the planes of the shearing stress in one direction and a tensile stress of the same intensity at an angle of 45 degrees in the opposite direction. These are shown in Fig. 19. With the shearing toward the left at the bottom, as indicated by the arrow, the maximum tensile stress is normal to the plane which makes an angle of 45 degrees to the left of the vertical upward, and the maximum compressive stress is normal to the plane which makes an angle of 45 degrees to the right of the vertical upward.

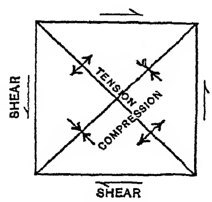


FIG. 19.—Tension, compression, and shear.

Problems

1. The unit shearing stress in a block is 200 lb. per sq. in. and is directed toward the right at the bottom. Find the unit compressive stress across a plane which makes an angle of 25° with the horizontal toward the right. *Ans.* 153.6 lb./in.² tensile stress.
2. A block 8 in. long, 6 in. wide, and 5 in. high is glued to a horizontal surface at the bottom. A horizontal force of 1,152 lb. parallel to the length is applied near the top. Find the unit compressive stress caused by shear across a plane which makes an angle of 30° with the horizontal toward the right.
3. A block is subjected to a horizontal and vertical shearing stress of intensity s_s. The resultant tensile stress across a plane which makes an angle of 17° with the horizontal toward the left is 100.66 lb. per sq. in. Find s_s.
4. A block is 15 in. long and 8 in. high. It is subjected to shearing stress of 225 lb. per sq. in. on the ends and on the horizontal surfaces. Find the tensile or compressive stress across a plane which is perpendicular to the plane of the front face and passes through the diagonal of this face. Solve without using Formula VI with one-half of the block as a free body.
5. A tensile stress of 221 lb. per sq. in. is applied at the ends of the block of Problem 4 parallel to the length. Find the shearing stress along the diagonal of the front face. Solve without using the derived equations. What is the tensile stress normal to this diagonal?

6. Solve Problem 1 by moments. Use a triangle similar to Fig. 18, II, with arrows to suit the problem.

23. Relation of Shearing to Linear Elasticity.—The modulus of shearing elasticity may be calculated from the modulus in tension or compression if Poisson's ratio is known.

Figure 20 is the front elevation of a block of square section subjected to shearing forces. The unit shearing displacement is the tangent of the angle θ between the lines $H\,I$ and $H\,I'$ of Fig. 20, II. In Fig. 20, III, the figure has been rotated an amount equal to one-half the angle of shear. In this position,

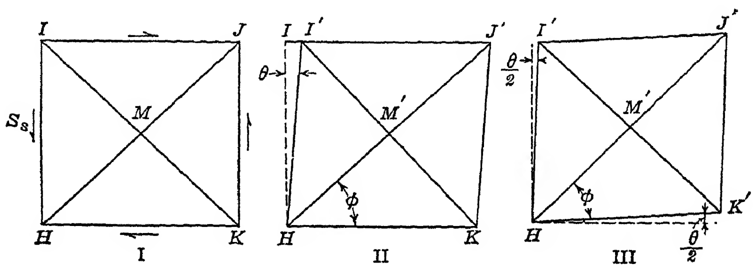


FIG. 20.—Shearing deformation.

the diagonals make angles of 45 degrees with the horizontal. The lengths of the diagonals are changed, while their directions remain the same as in Fig. 20, I. This kind of deformation is called *pure shear*. The deformation shown in Fig. 20, II, in which both length and direction of the diagonals are changed, while the direction of one pair of faces is unchanged, is called *simple shear*.

To find the ratio of shearing deformation to linear deformation, it is necessary to find the relation of the diagonals $H\,J'$ and $I'K'$ of Fig. 20, III, to the angle $\frac{\theta}{2}$. The shearing forces lengthen the diagonal $H\,J$ to $H\,J'$ and shorten the diagonal $I\,K$ to $I'K'$. The half diagonals $H\,M$ and $M\,K$ suffer the same relative deformation.

If δ is the unit deformation caused by unit tensile stress s, the unit elongation along $H\,J$ is $\delta(1 + \mu)$. This elongation is made up of the elongation δ caused by the tensile stress along this diagonal and an elongation $\mu\,\delta$ caused by the compressive stress along the diagonal $I\,K$. In a similar manner, the unit compression along the diagonal $I\,K$ is found to be $\delta(1 + \mu)$.

If Fig, 20, III, the angle $\frac{\theta}{2}$ is the difference between 45 degrees

and the angle ϕ. The tangent of ϕ is the ratio of the half diagonal $M'K'$ to the half diagonal $H\,M'$

$$\tan \phi = \frac{\text{length of } M'K'}{\text{length of } H\,M'} = \frac{\text{length of } M\,K\,[1 - \delta\,(1 + \mu)]}{\text{length of } H\,M\,[1 + \delta\,(1 + \mu)]}, \quad (1)$$

$$\tan \phi = \frac{1 - \delta(1 + \mu)}{1 + \delta(1 + \mu)}. \quad (2)$$

Since $$\frac{\theta}{2} = 45° - \phi,$$

$$\tan \frac{\theta}{2} = \frac{1 - \tan \phi}{1 + \tan \phi} = \frac{1 + \delta(1 + \mu) - 1 + \delta(1 + \mu)}{1 + \delta(1 + \mu) + 1 - \delta(1 + \mu)} = \delta(1 + \mu). \quad (3)$$

For a small angle, $\tan \theta = 2 \tan \frac{\theta}{2} = 2\,\delta(1 + \mu). \quad (4)$

Since $$\delta = \frac{s}{E}, \tan \theta = \frac{2\,s(1 + \mu)}{E},$$

$$E_s = \frac{s_s}{\delta_s} = \frac{s_s}{\tan \theta} = \frac{s_s E}{2\,s(1 + \mu)}. \quad (5)$$

At 45 degrees the unit tensile and the unit compressive stress caused by shear are each equal to the unit shearing stress; therefore, $s = s_s$ and Equation (5) becomes

$$E_s = \frac{E}{2(1 + \mu)}. \quad (6)$$

Problems

1. If Poisson's ratio is $\frac{1}{4}$, show that $E_s = \frac{2\,E}{5}$.
2. If the modulus of elasticity of steel is 29,800,000 lb. per sq. in. and Poisson's ratio is 0.275, what is the modulus of rigidity? *Ans.* $E_s = 11{,}690{,}000$ lb./in.²
3. Find Poisson's ratio if the modulus of rigidity is 11,500,000 and the modulus of elasticity in tension is 29,300,000 lb. per sq. in.
4. Landolt and Börnstein give the following values, in kilograms per square millimeter, for cast steel: $E = 20{,}400$, $E_s = 8{,}070$, $E_v = 14{,}600$. Find Poisson's ratio from Eq. (6) and also from Eq. (9) of Art. 14.

Miscellaneous Problems

1. Figure 21 shows a block which is subjected to horizontal tension and vertical compression. If the unit tensile stress is 240 lb. per sq. in. and the unit compressive stress is 200 lb. per sq. in., what is the unit shearing stress along the diagonal? *Ans.* $s_s = 220$ lb./in.²
2. A rectangular block is 15 in. wide, 8 in. high, and 4 in. long. A horizontal compression force of 27,744 lb. in the direction of its width is applied to the sides. A vertical tension of 34,680 lb. is applied to the top and

bottom surfaces. Find the unit shearing stress along a plane which passes through the lower left edge and the upper right edge. Solve without using formula. *Ans.* $s_s = 600$ lb./in.^{2}

3. In Fig. 22, the block *A B C* is 8 in. long perpendicular to the plane of the paper. Find the unit shearing stress and the unit compressive stress in the glue which fastens the block to its base.

4. In Fig. 23, *A* and *B* are short compression members or struts of yellow pine, joined together by a bolt or pin at the top. The lower ends are set

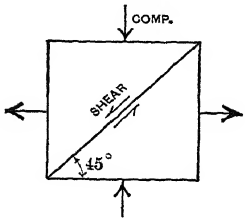


FIG. 21.—Shear caused by tension and compression.

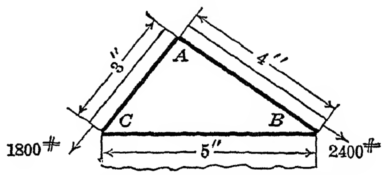


FIG. 22.—Stress due to shear.

in notches in the bottom chord *C*. If the load *P* is 7,200 lb., what is the unit compressive stress in *A* and *B*? What is the maximum unit tensile stress in *C*? What must be the length of the section *d* to avoid shearing if *C* is made of yellow pine? What must be the length of *d* if *C* is made of oak?

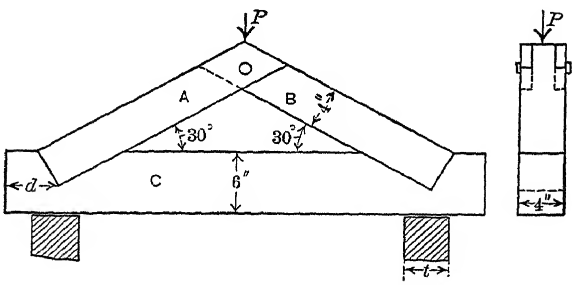


FIG. 23.—Stresses in a truss.

5. What must be the thickness *t* of the supports of Fig. 23 if *C* is yellow pine? If *C* is oak?

6. A horizontal beam is 5 ft. long and is hinged at the left end *A*. The beam carries a load of 1,800 lb. 3 ft. from *A* and is supported by a steel rod at the right end *B*. The rod is attached to a point which is 6 ft. above the hinge. What should be the minimum diameter of the rod?

CHAPTER III

STRESS BEYOND THE ELASTIC LIMIT

24. Ultimate Strength.—The *ultimate* strength of any material is the maximum unit stress which it can exert. In this sense *ultimate* means the greatest, which is not necessarily the last load before failure. In most testing machines, the load is applied by a screw or by a hydraulic press. When the maximum load is reached, the material is deformed considerably. Since the application of the load is not instantaneous, the stress drops. The final or breaking load may be much less than the ultimate. For instance, a steel rod of 1 square inch cross section may have an ultimate strength of 58,000 pounds. If an actual weight were hung on this rod and additional loads were added until 58,000 pounds was reached, the load would remain 58,000 pounds, no matter what the deformation might be. With a load of this kind, which follows up the deformation, the ultimate load is the last load. On a testing machine, the maximum load is read at 58,000 pounds. The machine then slowly elongates the material at smaller stress. Just before the rupture, the load may read 40,000 pounds. This is called the *breaking load.*

25. Factor of Safety.—In Art. 6, the allowable unit stress was said to be based on the judgment of some competent authority. These judgments depend upon tests of the materials and upon experience in actual use.

Working stresses should never exceed the elastic limit and should be only a small fraction of the ultimate strength. The ratio of the ultimate strength of a material to the allowable working stress is called the *factor of safety.* If the ultimate tensile strength of a given grade of steel is 64,000 pounds per square inch and the elastic limit is 32,000 pounds per square inch, while the allowable stress is 16,000 pounds per square inch, the factor of safety based on the ultimate strength is 4, and based on the elastic limit is 2.

The value of the factor of safety depends upon a great number of conditions. Some of these are

Repeated stresses slightly beyond the elastic limit will finally cause failure; therefore a body subjected to a variable load should have its allowable stresses well below that limit. The greater the variation of stress, the smaller should be the allowable unit stress.

The factor of safety must be sufficiently large to allow for any deterioration of the material during the time which it is to be used. This includes the decay of timber, the rusting of metal, injury from frost, and electrolysis.

The uniformity of the material must be taken into account in deciding what factor of safety to use. If test pieces from a batch of structural steel manufactured under well-controlled conditions give an ultimate strength of 57,000 pounds per square inch, none of the steel of this batch will vary more than a few hundred pounds from this figure. On the other hand, the variation of timber sufficiently good to pass a reasonable inspection may be as much as 50 per cent of the average ultimate strength. An engineer, in designing a structure to be built under competent supervision, may use considerably higher unit stresses than he would risk when such supervision is wanting.

The factor of safety must depend also upon the damage which would occur if the material should fail. A workman might use a plank with a small factor in a scaffold 3 feet above ground but would demand an ample factor if failure meant a fall of 100 feet.

The factor of safety must allow some margin for unexpected and unreasonable loads. That part of the factor of safety which makes allowance for lack of ordinary judgment on the part of persons using the machine or structure is called the "fool factor."

26. Ultimate Strength in Shear.—Figure 24 shows one type of apparatus for determining the ultimate shearing strength of metal rods. The test rod *A* fits closely in two hollow cylinders, *B* and *B′*, which are made of hardened steel. The hard-stee shear plate *C* is placed across the test piece between adjacent ends of the hollow cylinders, with its semicircular opening fitting around the upper half of the specimen. The hollow cylinders are mounted in a heavy, rigid block of cast iron or steel *D*. A rectangular slot, a little wider than the thickness of the shear plate, is cut across the block. A cylindrical hole, at right angles to the slot, is bored lengthwise through the block. The inner portion of each half of the hole for a length a little less than the length of a hollow cylinder *B* is finished to fit closely to

the hollow cylinder. The outer ends of the hole are threaded. A hollow threaded cylinder E, in each end, adjusts the smooth hollow cylinder B (or B') and holds it firmly against pressure outward.

The apparatus is placed on the weighing table of a universal testing machine and the load is applied by the movable crosshead

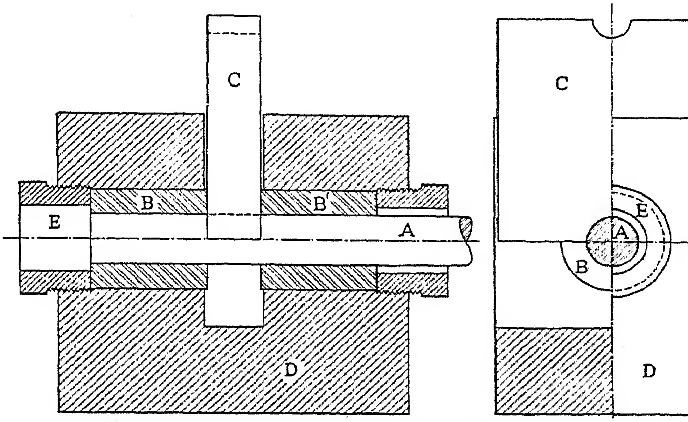


FIG. 24.—Apparatus for testing shearing stress of metal rods.

to the shear plate C. Before applying the load, the threaded cylinders are turned to bring the ends of the smooth cylinders B and B' against the shear plate. The adjustment is made to bring the plate near the middle of the slot, so that it will touch only the test rod and the smooth ends of the shear cylinder B and B'.

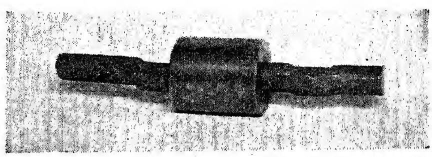


FIG. 25.—Cylinder of shear apparatus and tested steel rod.

For double shear the rod extends entirely through each hollow cylinder, to prevent bending as much as possible. For single shear the rod extends entirely through one hollow cylinder but does not quite reach the other hollow cylinder.

Figure 25 shows one of the hollow cylinders with a ¾-inch soft-steel rod which has been loaded to its ultimate strength at two places. With the deformation shown, the rod is still able to support almost its ultimate load. For a similar rod, tested July 17, 1933, the last loads, with the testing machine running constantly at minimum speed, were 37,000, pounds 37,300 pounds, 37,450 pounds, 37,600 pounds, 37,700 pounds, and 37,500 pounds. The maximum load was 37,700 pounds. The machine was

stopped after the load had dropped to 37,500 pounds. A straight line which had been drawn longitudinally on the side of the rod before loading was found to be displaced downward on the slug a distance of 0.11 inch below its position on the rod. The cleanly cut faces on the rod at the top were 0.06 inch high. The difference of 0.05 inch was principally due to bending. Although the rod fitted closely in the hollow cylinder, the compressive stress at the bottom pushed some of the material outward and afforded an opportunity for some bending. Since the shear plate has a semicylindrical surface of contact, the slug which is cut off has free opportunity to bend. This is clearly shown by the form assumed by the straight line drawn on the rod before loading. A shear plate with a complete cylindrical opening would be better but would be inconvenient to use. R. R. Moore, chief metallurgist of the Wright Aeronautical Corporation, has developed this form of shear plate and has found it to give more consistent results.

Problems

1. What was the ultimate shearing strength of the rod mentioned above? *Ans.* Ultimate $s_s = 42{,}670$ lb./in.^{2}
2. The rod of Problem 1 tested in single shear gave for the last readings 19,000 lb., 19,100 lb., 19,100 lb., and 19,000 lb. Find the ultimate shearing strength. *Ans.* $s_s = 43{,}230$ lb./in.^{2}
3. With the average results from Problems 1 and 2, what would be the allowable unit shearing stress with a factor of safety of 5?

Figure 26 shows the apparatus for measuring shearing strength of timber parallel to the grain which was developed by the

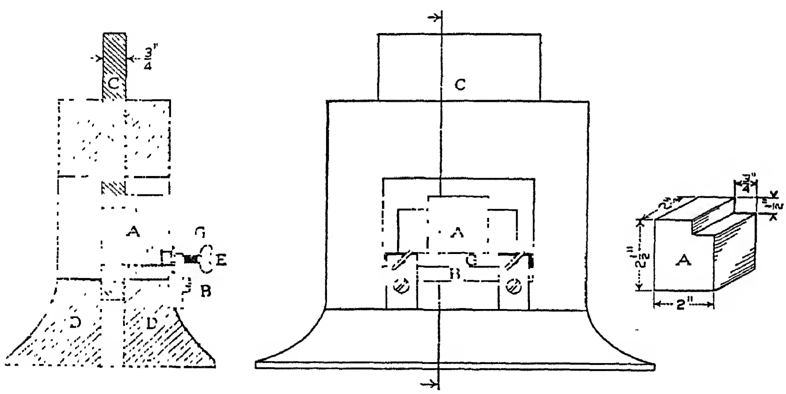


FIG. 26.—Shear apparatus for timber.

U. S. Forest Service Bureau. Two large cast-iron blocks, D and D', rigidly fastened together, act as guides for the steel shear

plate C. The second shearing surface is on the hardened steel plate B. The shearing surface on B is set back $\frac{1}{8}$ inch from the shearing surface of C, instead of being in direct contact with it as in the apparatus for shearing steel. If the grain of the block is not exactly perpendicular to the plane of the base, some fibers might begin on B and end under C. These fibers would be tested in compression instead of shear, and the measured results would be too high. With a clearance of $\frac{1}{8}$ inch, and reasonable care in the preparation of the specimen, this error will not occur.

Figure 26 also shows the form of test piece specified by the American Society for Testing Materials (A.S.T.M. Standards, 1930, Part II, p. 837). A notch of $\frac{3}{4}$ inch gives room for the entire thickness of the test plate to act in bearing. Satisfactory results may be obtained by a simple rectangular parallelepiped, if care is taken to have the full bearing width of $\frac{3}{4}$ inch and the thickness of the block is not less than $1\frac{5}{8}$ inches, which gives $\frac{3}{4}$-inch bearing width on the lower shear block B. The rectangular bar B, which is adjusted by the screws E, holds the lower end of the test block in the desired position. If the test block is held vertically until a small load is applied, the friction of the shear plate will continue to hold it.

Problems

4. A test piece of longleaf yellow pine (Fig. 26) was 1.89 in. wide and 1.92 in. high up to the notch. It failed in shear under a load of 5,300 lb. Find the ultimate shearing strength of this piece. *Ans.* s_s = 1,460 lb./in.^{2}

5. A second test piece cut from the same stick as that of Problem 4 was 1.89 in. wide and 1.90 in. high to the notch. The ultimate load in shear was 5,150 lb. Find the shearing strength parallel to the grain. *Ans.* s_s = 1,435 lb./in.^{2}

6. A rectangular parallelepiped from the same stick as Problems 4 and 5 was 1.88 in. long and 1.62 in. wide. (The direction of shear was at right angles to that of the preceding test pieces.) The ultimate load was 5,200 lb. Find the unit shearing strength of this sample. *Ans.* s_s = 1,707 lb./in.^{2}

7. A rectangular parallelepiped of white pine, 2.18 in. long and 2.05 in. wide, failed by shear under a load of 4,350 lb. Find the shearing strength. *Ans.* s_s = 973 lb./in.^{2}

8. A second block of the same white-pine piece was 2.18 in. long and 1.81 in. wide. It failed under a load of 4,250 lb. Find the ultimate shearing strength of this white pine. *Ans.* s_s = 1,077 lb./in.^{2}

27. Stress-strain Diagrams.—Stresses below the elastic limit are considered in Chapter I. Below that limit unit stress is proportional to unit deformation; Formula III of Art. 9 and the

TABLE IV.—COMPRESSION TEST OF LONGLEAF YELLOW PINE

Length, 12.5 inches; cross section 1.62 inches by 1.48 inches = 2.40 square inches. Weight of piece, 12.25 ounces. Gage length, 8 inches. Lever extensometer magnifies five times; dial readings, 0.0001 inch.

Total load, pounds	Unit stress, per square inch	Dial reading, $\frac{1}{50,000}$ inch	Compression: In gage length, $\frac{1}{50,000}$ inch	Compression: In gage length, Inches	Compression: Unit, inches per inch
10	4	4,728	0	0	0
480	200	4,700	28	0.00056	0.000070
960	400	4,669	59	118	147
1,440	600	4,638	90	180	225
1,920	800	4,607	121	242	302
2,400	1,000	4,575	153	0.00306	0.000382
2,880	1,200	4,546	182	364	455
3,360	1,400	4,518	210	420	525
3,840	1,600	4,488	240	480	600
4,320	1,800	4,459	269	538	672
4,800	2,000	4,426	302	0.00604	0.000755
5,280	2,200	4,398	330	660	825
5,760	2,400	4,368	360	720	900
6,240	2,600	4,338	390	780	975
6,720	2,800	4,309	419	838	0.001047
7,200	3,000	4,278	450	0.00900	0.001125
7,680	3,200	4,248	480	960	1200
8,160	3,400	4,216	512	0.01024	1280
8,640	3,600	4,187	541	1082	1352
9,120	3,800	4,157	571	1142	1427
9,600	4,000	4,122	606	0.01212	0.001515
10,080	4,200	4,089	639	1278	1597
10,560	4,400	4,058	670	1340	1675
11,040	4,600	4,026	702	1404	1755
11,520	4,800	3,996	732	1464	1830
12,000	5,000	3,960	768	0.01536	0.001920
12,480	5,200	3,928	800	1600	2000
12,960	5,400	3,897	831	1662	2077
13,440	5,600	3,866	862	1724	2155
13,920	5,800	3,830	898	1796	2245
14,400	6,000	3,799	929	0.01858	0.002322
14,880	6,200	3,760	968	1936	2420
15,360	6,400	3,731	997	1994	2492
15,840	6,600	3,700	1,028	2056	2570
16,320	6,800	3,668	1,060	2120	2650
16,800	7,000	3,628	1,100	0 02200	0.002750
17,280	7,200	3,589	1,139	2278	2847
17,760	7,400	3,556	1,172	2344	2930
18,290	7,622	3,515	1,213	2426	3032
18,720	7,800	3,478	1,250	2500	3125
19,200	8,000	3,439	1,289	0.02578	0.003222
19,680	8,200	3,400	1,328	2656	3320
20,160	8,400	3,362	1,366	2732	3415
20,700	8,625	3,310	1,418	2836	3545
		Extensometer removed. Beam kept in balance			
26,650	11,104				
27,050	11,271				
27,150	11,312	Ultimate strength			
26,700	11,125				

equations of resilience and change of volume hold good. Unit stress below the elastic limit is most important from the standpoint of the engineer, for allowable stresses in correctly designed structures are kept well below that limit.

It is desirable, however, to know what takes place above the elastic limit, and to understand the conditions of complete failure for the various structural materials. To gain this knowledge, experiments are made in which a series of loads are applied to a test piece of the material, and the corresponding deformations are observed with suitable measuring apparatus.

Table IV gives the results of a compression test of a stick of longleaf yellow pine. Loads were applied and the resistance weighed by means of a 50,000-pound testing machine. The deformation in an 8-inch gage length was measured by a lever micrometer with an arm ratio of 1:5. The magnified deformation was read on an Ames dial graduated to $\frac{1}{1,000}$ inch for each division. By estimating tenths of a division the readings become $\frac{1}{10,000}$ inch on the dial, which corresponds to a deformation of $\frac{1}{50,000}$ inch for the gage length of the test piece.*

The first column of Table IV gives the total load. The second column gives the unit stress. *Unit stress* is the most *important quantity* which should be kept in mind. It is customary to apply total loads which give convenient equal increments of unit stress. In this test the increment was 200 pounds per square inch.

The third column gives the dial reading, each integer corresponding to a deformation of 0.00002 inch in the 8-inch gage length. The fourth column gives the deformation obtained by subtracting the dial reading at the initial 10-pound load from each of the others. The fifth column gives the deformation in inches. The last column is the unit deformation.

Figure 27 is the *stress-strain diagram* for Table IV. Each vertical interval corresponds to a unit stress of 2,000 pounds per square inch. Each horizontal interval corresponds to a unit deformation (strain) of 0.0005. (It is customary in America to plot strain horizontally and stress vertically. Some British writers take stress horizontally and strain vertically.)

* This extensometer reads the same as the well-known Berry strain gage, except that increased reading means positive elongation.

When the data have been plotted on cross-section paper, it is found that the points up to a stress of nearly 4,000 pounds per square inch lie approximately on a straight line. From thousands of tests it has been shown that the stress-strain diagrams of the best elastic materials are smooth curves which approximate straight lines for a considerable range (Hooke's law). With the points plotted, the problem is that of locating *by eye* a straight line which represents the best average. Points that fall off the line on either side indicate accidental or other errors.

The location of the straight line of Fig. 27 after the points have been plotted is comparatively easy. A straight line through the

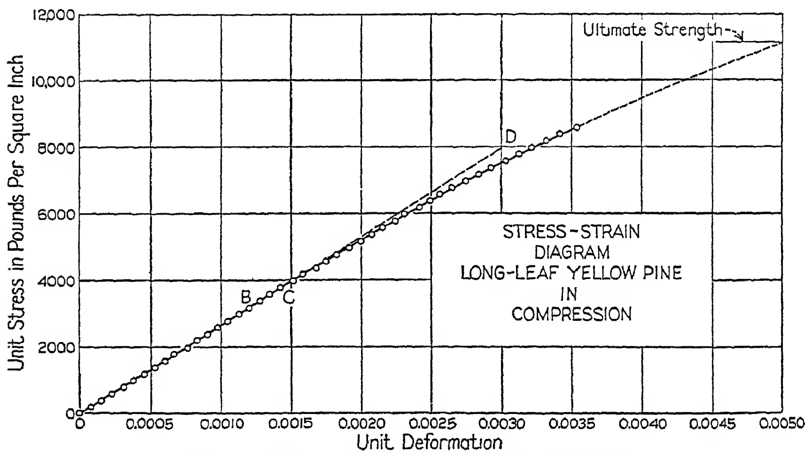


Fig. 27.—Stress-strain diagram for Table IV.

origin of coördinates and the point B at which the unit stress is 3,200 pounds per square inch is found to pass directly through seven of the intervening points. The points at 200, 400, 1,800, and 2,800 pounds per square inch are found to lie to the left of this straight line and the points at 800, 1,000, 1,200, and 2,000 pounds per square inch are found to lie on the right. (The initial point at 4 pounds per square inch, with a corresponding *calculated* unit deformation of less than 0.000002, lies so close to zero that the difference cannot be shown on the drawing.)

The straight line to B is extended as a broken line from C to D (the portion from B to C omitted to avoid crowding) and a smooth curve is drawn beyond B. The points at 3,400, 3,600, and 3,800 pounds per square inch are all slightly to the right of the straight line. The point at 4,000 pounds per square inch and all

points beyond that stress are decidedly to the right of the straight line. The stress of 3,200 pounds may be regarded as the point of tangency at which the curve leaves the straight line. However, a straight line of slightly different slope might have been drawn from a point a little above the origin through a point just to the right of the 3,800-pound circle. The point of tangency of this straight line to the curve would lie at about 3,800 pounds per square inch.

28. Proportional Elastic Limit.—The point of tangency, at which the curved portion of the stress-strain diagram leaves the straight line, is called the *proportional elastic limit*. Where the

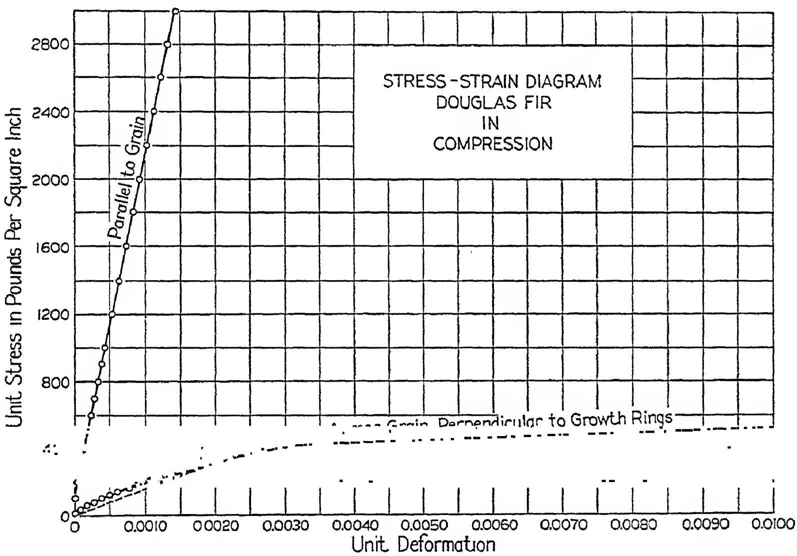


FIG. 28.—Stress-strain diagrams for Tables V and VI

deviation is very gradual, as in Fig. 27, it is difficult to locate this limit accurately. Below the proportional elastic limit the equations

$$E = \frac{s}{\delta} \quad \text{and} \quad U_2 - U_1 = \frac{s_2^2 - s_1^2}{2\,E}$$

apply. Above the proportional elastic limit these equations are no longer valid.

The elastic limit was defined in Art. 8 as the maximum stress which may be applied without *permanent set*. This limit also is difficult to locate, especially when very *precise* measurements are

made. Roughly the two definitions of elastic limit give the same values.

Table V gives the results of a compression test of Douglas fir, which was made at the Watertown Arsenal. The initial load was

Table V.—Compression Test of Douglas Fir, Lengthwise the Grain*

Length, 59.6 inches; cross section, 4.1 inches by 11.96 inches = 49.04 square inches. Gage length, 50 inches. Weight, 39.8 pounds per cubic foot.

Total load, pounds	Unit stress, pounds per square inch	Total compression, inches	Compression for 200-lb. stress, inches	Unit compression, inches per inch
4,904	100	0	......	0
9,808	200	0.0021	......	0.000042
14,712	300	44	0.0044	88
19,616	400	68	47	0.000136
24,520	500	92	48	184
29,424	600	0.0117	0.0049	0.000234
34,328	700	141	49	282
39,232	800	166	49	332
44,136	900	191	50	382
49,040	1,000	216	50	432
58,848	1,200	0.0261	0.0045	0.000522
68,656	1,400	314	53	628
78,464	1,600	364	50	728
88,272	1,800	415	51	830
98,080	2,000	466	51	932
107,888	2,200	0.0514	0.0048	0.001028
117,696	2,400	568	54	1136
127,504	2,600	620	52	1240
137,312	2,800	674	54	1348
147,120	3,000	726	52	1452
4,904	100	0.0004	......	0.000008
305,050	6,220 Ultimate strength			

"Failed by triple flexure. Fibers crushed 12 inches from one end of stick."

* Table V is taken from "Tests of Metals," 1896, p. 415.

100 pounds per square inch. It is customary, generally, to start the deformation readings at some conveniently low stress rather than at zero. (The machine at Watertown Arsenal is horizontal.

It is imperative, therefore, to apply a compression sufficient to hold the test piece from falling before the extensometers are attached.)

The results of Table V are plotted as Fig. 28, parallel to the grain. Inspection shows that the best straight line intersects zero deformation at unit stress of about 140 pounds per square inch, instead of 100 pounds per square inch, which is the location of the point representing the initial load. On account of "lost motion" in the extensometers or residual deformations in the test piece, one or more deformations at the beginning are frequently off the straight line.

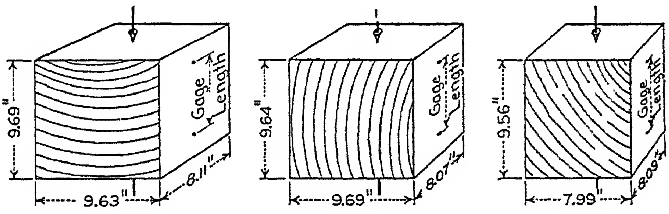


FIG. 29.—Grain of Douglas fir-blocks.

A straight line which passes through the point (0,140) and the center of the circle at 2,600 pounds per square inch is very close to all the points except the first and last. The point at 2,800 pounds is slightly to the right of the straight line. Since the point at 3,000 pounds is decidedly to the right of the line, it is safe to assume that the proportional elastic limit has been passed. The elastic limit may be taken as somewhere between 2,600 pounds per square inch and 2,800 pounds per square inch.

Table VI is the record of a compression test on a block of Douglas fir. The pressure was applied across the grain (Fig. 29, No. 1430) in a direction perpendicular to the growth rings. The initial load was 20 pounds per square inch. At every hundred pounds the load was reduced to the initial 20 pounds per square inch. The set was found to be very small up to 400 pounds per square inch but became very large with increasing loads.

The data of Table VI up to a stress of 500 pounds per square inch are plotted as Fig. 28 (across grain). A straight line may be drawn which approximates the points from 60 pounds per square inch to 140 pounds per square inch. Another straight line extends from 160 pounds per square inch to 320 pounds per square inch. A third straight line runs from 120 pounds per square

TABLE VI.—COMPRESSION TEST OF DOUGLAS FIR ACROSS THE GRAIN, PERPENDICULAR TO GROWTH RINGS (FIG. 29, No. 1430)*

Length, 8.11 inches; cross section, 9.63 inches by 9.69 inches. Area perpendicular to 9.69-inch dimension, 78.1 square inches. Gage length, 6 inches.

Total load, pounds	Unit stress, pounds per square inch	Total compression, inches	Unit compression, inches per inch
1,562	20	0	0
3,124	40	0.0004	0.00007
4,686	60	0.0011	0.00018
6,248	80	16	27
7,810	100	23	38
1,562	20	0 set	0
9,372	120	0.0030	0.00050
10,934	140	37	62
12,496	160	47	78
14,058	180	54	90
15,620	200	64	0.00107
1,562	20	0.0001 set	0.00002
17,182	220	0.0073	0.00122
18,744	240	83	138
20,306	260	93	155
21,868	280	0.0103	172
23,430	300	113	188
1,562	20	0.0004 set	0.00007
24,992	320	0.0123	0.00205
26,554	340	133	222
28,116	360	149	248
29,678	380	162	270
31,240	400	175	292
1,562	20	0.0010 set	0.00017
32,802	420	0.0187	0.00312
34,363	440	288	480
35,926	460	373	622
37,488	480	426	710
39,050	500	504	840
1,562	20	0.0219 set	365
40,612	520	0.0690	0.01150
42,174	540	0.13	0.022
43,736	560	0.28	47
45,298	580	0.42	70
46,860	600	0.60	0.100
54,670	700	1.24	0.207
62,480	800	1.80	0.300
70,290	900	2.03	0.338
78,100	1,000	2.24	0.373
156,200	2,000	2.75	0.458
234,300	3,000	2.94	0.490

"Fibers crushed laterally, but wood did not split along the grain."

* From "Tests of Metals," Watertown Arsenal, 1896, p. 389.

inch to 420 pounds per square inch. Apparently the proportional elastic limit lies between 140 pounds per square inch and 160 pounds per square inch. The measurements show zero set at 100 pounds per square inch and a very small set at 200 pounds per square inch. These seem to indicate an elastic limit between 100 pounds and 200 pounds.

29. Calculation of E.—Since instrumental errors, internal stresses, and inaccuracy of the experimenter combine to give results which do not fall exactly on a mathematically straight line, practical methods must be devised for getting a fairly correct value of the modulus of elasticity without too much labor. The slope of the stress-strain diagram affords one method. The straight line of Fig. 27 passes through the origin. The broken-line extension passes through the point D at which the unit stress is 8,000 pounds per square inch and the unit deformation is 0.003 inch per inch.

$$E = \frac{8{,}000 - 0}{0.003 - 0} = \frac{8{,}000}{0.003} = 2{,}667{,}000 \text{ pounds per square inch.}$$

The straight line of Fig. 28, for Douglas fir parallel to the grain, intersects zero deformation at unit stress of about 140 pounds per square inch and passes through the line of 0.001 deformation at about 2,140 pounds per square inch.

$$E = \frac{2{,}140 - 140}{0.001 - 0} = \frac{2{,}000}{0.001} = 2{,}000{,}000 \text{ pounds per square inch.}$$

For Douglas fir in compression across the grain, perpendicular to the growth rings, the curve of Fig. 28 gives

$$E = \frac{210 - 35}{0.001 - 0} = 175{,}000 \text{ pounds per square inch.}$$

Modulus of elasticity calculated from the slope of the straight line may be regarded as an approximate average, the accuracy of which depends upon the exactness with which the straight line has been located with reference to the experimental points. Two students, from the same data, may get somewhat different values for the modulus of elasticity and elastic limit, and neither be wrong. The difference is one of judgment in locating the straight line.

The modulus of elasticity may be computed directly from the experimental table. While the plotted curve is not absolutely

necessary, it should be drawn to give the limiting values of the readings that may be used. To get an average result several equal intervals should be calculated. Since the errors in balancing the scale beam and reading the extensometer are practically constant, the *relative* errors are proportional to the length of the interval. To obtain the maximum accuracy, the intervals of unit stress should be made as large as possible. No interval should extend above the elastic limit. If one or more points at the lower end of the line are distinctly off the line in the *same direction*, these points should not be used.

Example

Find the modulus of elasticity of the longleaf yellow-pine stick of Table IV from four intervals of 2,000 pounds each, beginning with the interval from 200 pounds to 2,200 pounds.

Stress interval	Unit-deformation interval	Modulus of elasticity
200 to 2,200	0.000825 − 0.000070 = 0.000755	2,649,000
400 to 2,400	0.000900 − 0.000147 = 0.000753	2,656,000
600 to 2,600	0.000975 − 0.000225 = 0.000750	2,667,000
800 to 2,800	0.001047 − 0.000302 = 0.000745	2,684,000
		256 ÷ 4 = 64
	Average	2,664,000

Instead of calculating E for each interval and averaging the results, the average value of the deformation intervals might be calculated, and a single value of E computed from this average. It is best, however, to calculate each modulus separately in order to see how great is the variation in a single test and to be able to compare the relative accuracy of different tests in terms of the desired quantity. The labor of division is negligible if a table of reciprocals is used.

Problems

1. Find the modulus of elasticity of the Douglas fir of Table V from four intervals of 1,000 lb. unit stress, beginning with the interval from 200 lb. to 1,200 lb. Use total deformation intervals from the third column of the table. Average these and find the average unit deformation.
Ans. $E = 1{,}000 \div 0.00049 = 2{,}041{,}000$ lb./in.^{2}
2. Solve Problem 1 from four intervals of 1,000 lb. unit stress beginning with the interval from 1,200 lb. to 2,200 lb. and ending with the interval from 1,800 lb. to 2,800 lb. *Ans.* $E = 1{,}000 \div 0.00051 = 1{,}961{,}000$ lb./in.^{2}

Problems 1 and 2 indicate that the line from 200 pounds per square inch to 2,800 pounds per square inch should not be drawn exactly straight. It is evident that the deformation increases slightly with increase of stress, giving a curve which is concave toward the right. This is seen from the fourth column of Table V, in which the deformation difference for 200 pounds unit stress increases from an average of less than 49 to an average of more than 51.

Problems

3. Find E for Douglas fir across the grain, perpendicular to the growth rings, from Table VI. Use three intervals of 60 lb., beginning with the interval from 40 lb. to 100 lb.
4. Find E for the longleaf yellow pine of Table IV, using four intervals of 1,000 lb., beginning with 400 lb. per sq. in., and four intervals of 1,000 lb. ending with 3,200 lb. per sq. in. Solve as in Problem 1.
Ans. 2,676,000; 2,676,000.

Modulus of elasticity is sometimes determined by returning from a given load to the initial load and subtracting the set at the initial load from the deformation at the given load to get the required deformation. For instance, in Table VI the reading at 300 pounds is 0.0113 inch and the set at 20 pounds is 0.0004. The unit deformation from the difference is 0.001817.

$$E = 280 \div 0.001817 = 154{,}000 \text{ pounds per square inch.}$$

The slope of the line through the point C of Fig. 28 is a measure of this modulus.

30. Failure of Timber.—Timber in compression parallel to the grain usually fails by shear at approximately 45 degrees with the direction of the compressive force. The length of the specimen should be greater than the maximum transverse dimension along any diagonal, in order that any shear plane at 45 degrees may extend from one side to the other without ending on one of the compression heads of the testing machine. A cube is not a desirable form for a test piece.

Figure 30 shows two blocks of longleaf yellow pine and one block of white oak which have been tested to failure. The longer pine block sheared along two planes and split longitudinally from one plane to the other. Since the planes of shear were normal to neither of the faces of the block which show in the photograph, the shear does not *seem* to be at 45 degrees with the length. The shorter pine block failed along several planes.

The oak block failed along a single plane which was inclined about 45 degrees with one diagonal of the cross section.

When a compressive load is applied at right angles to the fibers of a timber block, if the width in the direction of the force is appreciably greater than the

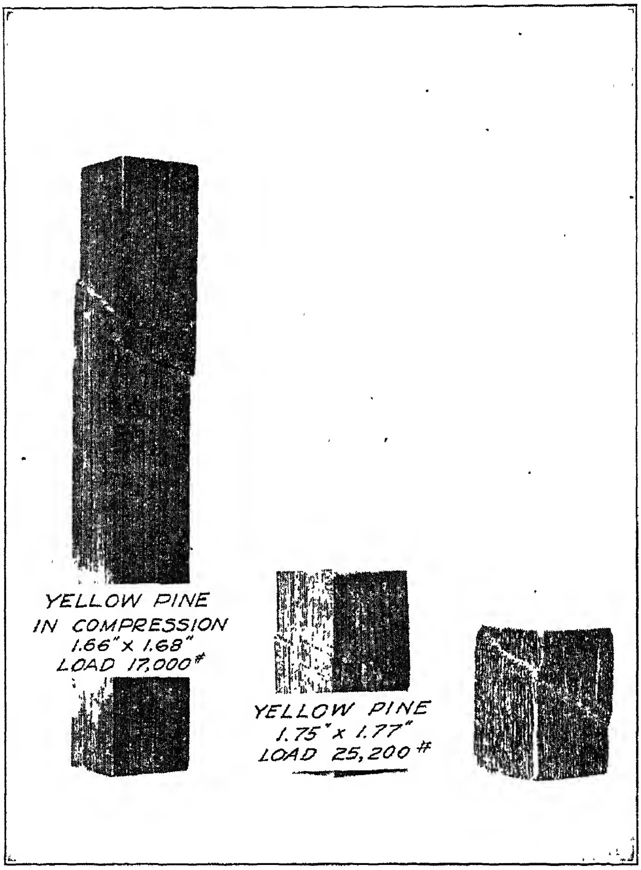


FIG. 30.—Timber in compression.

thickness, the failure usually is by shear. On the left of Fig. 31 are three blocks of white pine, each of which was originally 3 inches long in the direction of the grain, 1.81 inches thick, and 4 inches wide. The block at the left was loaded vertically, parallel to the width, as it stands in the figure. The readings were

Load, pounds......	0	2,200	2,400	2,400	2,450	2,520	2,600	2,630	2,680
Breadth, inches....	4.00	3.79	3.72	3.56	3.52	3.42	3.37	3.28	3.18

Failed by shear along growth ring. After load was released, breadth increased to 3.45 inches. Maximum thickness at ½ inch from the bottom was 2.05 inches.

Since there was a large deformation without appreciable change of load at 2,400 pounds, this load may be regarded as the ultimate. While the timber did not fail completely, this deformation is so great as to ruin any structure. The second block of Fig. 31 was not loaded. The third block was placed in the machine with its 3-inch by 4-inch faces horizontal and loaded parallel to the smallest dimension. Some of the readings were

Load, pounds	0	6,000	6,500	6,700	7,300	8,400	9,000	10,500
Thickness, inches	1.81	1.76	1.71	1.56	1.40	1.21	1.12	0.98

After load was released, thickness increased to 1.20 inches near the middle, 1.25 inches at one side, and 1.35 inches at the other.

The maximum width was 4.5 inches. There were no fractures.

This is characteristic of soft wood when loaded across the grain along the smallest thickness.

Of the three larger pieces of Fig. 31, the left is an oak block, originally 5.72 inches high (wide), 2.90 inches thick, and 5 inches long in the direction of

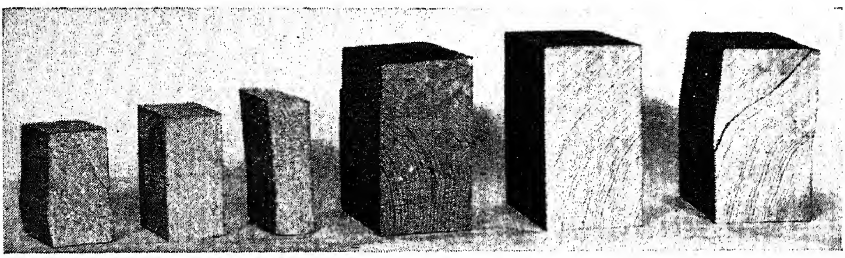


FIG. 31.—Timber tested across the grain.

the grain. A vertical load was applied to this block in the position shown. Some readings were

Load, pounds	0	4,000	10,000	12,000	14,000	18,000	20,000	21,000
Height, inches	5.72	5.70	5.66	5.64	5.63	5.60	5.59	5.57

Cracking noise at 21,000 pounds. Load fell to 14,000 pounds. Load was increased to a maximum of 16,000 pounds when failure occurred along one growth ring.

The second of the larger pieces of Fig. 31 is a block of Douglas fir, 6 inches high (wide), 3.12 inches thick, and 5 inches long. This block was not loaded. The block at the extreme right of Fig. 31 is a second piece of Douglas fir cut from the same stick as the one adjacent to it. The rear end of this sixth block corresponds to the front end of the fifth. The difference of the growth rings is principally the distortion caused by loading. There were 11 growth rings to the inch. Some of the readings were

Load, pounds	500	6,000	5,500	5,800	6,100	6,250	5,900	6,050
Height parallel to load, inches	6.00	5.81	5.74	5.66	5.56	5.54	5.52	5.44

First cracking noise at 6,250 pounds. Sudden shear failure along one growth ring at 6,050 pounds.

The other Douglas-fir block was later loaded in compression parallel to the grain. When the load reached the 100,000 pounds, the total compression in a length of 5 inches, as read by callipering the distance between compression heads, was 0.02 inch. There were no evidences of failure. The block was next loaded with the thickness vertical. Some of the readings were

Load, pounds.....	500	10,000	14,000	15,000	15,200	15,200	20,900	28,500
Height, inches....	3.12	3.12	3.07	3.05	3.01	2.99	2.45	1.96

First cracking noise at 14,000 pounds. Flattened without splitting.

At the left of Fig. 32 are two blocks of longleaf yellow pine which had been tested in shear parallel to the grain. These blocks are parallelepipeds instead of the American Society for

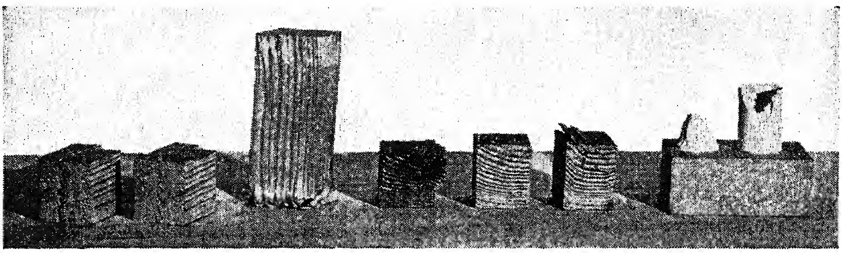


FIG. 32.—Shear and compression tests.

Testing Materials standard form shown in Fig. 26. Since the shearing strength along the growth rings was relatively small, the material sheared along one growth ring and then took the shortest distance across to the next ring. The resultant failure looks like saw teeth with points close to the planes of the shear plate and shear block, respectively.

The tallest block of Fig. 32, of longleaf yellow pine, illustrates shear near the top and "brooming" at the bottom. The lower failure began with shear about ¼ inch from the bottom on the right face shown. This failure threw additional load on other fibers. The fibers under the greatest stress were bent and split off from the remainder of the block. Unequally distributed load at the ends is a frequent cause of brooming.

The next three blocks to the right of the tall piece are very dense longleaf yellow pine, which were originally 2 inches long, parallel to the grain; 2.06 inches high, perpendicular to the growth rings; and 1.56 inches thick, parallel to the growth rings. The middle block of the three was not loaded. The left block was loaded perpendicular to the growth rings. Some readings were

Load, pounds	0	2,000	6,000	6,200	6,300	6,750	6,300
Height, inches	2.06	2.06	2.01	1.92	1.90	1.81	1.70

Failed by splitting.

The right block of the three was loaded parallel to the growth rings, horizontally as shown in the photograph. Some readings were

Load, pounds	500	3,000	4,000	5,000	6,000	8,000	8,300	10,200
Height, inches	1.56	1.55	1.54	1.54	1.53	1.53	1.41	1.40

Failed by shear parallel to growth rings.

The results of tests show a great variation in the ultimate strength of timber. The longleaf yellow-pine block of Table IV and Fig. 27 gave an ultimate strength of 11,300 pounds per square inch. This ultimate strength is represented by the short horizontal line at the top of Fig. 27. The dotted continuation of the stress-strain diagram from E to the ultimate strength represents *approximately* the last of the curve, for which no deformation readings were taken.

For the Douglas fir of Table V, the ultimate strength was 6,220 pounds per square inch. These two tests do not prove that longleaf yellow pine is superior to Douglas fir. The stress of 11,300 pounds is exceptionally high for yellow pine. Selected small blocks of timber usually show much higher strength than larger pieces. A large piece is likely to have some defect, such as a small knot, at which failure begins. The stick of Table V was relatively slender, its length being nearly fifteen times its smallest transverse dimension. As a result of this slenderness, the stick buckled and failed by triple flexure. This kind of failure means that bending stress (see Chapter XIV) was added to the direct stress and that the actual unit stress at the ultimate load was considerably greater than 6,220 pounds per square inch.

The block of Douglas fir of Table VI (Fig. 29), which was tested across the grain perpendicular to the growth rings, resisted a maximum load of 3,000 pounds per square inch. However, the unit deformation at this load was nearly 50 per cent. At 600 pounds per square inch the unit deformation was 10 per cent, which is much more than would be permitted in a structure. For practical purposes, the ultimate strength of this piece is about 520 pounds per square inch. For some soft woods across the grain, the ultimate compressive strength is quite indefinite.

Problems

1. Calculate the ultimate strength of the two pine blocks of Fig. 30.
2. If 2,400 lb. is taken as the ultimate load of the white pine of Fig. 31 when loaded parallel to its breadth, what is the ultimate strength in compression across the grain? *Ans.* s_c = 442 lb./in.^{2}
3. If 6,000 lb. is taken as the ultimate load of the white pine of Fig. 31 when loaded parallel to its thickness, what is the ultimate strength and what is the approximate modulus of elasticity?
Ans. s_c = 500 lb./in.^{2}; E = 18,100 lb./in.^{2}
4. If 21,000 lb. is taken as the ultimate load of the oak block of Fig. 32, what is the ultimate strength across the grain parallel to the growth rings? What should be the allowable stress with a factor of safety of 4?
Ans. 1,448 lb./in.^{2}; 362 lb./in.^{2}
5. Figure 29, No. 1431, is a second block of Douglas fir, cut from the same stick as Fig. 29, 1430, and tested in compression across the grain parallel to the growth rings. For a gage length of 6 in., for total loads of 3,128 lb., 4,692 lb., 6,256 lb., 10,948 lb., 12,512 lb., and 14,076 lb. the gage readings were 0.0014 in., 0.0030 in., 0.0048 in., 0.0105 in., 0.0119 in., and 0.0140 in., respectively. The ultimate load was 54,740 lb., at which the fibers split along the grain. As loading was continued, the total stress dropped to 26,000 when the thickness had been reduced from 9.64 in. to about 6 in. Find three values for E, and find the ultimate strength.
Ans. Average E = 107,000 lb./in.^{2}
6. Figure 29, No. 1432, represents a third block of Douglas fir, which was tested in compression across the grain at about 45° with the growth rings ("Tests of Metals," 1896, p. 391). For total loads of 2,586 lb., 3,878 lb., 5,171 lb., 6,464 lb., 7,757 lb., and 9,050 lb. the gage readings were 0.0028 in., 0.0062 in., 0.0095 in., 0.0130 in., 0.0165 in., and 0.0208 in., respectively. The stick failed by splitting along the growth rings at an ultimate load of 28,400 lb. Find E and the ultimate strength.
Ans. Average E = 94,300 lb./in.^{2}; s_c = 439 lb./in.^{2}
7. A white-oak block, 8.10 in. long parallel to grain, 9.63 in. thick perpendicular to growth rings, was tested in compression parallel to growth rings ("Tests of Metals," 1896, p. 427). When the load changed from 1,939 lb. to 60,115 lb., the compression in a 6-in. gage length changed from 0.0011 to 0.194. Find the modulus of elasticity. The maximum load was 155,136 lb. Find the ultimate strength.

31. Steel in Tension.—Table VII gives the results of the tension test of a rod of low-carbon steel. Figure 33, I, shows the entire stress-strain diagram. Figure 33, II, gives a small portion of the diagram with the horizontal scale one hundred times as great as that of Fig. 33, I. The elastic limit is located at B at a unit stress of 35,000 pounds per square inch. From C to C_1 there is a relatively large elongation with very little increase of stress. From C_1 to C_2 the unit stress drops from 36,520 pounds per square inch to 36,490 pounds per square inch (see Table VII).

TABLE VII.—TENSION TEST OF LOW-CARBON STEEL

Gage length, 8 inches; mean diameter, 0.5993 inch; area, 0.28208 square inch; approximate area, 0.282 square inch. The crosshead speed of the testing machine was 1/60 inch per minute.

Total load, pounds	Unit stress, pounds per square inch	Dial reading, 0.00002 inch	Elongation: In gage length, 0.00002 inch	Elongation: In gage length, Inches	Elongation: Unit, inches per inch
0	0	16	0	0	0
282	1,000	30	14	0.00028	0.000035
564	2,000	40	24	48	60
846	3,000	52	36	72	90
1,128	4,000	66	50	0.00100	0.000125
1,692	6,000	93	77	0.00154	0.000192
2,256	8,000	120	104	208	260
2,820	10,000	146	130	260	325
3,384	12,000	175	159	0.00318	0.000397
3,948	14,000	200	184	368	460
4,512	16,000	229	213	426	532
5,076	18,000	256	240	480	600
5,640	20,000	282	266	532	665
6,204	22,000	309	293	0.00586	0.000732
6,768	24,000	336	320	640	800
7,332	26,000	364	348	696	870
7,896	28,000	391	375	750	937
8,640	30,000	427	411	822	0.001027
8,742	31,000	434	418	0.00836	0.001045
9,024	32,000	448	432	864	1080
9,306	33,000	456	440	880	1100
9,588	34,000	467	451	902	1127
9,870	35,000	482	466	932	1165
10,000	35,460	491	475	0.00950	0.001187
10,200	36,170	502	486	972	1215
10,300	36,520	607	591	0.01182	1477
10,290	36,490	665	649	1298	1622
10,540	37,380	780	764	1528	1910
10,530	37,340	890	874	0.01748	0.002185
10,530	37,340	980	964	1928	2410
10,640	37,730	1,127	1,111	2222	2777
10,450	37,060	1,431	1,415	2830	3537
10,610	37,624	1,553	1,537	3074	3842
10,500	37,230	1,750	1,734	0.03468	0.004335
10,630	37,700	2,080	2,064	4928	5160
10,600	37,590	2,480	2,464	4928	6160
10,410	36,910	2,945	2,929	5858	7322
10,580	37,520	3,310	3,294	6588	8235
10,460	37,090	3,810	3,796	0.07592	0.009490
10,700	37,940	4,070	4,054	8108	0.010135
10,440	37,020	4,360	4,344	8688	10860
10,560	37,450	4,670	4,654	9308	11635
10,520	37,310	4,740	4,724		
		4,765	4,749		

After 30 sec. reset to 376 new zero = 376 − 4,749 = −4,373

Total load, pounds	Unit stress, pounds per square inch	Dial reading, 0.00002 inch	Elongation: In gage length, 0.00002 inch	Elongation: In gage length, Inches	Elongation: Unit, inches per inch
10,400	36,880	695	5,068	0.10136	0.012670
10,500	37,230	960	5,333	.10666	13442
10,450	37,060	1,227	5,600	.11200	16345
10,630	37,700	2,165	6,538	.13076	16345
10,680	37,870	2,755	7,128	.14256	17820
10,550	37,410	3,160	7,533	0.15066	0.018882
10,440	37,020	4,060	8,433	.16866	21082
10,300	36,520	4,695	9,068	.18136	22670
		4,730	9,103		

TABLE VII.—TENSION TEST OF LOW-CARBON STEEL.—(*Continued*)

Total load, pounds	Unit stress, pounds per square inch	Dial reading, [illegible]	Elongation		
			In gage length		Unit, inches per inch
			0.00002 inch	Inches	
		After standing reset to 600 new zero = −8,503			
10,550	37,410	1,170	9,678	0.19356	24182
10,720	38,010	1,270	9,773	.19546	24432
10,990	38,970	1,550	10,053	0.20106	0.025132
11,100	39,360	1,850	10,353	.20706	25882
11,200	39,720	2,081	10,584	.21168	26435
11,300	40,070	2,370	10,873	.21746	27182
11,460	40,640	2,850	11,353	.22706	28382
11,590	41,100	3,300	11,803	0.23606	0.029507
11,700	41,490	3,550	12,053	.24106	30132
11,850	42,020	4,120	12,623	.25246	31557
11,940	42,340	4,507	13,010	.26020	32525

Removed extensometer. Reading with dividers and scale at 12,000 pounds load was 0.26 inch.

Total load, pounds	Unit stress, pounds per square inch	Elongation		
		In 8 inches	Unit	
12,180	43,190	0.30	0.0375	*Machine run at 1 in. per minute for about 0.08 in. Then run at 1/60 in. per minute for the remaining 0.02 in. of each interval*
13,150	46,630	.40	500	
14,000	49,650	.50	625	
14,570	51,670	.60	750	
14,970	53,080	.70	975	
15,330	54,360	0.80	0.1000	
15,600	55,320	.90	.1125	
15,790	55,990	1.00	.1250	
15,940	56,250	1.10	.1375	
16,050	56,920	1.20	.1500	
16,100	57,090	1.30	0.1625	
16,220	57,520	1.40	.1750	
16,270	57,700	1.50	.1875	
16,315	57,850	1.60	.2000	
16,330	57,910	1.70	.2125	
16,345	57,960	1.80	0.2250	
16,365	58,030	1.90	.2375	
16,350	57,980	2.00	.2500	
16,350	57,980	2.10	.2625	
16,330	57,910	2.20	.2750	
16,280	57,730	2.30	0.2875	
16,200	57,450	2.40	.3000	
16,120	57,160	2.50	.3125	
16,150	57,270	2.55	.3187	Diameter of neck 0.483 in
15,720	55,740	2.60	.3250	

Total load, pounds	Unit stress, pounds per square inch	Elongation		Diameter of neck, inches	
		In 8 inches, inches	Unit, inches per inch		
15,300	54,260	2.65	0.3312	0.457	Ran machine entirely at 1/60 in. per minute after necking began
14,580	51,702	2.68	.3350	.441	
14,150	50,180	2.70	.3375	.428	
13,800	48,940	2.72	.3400	.413	
13,130	46,560	2.74	.3425	.395	
12,700	45,040	2.75	0.3432	0.385	
11,520	40,850	2.76	.3450	.339	

The point at which the deformation increases with no increase of stress is called the *yield point* of the material. The yield-point stress of Table VII may be taken at 36,520 pounds per square inch.

Not all of a test bar reaches the yield point at the same time. A small portion yields; the stress drops and then gradually rises. Another portion yields and there is another sudden drop. With the testing machine elongating the specimen with a constant

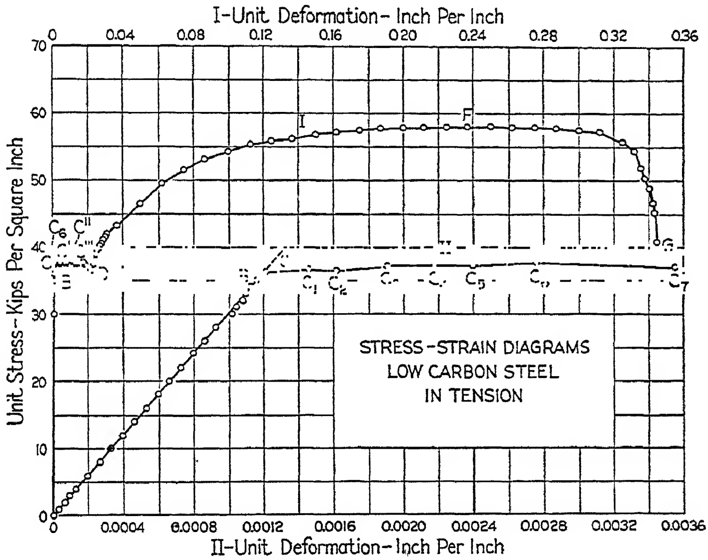


FIG. 33.—Stress-strain diagrams for Table VII.

slow speed and the beam balanced, there is a sudden "drop of the beam" when a new portion reaches the yield point. Before the beam can be balanced again, there is a considerable elongation; the curves, therefore, fail to show the true abruptness of reduction of tension.

On Fig. 33, I, the circles are omitted between the points C and C'''; the principal points are shown at the angles of intersection. The maximum stresses are

Point	Stress	Unit deformation
C_6	37,730	0.00278
C'	37,700	0.00516
C''	37,940	0.01013
C'''	37,870	0.01782

Minimum stresses are

	37,060	0.00354
	36,910	0.00732
	36,880	0.01267
D	36,520	0.02267

The largest stress is called the *upper yield point;* the minimum is the *lower yield point.* For this experiment, the range is 1,520 pounds per square inch—about 4 per cent. Under some conditions, the initial upper yield point may be considerably higher than any which immediately follow. There is a large initial hump in the curve. The stress then drops suddenly and continues to vary through a moderate range similar to that shown in Fig. 33.

At D, which is the lowest minimum for this particular test piece, the curve suddenly turns upward. From C to D for a unit deformation of more than 2 per cent, the steel is in the *plastic condition.* There is a considerable flow and rearrangement of the material. This is *cold-working* by stretching. Cold-working may be accomplished also by hammering, cold-rolling, or drawing. Cold-working low-carbon steel and some other materials raises the elastic limit. The material is said to be *work hardened.*

Yield point is determined by "drop of the beam." With the testing machine running at a constant slow speed, and the poise kept balanced automatically or by hand, a point is reached at which there is a sudden drop of the beam, and the poise must be run backwards to balance. This maximum stress is one upper yield point. If the material is covered with scale, this scale begins to loosen and fall at any portion which has reached the yield point.

Figure 34 shows stress-strain diagrams for three steel bars of quite different carbon content. The bar of Fig. 34, I (Appendix A) has 0.20 per cent carbon. The curve is much like that of Fig. 33. The bar of Fig. 34, II (Appendix B), has 0.44 per cent carbon. The yield-point stress is much higher than that of Fig. 34, I; the plastic deformation is less; and the maximum unit elongation is smaller. Figure 34, III, is for steel of nearly 1 per cent carbon. There is no *true yield point.* The elastic limit is at about 75,000 pounds per square inch. The maximum elongation is less than one-half as great as that of steel of 0.44 per cent carbon, and less than one-fourth as great as the elongation of the softest bar.

Problems

1. Calculate the modulus of elasticity of the steel of Fig. 33 from the slope of Curve II.
2. Calculate the modulus of elasticity of the steel of Fig. 33 from the data of Table VII, using the first five intervals of 30,000 lb. per sq. in.
   Ans. Average $E = 29{,}590{,}000$; max. $E = 29{,}900{,}000$; min. $E = 29{,}210{,}000$.
3. From the data of Table VII find the total work in the 8-in. gage length when the unit stress changed from 4,000 lb. per sq. in. to 34,000 lb. per

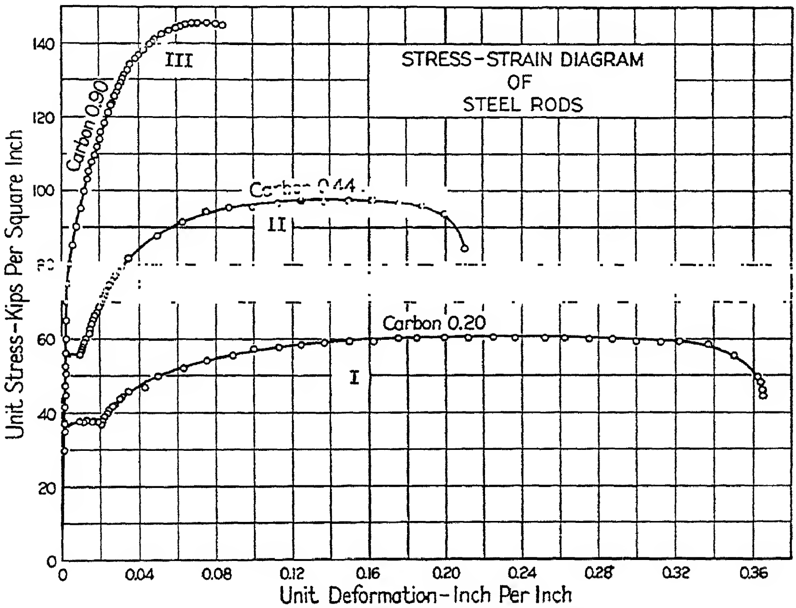


FIG. 34.—Stress-strain diagrams for three kinds of steel.

   sq. in. Use total load and total deformation. Divide by the volume to get the work per cubic inch. Check work per unit volume by Eq. (3) of Art. 12. What value of E must be used to secure an exact check? Why?
4. Find the modulus of elasticity of the steel of Fig. 34, I, for three intervals of 25,000 lb. per sq. in. from Appendix A.
   Ans. Average $E = 29{,}550{,}000$ lb./in.^{2}
5. Find E for steel of 0.44 per cent carbon for the first four intervals of 25,000 lb. per sq. in. from Appendix B. Solve also for the first five intervals of 30,000 lb. per sq. in. *Ans.* 29,580,000; 29,360,000.
6. Find E for the high-carbon steel of Fig. 34, III, for the first three intervals of 30,000 lb. from Appendix C. Solve also for the first five intervals of 40,000 lb. and the first three intervals of 60,000 lb.
   Ans. 28,890,000; 29,290,000; 29,070,000.
7. Solve Problem 3 for the steel of Appendix B when the load changed from 2,205 lb. to 17,640 lb.

8. Solve Problem 3 for the steel of Appendix C when the unit stress changed from 5,000 lb. per sq. in. to 70,000 lb. per sq. in.

9. How high will the total energy in the steel of Appendix C at total load of 30,646 lb. lift its weight?

32. Breaking Strength.—The point F of Fig. 33, I, at unit stress of 58,030 pounds per square inch and unit elongation of 0.2375, indicates the ultimate strength of this test bar of low-carbon steel. The point G at the end of the curve is the breaking strength or stress at rupture. For brittle materials, breaking strength and ultimate strength practically coincide. For ductile materials, the ultimate strength is much higher than the breaking strength. Used in connection with stress, the word *ultimate* means the *greatest stress*, not the last stress.

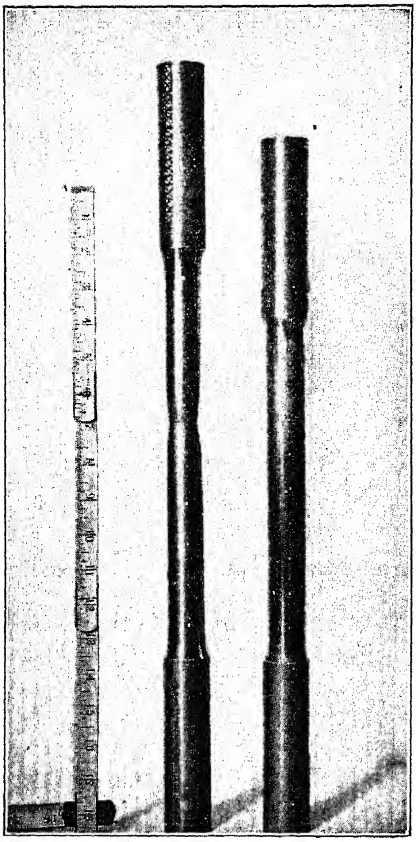


FIG. 35.—Steel rod tested in tension.

When a bar is tested in tension, the cross section decreases uniformly throughout the entire gage length until the ultimate strength is reached. Finally some section begins to decrease much faster than the remainder of the gage length. This reduced section is called the *neck*. Figure 35 shows two bars of 0.42 per cent carbon, which had the same original dimensions. The right bar has not been deformed. The left bar has been broken in tension. The pieces were replaced in their original relative positions before the photograph was taken. The line of fracture is shown at the smallest section of the neck.

On account of the smaller section's having to carry the load after the bar begins to neck, the total load is reduced. After the diameter at the neck has become somewhat smaller than that of the remainder of the gage length, the total stress becomes

so low that there is no additional elongation except in the portion at the neck. For the last part of the test, the uniform portions of the bar shorten slightly under the reduced load.

Questions

1. What are the ultimate strength and the breaking strength of each bar of Fig. 34? Read values from the curves.
2. From measurements of Fig. 35, find the ratio of the diameter of the neck to the diameter of the remainder of the gage length. Find the ratio of the diameter of the neck to the original diameter of the bar.
3. Find the ratio of the gage length of the bar of Fig. 35 after testing to the original gage length.

33. Percentage of Elongation and Reduction of Area.—For ductile materials, such as steel or wrought iron, the percentage of elongation is an important factor. The bar of Fig. 33 (Table VII) was elongated 2.76 inches in a gage length of 8 inches. The percentage of elongation was 34.5. The greatest relative elongation is in the portion of the bar which contains the neck. To

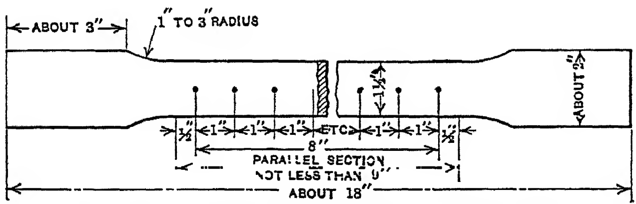


FIG. 36.—Tension test bar—8-inch gage length.

show this, it is customary to subdivide the gage length into 1-inch intervals by punch marks. For the bar of Fig. 35, the percentage of elongation in the 8-inch gage length was 24.7. The original 1-inch intervals showed the following elongations after fracture:

Interval	Elongation, Inches
0–1	0.17
1–2	0.19
2–3	0.31
3–4	0.54 (included neck)
4–5	0.25
5–6	0.19
6–7	0.17
7–8	0.17

If the elongation is taken from the single inch interval 3–4, which included the neck, the result is 54 per cent. From the 4-inch interval 0–4, the elongation is 30.2 per cent. From the

other 4-inch interval 4–8, the elongation is only 19.5 per cent. In order to make the results of different tests comparable with each other, the American Society for Testing Materials has adopted 8 inches as the standard gage length of test bars from most rolled stock. Figure 36 shows the dimensions of a standard test bar of this length as made from a plate.

The entire length of a test bar having a gage length of 8 inches is about 18 inches. It is not possible to take a test piece of this

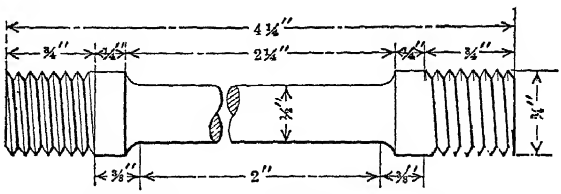


Fig. 37.—Tension test bar—2-inch gage length.

length from small forgings or castings. Test bars of 2-inch gage length are used under these conditions, as shown in Fig. 37. Specifications, such as those of Table VIII sometimes give the percentage of elongation for both 2-inch and 8-inch gage lengths.

To get the percentage of elongation, it is customary to fit the two pieces together after rupture and measure the total elongation of the gage length. For hard material with high ultimate strength, which breaks with inappreciable necking, this method gives an incorrect value for the percentage of elongation. The upper 4 inches of the steel bar of Appendix C was stretched 0.34 inch under unit stress of 145,040 pounds per square inch. The percentage of elongation before fracture was 8.5. After fracture, which took place in the lower 4 inches, the elongation of the upper 4-inch length had been reduced to 0.26 inch, or 6.5 per cent. A bar of this steel, when broken suddenly at almost the ultimate stress, shortens 2 per cent on the rebound. A bar of the same material shortened less than 1 per cent when the load was released gradually by reversing the machine. For softer steel the unit stress is materially reduced before fracture as the test bar necks. The rebound is relatively small and makes no appreciable error in the percentage of elongation. For a cold-rolled bar the rebound is about one-half of 1 per cent in the portion which is not fractured.

The percentage of reduction of area at the neck is an important constant of steel specifications. The final diameter of the neck

TABLE VIII.—SPECIFICATIONS FOR ROLLED STEEL ADOPTED BY AMERICAN SOCIETY FOR TESTING MATERIALS, STANDARDS, 1933

Material		Tensile strength, lb. per sq. in.	Yield point, lb. per sq. in.	Elongation in 8 in., per cent	Elongation in 2 in., per cent	Reduction of area, per cent
Steel for bridges	Structural	55,000 to 65,000	0.5 tens. str.	$\frac{1,500,000}{\text{tens. str.}}$	22	
	Rivet	52,000 to 62,000	0.5 tens. str. but in no case less than 28,000	$\frac{1,500,000}{\text{tens. str.}}$		
Structural nickel steel	Rivet	70,000 to 80,000	45,000	$\frac{1,500,000}{\text{tens. str.}}$	..	40
	Plates, shapes, and bars	85,000 to 100,000	50,000	$\frac{1,500,000}{\text{tens. str.}}$	18	25
	Eyebars, flats, and pins, annealed	90,000 to 105,000	52,000	20	20	35
Rail steel for concrete reinforcement bars	Plain bars	80,000 minimum	50,000	$\frac{1,200,000}{\text{tens. str.}}$		
	Deformed and hot-twisted bars	80,000 minimum	50,000	$\frac{1,000,000}{\text{tens. str.}}$		

of the soft steel of Table VII was 0.339 inch. The corresponding area is 0.0903 square inch.

$$\text{Reduction of area} = \frac{0.2820 - 0.0903}{0.2820} = 0.68 = 68 \text{ per cent.}$$

Problems

1. Find the percentage of reduction of area and of elongation for the bar of Appendix A. Use the square of the diameter instead of the area. *Ans.* 66.5 per cent.
2. Find the reduction of area for the bar of Appendix B and the bar of Appendix C. *Ans.* 44.5 per cent; 11.6 per cent.
3. In Appendix A, what is the percentage of elongation of the upper 1-in. interval? What is the average of the second, third, and fourth intervals? What is the average of the lower two intervals?

4. A steel bar has a tensile strength of 62,000 lb. per sq. in.; what are the minimum value of the yield point and percentage of elongation in order to meet the minimum requirements of structural steel for bridges?

Ans. 31,000 lb./in.^{2}; 24.2 per cent.

5. A ¾-in. bar of structural nickel steel has an ultimate strength of 35,200 lb. An 8-in. gage length is elongated 1.44 in. and the final diameter of the neck is 0.64 in. Does this steel meet the minimum requirements of the A.S.T.M. Specifications?

6. A test bar from an annealed pin of structural nickel steel is 0.505 inch in diameter. The ultimate load is 19,200 lb. The elongation of 2 in. is 0.42 in.; and the final diameter of the neck is 0.409 in. Does this steel meet the minimum requirements of the A.S.T.M. Specifications?

Ans. No.

7. Would the steel of Problem 6 meet the minimum requirements if the elongation of 2 in. were 0.42 in. and the minimum diameter of the neck were 0.401 in.?

34. Actual and Nominal Unit Stress.—The unit stresses of the preceding tables and curves were calculated by dividing the

TABLE IX.—TENSION TEST OF LOW-CARBON STEEL

Mean diameter, 0.59905 inch. Area, 0.2818 square inch; approximate area, 0.282 square inch. Gage length, 8 inches.

Total load, lb.	Unit stress, lb. per sq. in.	Elongation		Total load, lb.	Unit stress, lb. per sq. in.	Elongation		Diameter of neck, inches
		In 8 in., inches	Unit, inches per inch			Total, inches	Unit, inches per inch	
0	0	0	0	12,920	45,820	0.30*	0.0375	
1,410	5,000	0.00126	0.00015[illegible]	13,710	48,620	.40	50	
2,820	10,000	264	330	14,450	51,240	.50	625	
4,230	15,000	400	500	14,900	52,840	.60	750	
5,640	20,000	534	667	15,360	54,470	.70	875	
7,050	25,000	0.00666	0.000832	15,610	55,350	0.80	0.100	
8,460	30,000	798	997	15,880	56,310	.90	.1125	
9,870	35,000	962	0.001202	16,020	56,810	1.00	.125	
10,900	38,650	0.01168	1460	16,190	57,410	1.10	.1375	
11,220	39,790	1808	2235	16,350	57,980	1.20	.150	
11,250	39,890	0.02850	0.003562	16,395	58,140	1.30	0.1625	
11,150	39,540	4458	5572	16,460	58,370	1.40	.175	
11,400	40,430	5348	6685	16,500	58,510	1.50	.1875	
11,490	40,740	7298	9122	16,560	58,720	1.60	.200	
11,170	39,610	8128	0.010160	16,565	58,740	1.70	.2125	
11,450	40,600	0.09698	0.012122	16,572	58,770	1.80	0.225	
11,350	40,250	0.12198	15247	16,560	58,720	1.90	.2375	
11,150	39,540	.13894	17367	16,520	58,580	2.00	.250	
11,350	40,250	.14848	18560	16,550†	58,690	2.10	.2625	
11,280	40,000	.16258	20322	16,400	58,160	2.20	.275	
11,350	40,250	0.18592	0.023240	15,630	55,430	2.30	0.2875	0.468
11,200	39,720	.20028	25035	15,150	53,720	2.35	.2937	.447
11,400	40,430	.22448	28060	14,100	50,000	2.42	.3025	.410
11,800	41,840	.23468	29335	11,800	41,840	Broke		
11,970	42,450	.24028	30035	After fracture		2.47	0.3090	0.344
12,200	43,260	0.25648	0.032060					

* Measured elongation with dividers.
† Stood under load for several minutes after last reading.

TABLE IX.—TENSION TEST OF LOW-CARBON STEEL.—(*Continued*)

Diameter readings at the middle of each 1-inch interval. Upper line of each pair gives diameters in plane of punch marks and axis of rod. Lower line gives diameters perpendicular to these.

Elongation	Intervals								
	Top	2	3	4	5	6	7	8	Average
0.0	0.5997	0.5992	0.5992	0.5988	0.5986	0.5988	0.5990	0.5992	
	0.5999	0.5993	0.5992	0.5986	0.5985	0.5988	0.5988	0.5992	0.59905
0.5	0.5820	0.5820	0.5820	0.5828	0.5832	0.5840	0.5824	0.5820	
	0.5860	0.5837	0.5808	0.5810	0.5800	0.5796	0.5800	0.5830	0.58216
1.0	0.5668	0.5666	0.5664	0.5650	0.5658	0.5658	0.5658	0.5652	
	0.5710	0.5680	0.5648	0.5646	0.5630	0.5636	0.5636	0.5672	0.56582
1.5	0.5540	0.5536	0.5518	0.5486	0.5486	0.5486	0.5500	0.5504	
	0.5582	0.5540	0.5490	0.5484	0.5460	0.5462	0.5486	0.5530	0.55056
2.0	0.5470	0.5430	0.5380	0.5310	0.5260	0.5275	0.5350	0.5380	
	0.5500	0.5448	0.5370	0.5302	0.5240	0.5258	0.5350	0.5420	0.53591
2.3	Not taken								
	0.5498	0.5440	0.5350	0.5270	0.4680	0.5170	0.5318	0.5402	
	Length of each interval after failure, inches								
	1.19	1.22	1.26	1.29	1.56	1.44	1.29	1.22	

Fracture near bottom of fifth interval at 6.48 inches from top of gage length.

total load by the area of the cross section at the beginning of the test. This is the usual method of calculation and unit stress always is so understood, unless otherwise designated. When it is desirable to distinguish from unit stress calculated in some other way, the unit stress obtained from the original area may be called the *nominal unit stress*.

On account of the permanent reduction of area of a ductile material after passing the yield point, the *actual unit stress*, which is calculated by dividing the total load by the area of the cross section as loaded, may be much larger than the nominal unit stress from the original area. From the test piece of Table IX, when the elongation was 0.5 inch, the average diameter was found to be 0.5822 inch, and the corresponding area of cross section was 0.2662 square inch. The *actual unit stress* was 14,450 divided by 0.2662, which equals 54,280 pounds per square inch. When this load is divided by the original area, the *nominal unit stress* is found to be 51,240 pounds per square inch.

Problems

1. From Table IX calculate the actual unit stress when the elongation was 1 in. *Ans.* 63,720 lb./in.^{2}

2. From Table IX calculate the actual unit stress in terms of the average cross section and in terms of the minimum cross section when the elongation was 2 in. Use area to four significant figures.

Ans. 73,230; 76,350 lb./in.^{2}

Before necking begins, the actual unit stress may be calculated from the nominal unit stress and the unit deformation. After reaching the yield point, the volume remains practically constant. If A represents the original area of cross section, the volume of a portion 1 inch in length is equal to A cubic inches. If A' is the

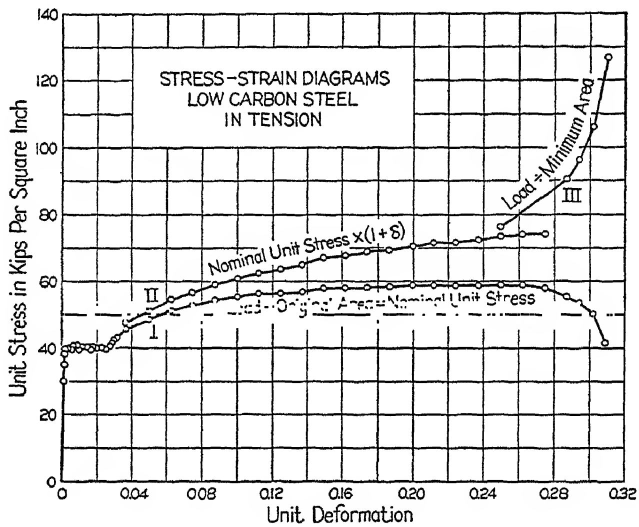


FIG. 38.—Nominal and actual unit stress.

area of the cross section when the original inch of length has been stretched to a length of $1 + \delta$, the volume is $A'(1 + \delta)$.

$$A = A'(1 + \delta), \quad A' = \frac{A}{1 + \delta}.$$

$$\text{Actual unit stress} = \frac{P}{A'} = \frac{P}{A}(1 + \delta).$$

Actual unit stress = nominal unit stress multiplied by $(1 + \delta)$.

Problems

3. From Table IX calculate the actual unit stress when the nominal unit stress was 52,840 lb. per sq. in. and when the nominal unit stress was 58,140 lb. per sq. in. Compare with Curve II of Fig. 38.

Ans. $52{,}840 \times 1.075 = 56{,}803$; 67,588 lb./in.^{2}

4. From Table IX calculate A' when the elongation was 0.5 in., 1.00 in., and 1.50 in. Compare with the area calculated from the average diameter.

Curve I of Fig. 38 was drawn from Table IX. It is the ordinary diagram of nominal stress. The ordinates for Curve II were calculated by multiplying the nominal unit stress by $(1 + \delta)$. This curve represents the *average actual* unit stress with the average unit elongation. When the total elongation had reached 2 inches, the minimum diameters measured were nearly 2 per cent smaller than the average diameter. The area

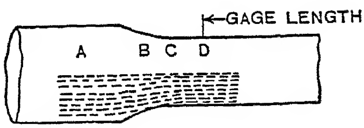


FIG. 39.—Stress distribution in test bar.

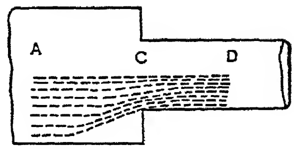


FIG. 40.—Abrupt change of section.

at the minimum section was about 4 per cent less than the average and the actual unit stress was 4 per cent greater than the average. The ordinates of Curve III give the actual stress at the smallest sections, with the *average* unit elongation.

35. Effect of Form on the Stress.—In calculating unit stress it has been assumed that the stress is uniform over the entire cross section. This is true in a rod of uniform cross section at some distance from the surface at which the external force is applied, provided the resultant force acts along the axis of the body. In order to have the stress uniformly distributed and the same at all sections, test bars are made of uniform section throughout a portion somewhat greater than the gage length. In Fig. 36, for instance, the gage length is 8 inches, while the parallel portion is required to be not less than 9 inches. The bar tapers gradually from the larger to the smaller sections.

The broken lines in Figs. 39 and 40 represent the flow of stress in the bar. At sections C the lines are crowded, indicating a greater intensity of stress near the surface. If the gage length extended to C at the end of the parallel portion, the measured elongation would be too great, since the stress in the outer surface (to which the gages are applied) is greater than the average of the section. The more abrupt the change in section, the greater the inequality of stress.

Figure 41 represents a plate of uniform width and thickness with a circular hole at the middle of its width. The plate is subjected to tension producing uniform tensile stress over any section located some distance from the hole. The arrows at the top represent this uniform stress at one section. The arrows which begin at the horizontal line through the center of the hole represent the stress at that section. The problem of the stress distribution of a plate of this kind has been solved, and the solution has been verified by photoelastic experiments.* For a plate of considerable width in comparison with the diameter of the hole, the maximum stress at the tangent is three times the stress in the uniform portion. This stress is shown in Fig. 41 by the two longest arrows. At one-fifth the radius from the tangent the stress is 2.070 times the stress in the uniform portions. The stress at a distance from the tangent equal to the radius is 1.219 times the stress in the uniform portions. The stress at a distance from the tangent equal to the diameter is 1.074 times that of the uniform portions.

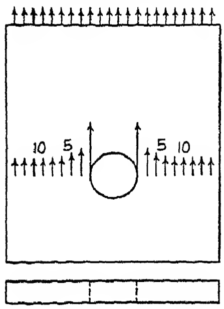


FIG. 41.—Variation of stress near a circular hole.

The ultimate strength of a body at a point where the section changes depends upon the ductility of the material. A rod of cast iron or other *non-ductile* material of the form of Fig. 40 will fail at section *C*, where there is a concentration of stress near the surface. The more abrupt the change of section, the greater the concentration and the easier the failure.

On the other hand, ductile material subjected to tension is not likely to fail at a section at which there is an abrupt change. While the stress at the hole in Fig. 41 is three times the average, this overstressed material soon reaches the yield point. The ductile metal receives a permanent set and the stress increases very little with any further deformation. When the load is removed, this material is under compression and produces some tension in the remainder of the section. A ductile substance necks before it fails. The larger portions of Figs. 39 and 40 prevent necking in the smaller portions to the right of *C*. Rods of ductile material with short reduced area, such as I and II of Fig. 42, may show a higher ultimate strength than a rod for which

* See TIMOSHENKO's "Strength of Materials," p. 617; SEELY's "Advanced Mechanics of Materials," Chap. XI.

the minimum section is longer, as in III (Fig. 42). In rods I and II the portion with the minimum section is close to the larger sections which hinder the necking. In rod III, on the other hand, most of the portion of minimum section is so far removed from the larger sections that necking takes place without hindrance.

It is not absolutely necessary to make test bars of the form of Figs. 36 and 37. Any bar of uniform section will do, and many tests are made with bars as they come from the rolls. The

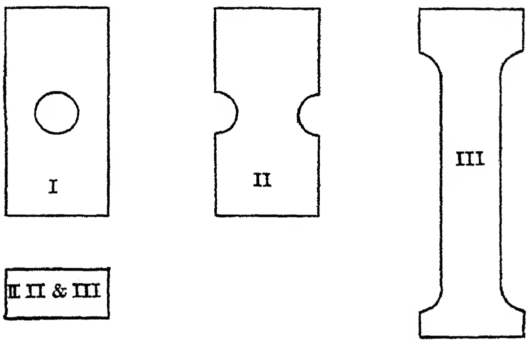


FIG. 42.—Reduced sections.

forms of Figs. 36 and 37 have the advantage that failure always occurs inside the gage length.

36. Yielding of Steel.—Permanent deformation of steel takes place by shearing deformations along microscopic planes. These planes of shear are called *slip planes.* The direction of the slip planes in a small portion of a steel bar varies greatly since the deformation depends upon the orientation of the steel crystals. A group of slip planes, however, follows the direction of maximum shearing stress. If the steel is covered with mill scale, the deformation caused by a group of slip planes fractures the scale and appears as a line on the surface. As the loading is continued and the area of the surface which has reached the yield point enlarges, a considerable portion of the scale may break from the bar. This is a definite indication that the yield point, for that part of the bar, has been reached.

Figure 43 shows the yielding of two 2½-inch by ¼-inch bars of soft steel in tension. Before testing, all of the plain bar and all of the notched bar except about 2 inches at the middle had been painted with a thin coat of whiting mixed with water. When tested, a fine line first appears and

extends gradually across the bar. As the stretching is continued, other lines form; wide bands of scale bend up and finally break off. This accounts for the relatively large areas which appear entirely dark. Frequently the scale breaks off on one side of an original fine line and holds on the other. The boundary of the dark area then locates the original line of the group of slip planes. A number of such boundaries at 45 degrees are shown on the figure. An area at the top of each bar has not yet reached the yield point. On the left side of the left bar is a fine line at an angle of 45 degrees to the left of the vertical upward. This line had just formed when the loading was stopped. The heavier line on the right side at right angles to the fine line

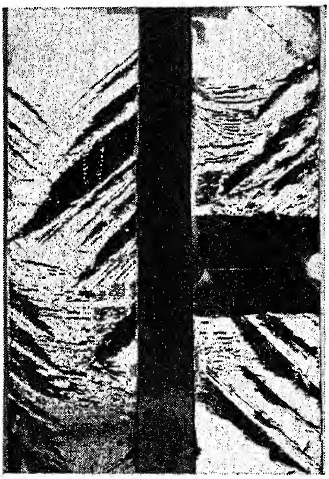


FIG. 43.—Plain bar and notched bar in tension.

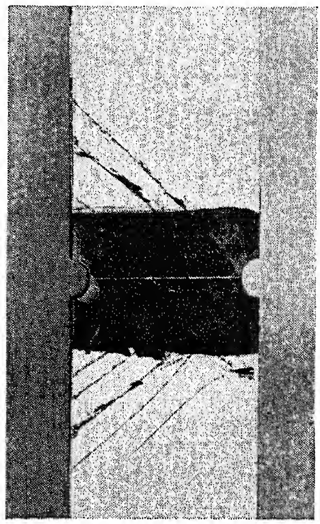


FIG. 44.—Notched bar beginning to yield.

is beginning to lose the scale. The still broader lines below these two have lost much of the adjoining scale, mostly on the upper side. At the top of the notched bar are two fine lines at 45 degrees to the right of the vertical, and one fine line at 45 degrees to the left of the vertical. At the bottom of the undisturbed area of the notched bar a fine line starts nearly horizontally from the left side and then curves upward until it intersects the upper one of the fine lines which make 45 degrees to the right of the vertical. Many fine lines may be seen between the heavy lines.

The load on the left bar at the end of the test was 23,300 pounds, which is 37,280 pounds per square inch. The final load on the notched bar was 23,200 pounds. This bar had previously been loaded to 22,000 pounds. The east side of this bar at 22,000 pounds total load is shown in Fig. 44. The jaws of the testing machine were slightly closer together on the south side than on the north. Consequently, the load was eccentric with the maximum stress on the south side until after some yielding had taken place. All of the first lines started at the south notch.

The drawings of Fig. 45 show four stages of a similar notched bar. At 12,000 pounds pull (an average stress of 24,000 pounds per square inch over minimum section) a horizontal crack, ¾ inch long, extended from the south notch. The crack on the east face was ¼ inch above that on the west face. Evidently the plane of shear made an angle of 45 degrees with the east face and the parallel west face and was perpendicular to the south face. This is true for all lines that run horizontally on the 2½-inch faces. At 14,000 pounds the line had widened into a triangular band. At 15,000 pounds the band was 1.45 inches long. Between 15,000 pounds and 16,000 pounds a line separated from the original triangle at a small angle with the horizontal and gradually curved until it became horizontal at about ¼ inch above the first line.

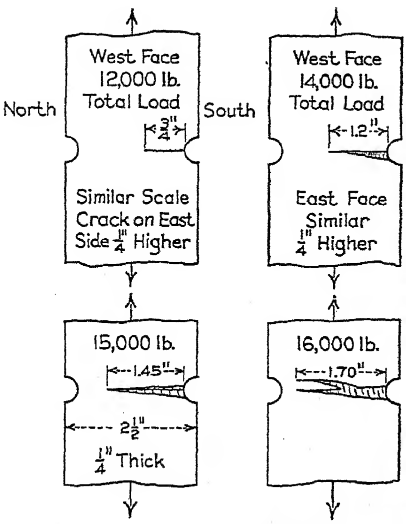


FIG. 45.—Four stages of yielding of notched bar.

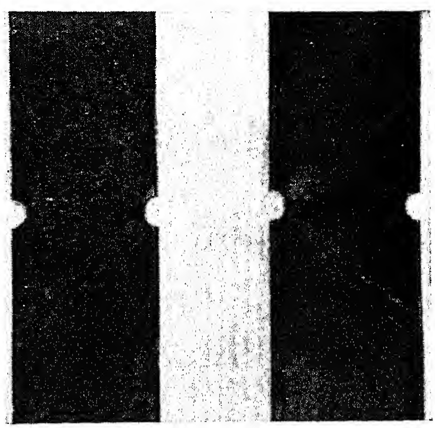


FIG. 46.—Opposite sides of similar bars at yield point.

Figure 46 shows the appearance of the scale on the west side (which was not painted) of a notched bar at 22,000 pounds load. The second bar of Fig. 46 is a similar test piece as viewed from the east.

Figure 47 shows a 2½-inch by ¼-inch bar with a ½-inch hole at the center. Part of the surface, including the central portion, has been coated with whiting. When the load reached 17,800 pounds, a fine line started at the right side of the circle a little below the center and extended about 0.4 inch slightly below the horizontal. At 18,700 pounds there were two lines on the right and one line on the left. At 20,000 pounds there were several lines extending to the edge on both sides, together with the five lines at about 45 degrees that show on the photograph. The two upper lines at the right start from the horizontal lines at a considerable distance from the hole. The other three lines radiate from the circumference of the circle. At a load a little above 20,000 pounds, the scale broke off over the horizontal lines.

37. Failure of Steel.—Failure of steel usually takes place by shear. In the tension test of a circular rod, a "crater" is formed. The material shears at 45 degrees with each element of the surface all around the circumference for about ¼ inch. This leaves a triangular wedge around the circumference which forms the wall of the crater. Occasionally the wall makes a complete circle on one portion of the piece. Usually, part of the crater wall is on one portion and part on the other. Between the crater walls the surface is perpendicular to the length of the rod. This surface, however, is made up of small hills and hollows with many slopes at about 45 degrees.

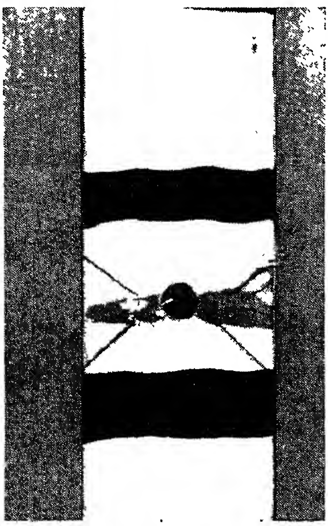


FIG. 47.—Yield point of bar with hole in center.

Figure 48 shows the failure of 2½-inch by ¼-inch low-carbon steel bars such as were shown at yield point in the figures of the preceding article. One bar of uniform cross section is shown before final failure. The maximum load was 33,800 pounds at an elongation of 1.92 inches in an 8-inch gage length. A neck then appeared and the load dropped to 31,000 pounds at an elongation of 2.37 inches. A line of minimum thickness developed at about 25 degrees with the horizontal. At 31,800 pounds pull the minimum was 0.204 inch near the south edge, 0.187 inch south of the middle, 0.184 inch north of the middle, and 0.199 inch near the north edge. At 31,000 pounds these had changed to 0.180, 0.165, 0.145, and 0.170, respectively. Rupture took place with a dull sound. When the machine was stopped and the tension balanced at 1,000 pounds, the bar was found to be still holding at the south (right) edge. The minimum width after failure was 1.90 inches. For about ½ inch at the left, shear at 45 degrees with the front and rear surfaces took place with the hollow above. Next there is a short shear surface perpendicular to the front face at 45 degrees with the vertical of the photograph. To the right of this, the shear again makes 45 degrees with the front and rear surfaces with the hollow above instead of below.

On the right of Fig. 48 is one portion of another plain bar. Some of the readings were

Load, Pounds	Elongation in 8 Inches, Inches
33,760	1.62
33,850	1.94
33,650	2.22
33,500	2.30
31,000	2.58
27,600	2.59 Broke suddenly with a sharp noise

The fracture of this piece was also at about 25 degrees. It was thinner at the middle than at the edges along the fracture. This bar sheared toward the middle from each broad face at about 45 degrees. The left half has the hollow below, while the right half has the ridge below. When the two

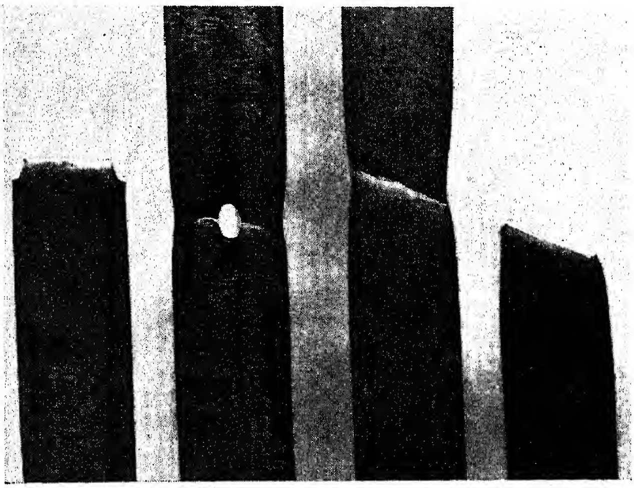


FIG. 48.—Failure of flat steel bars.

portions are placed together, they touch on the two edges and are at a considerable distance apart for most of the intervening section. The tension in each bar seems to have been greater at some little distance from the edges. It is probable that fracture started at the thinnest section near the middle.

A vertical line in the front face of both bars shows that the lower portion was displaced by shear toward the right.

At the left of Fig. 48 is the lower portion of a 2½-inch by ¼-inch bar with ½-inch semicircular notches opposite each other. At 29,720 pounds the elongation of 8 inches was 0.76 inch. At 29,730 pounds it was 0.82 inch. The load dropped to 25,800 pounds with an elongation of 0.84 inch when a horizontal crack appeared at the left notch. At 21,550 pounds the bar failed quietly. For most of the section, a single shear plane extends upward at 45 degrees from the front to the rear surface. For a short distance at the left end there are two planes with the hollow in the lower portion. Another bar of the same form, which had been loaded to 23,000 pounds several days before, had a maximum at 30,500 pounds. A crack

one-third of the way across from the right notch was seen when the load had dropped to 24,000 pounds. A crack appeared on the left side at 19,000 pounds. Load fell to 11,000. With the machine not running, the bar broke quietly after about 30 seconds. Most of left side fractured along a single plane upward from front to rear. Most of the right fractured along a single plane upward from rear to front. Near the middle shear planes have several directions. For both pieces, the width of the bar after fracture was 2.40 inches, and the width at the notch was 1.90 inches.

Another piece of Fig. 48 is a 2½-inch by ½-inch soft-steel bar with a ½-inch circular hole at the middle. At a load of 27,400 pounds, the elongation in 8 inches was 0.56 inch. At 27,500 pounds the elongation was 0.59 inch. At 24,750 pounds and an elongation of 0.64 inch, cracks appeared on both sides of the hole. The machine was stopped at 14,000 pounds, with an elongation of 0.70 inch. The transverse dimension of the hole was then 0.48 inch and the length (including the width of the cracks) was 0.88 inch. For another test, a ½-inch rod was forced into the hole before loading. At 25,000 pounds and an elongation of 0.34 inch in 8 inches, the rod was found to be loose. The maximum load was 27,200 pounds. Cracks appeared at the hole at 20,600 pounds with an elongation of 0.64 inch. The machine was stopped at 14,600 pounds. The transverse diameter of the hole was then 0.51 inch, and the length was 0.82 inch.

Failure took place by shear at 45 degrees with the front and rear surfaces—sometimes along a single plane, more frequently along two planes with a ridge or hollow at their intersection, sometimes along three or four planes.

The condition of the surfaces of these bars gives some indication of the final stresses. The uniform bars were subjected to high stress throughout. All of the scale has been broken off and the color is quite uniform. On the notched bar the maximum stress was smaller except at the reduced section. The maximum stress in the bar with the hole was still smaller. Many of the shear lines in the scale are still in evidence. An area on each side of the hole, in the form of a half ellipse about 2 inches long, shows the effect of intense stress. Above and below the hole are dark areas at which the stress was a minimum.

38. Steel Compression and Bearing.—Ductile metals in compression yield laterally. If the length of the test piece is small compared with its transverse dimensions, a metal of high ductility may be shortened indefinitely with a corresponding increase in cross section and total resisting stress. A relatively longer piece bends, as shown in the solid cylinder at the left of Fig. 49. Short tubes frequently fail by expansion at some one section followed by longitudinal splitting (Fig. 49).

Figure 50 shows two illustrations of bearing. The round cold-rolled steel rod at the right rests on a hard-steel plate. The load of 100,000 pounds was applied by a hard-steel plate, 1 inch thick, with a ¾-inch semicircular notch which exactly fitted the rod. (This is the plate of the shear apparatus of Fig. 24.)

Although the rod was originally straight and was loaded on a plane surface of the lower plate, it has been bent upward at each end. The circular steel plate, the shear plate, and the rod are shown together at the right of Fig. 51. The final surface of

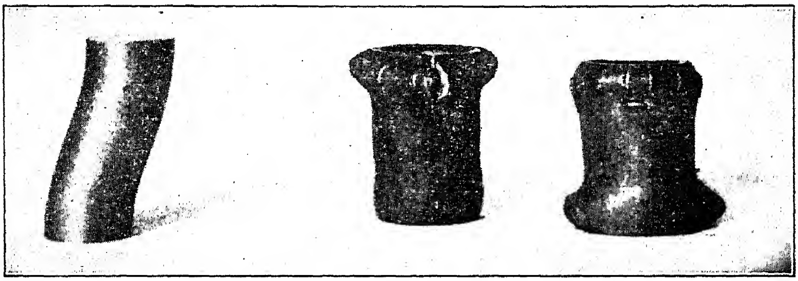


FIG. 49.—Metal in compression.

contact between the rod and the lower plate is diamond shaped. Its length is 2½ inches (the diameter of the plate). Its width at the middle is 0.56 inch, and its width at the ends is about 0.04 inch. The surface of this area is nearly plane. The maximum transverse thickness at the middle is 0.773, which shows that the material has been forced out laterally 0.023 inch. The depression on the opposite side, where it was compressed by the semicylindrical surface of the shear plate, is 0.06 inch at the ends. The minimum thickness is 0.628 inch. Evidently some of the material has been displaced longitudinally as well as laterally.

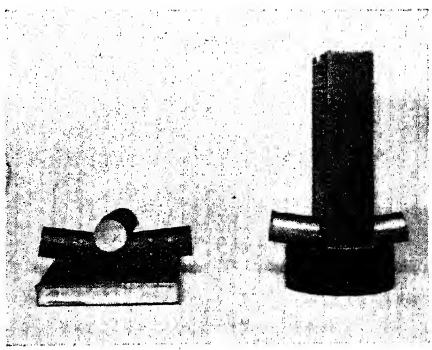


FIG. 50.—Bearing pressure.

The two ¾-inch rods at the left of Fig. 50 cross each other at right angles. The load was applied by medium-steel plate at the top which was similar to the plate at the bottom. The lower rod is 0.20 per cent carbon, while the upper rod is 0.44 per cent carbon (see Fig. 34, for the stress-strain diagrams for these rods in tension). The lower rod was flattened considerably at the bottom by the maximum load of 54,000 pounds, while the harder upper rod was flattened very little. The lower rod and plate are shown together in Fig. 51. The longer upper plate and the upper rod are shown with a short cast-iron block. The harder upper rod is deformed very little but remains nearly straight. On the other hand, there is a marked depression in the upper plate, which is of softer material than the upper rod. There is a very slight impression in the lower plate.

At the left of Fig. 51 are four slugs, each 1 inch long, which were cut out in double shear. The shear plate applies force by compression at the upper surface of the rod. With a semicircular notch exactly fitting the rod, this pressure is fairly uniformly distributed. The first slug at the left of Fig. 51 is cold-rolled steel, originally 0.749 inch in diameter, which was tested with the ¾-inch notch of the shear plate. The maximum load in double shear was 59,730 pounds. (Another test from the same rod failed at 59,450 pounds.) The vertical diameter of the slug after failure was 0.7486 inch and the maximum horizontal diameter was 0.751 inch. The third slug was sheared from the same cold-rolled rod.

The shear plate of Fig. 51 was inverted and the 1-inch semicircular notch was used on the ¾-inch rod. The maximum load was 59,500 pounds.

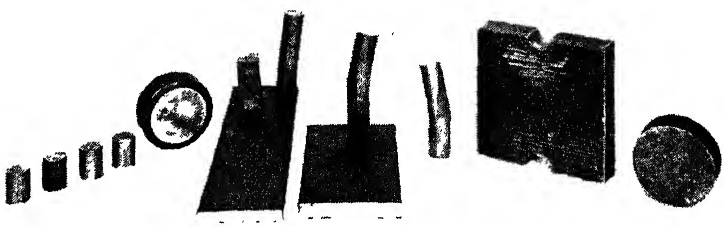


FIG. 51.—Bearing and shear failure with some equipment.

This shows that the concentration of bearing stress resulting from failure to have an exact fit between the rod and the shear plate causes very little error in the shearing strength. The vertical diameter of this slug after failure was 0.741 inch and the horizontal diameter was 0.753 inch. For the fourth slug, the plane side of the shear plate was used. The maximum load was 58,450 pounds, which shows that the concentration of compressive stress caused by using a plane shear plate on a round rod makes an appreciable error in the shearing strength. The transverse diameter of the slug was 0.760 inch. The vertical diameter at the middle was 0.723 inch. It was 0.704 inch at one end and 0.703 inch at the other end. An hourglass-shaped plane surface at the top showed the amount of material which had been displaced by compression. This surface was 0.45 inch wide at the ends and 0.26 inch wide at the middle. At the ends of the slug, about 0.02 square inch of the steel was cut off by bearing before the remainder failed by shear.

When the apparatus of Fig. 24 is used to test in single shear, if the slug is made too short, a considerable part of the cross section will be cut off by bearing stress, and the resulting shearing strength will be incorrect.

Problems

1. Calculate the shearing strength of the cold-rolled steel of the slug at the left of Fig. 51.
2. A test in single shear of the cold-rolled steel of Fig. 51 gave the following readings as the load was applied uniformly at a slow speed: 28,690 lb., 28,720 lb., 28,750 lb., and 28,700 lb. Calculate unit shearing strength.

   Ans. Max. $s_s = 65{,}700$ lb./in.^{2}

3. A second test of the steel of Problem 2 in double shear gave 59,100 lb., 59,250 lb., 59,450 lb., and 54,900 lb. Find the shearing strength.
Ans. s_s = 67,460 lb./in.^{2}
4. The double-shear tests of a 0.20 per cent $\frac{3}{4}$-in. steel rod gave maximum values of 44,000 lb. and 43,300 lb. Find the shearing strength.
Ans. s_s = 49,800 lb./in.^{2}; s_s = 49,000 lb./in.^{2}
5. Single-shear tests of the bar of Problem 4 gave maximum loads of 21,830 lb. and 22,150 lb. Find the maximum shearing stress.
Ans. s_s = 49,400 lb./in.^{2}; s_s = 50,100 lb./in.^{2}

The bearing strength of a solid depends upon the relative size of the surface of contact and the entire dimensions of the body. In the treatment of bearing stress there are two limiting cases. The first is that shown in Fig. 52, in which the surface of contact is equal to the entire cross section of the body *B*, and the length in the direction of the applied force is at least equal to the thickness of the body. In this case the bearing strength is equivalent to the compressive strength. Used in this way, a soft material like babbitt metal would show little bearing strength. Figure 53 shows a second case. Here the load is applied to a small portion of the body which is of unlimited extent or is confined laterally by another body. The portion outside the loaded area acts as a hoop to prevent the lateral expansion. In this form, a body composed of *separate particles* may have considerable bearing strength if the coefficient of friction is fairly high. Dry sand is an example. In a mass of wheat or flaxseed, where the coefficient of friction is small, the bearing strength is very low.

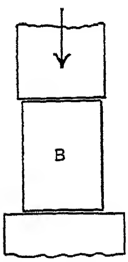


Fig. 52.—Bearing.

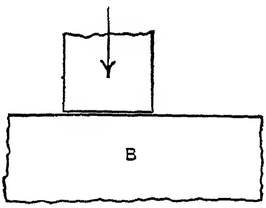


Fig. 53.—Bearing on large body.

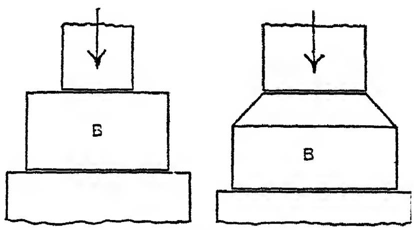


Fig. 54.—Cases of bearing pressure.

Figure 54 shows two types of bearing pressure intermediate between Figs. 52 and 53. When a post of wood or metal is placed on a concrete wall with no metal plate between them to transmit the pressure, the concrete wall or pier is given the form of one or the other support *B* of Fig. 54.

Cutting with a knife or chisel depends upon the bearing strength of the tool and of the material which is cut. The bearing strength of the tool under the conditions of Fig. 52 must be greater than the bearing strength of the material under the conditions of Fig. 53. At first there is a depression in the material under the edge of the tool, as shown in Fig. 55, I. When the unit stress under the edge of the tool exceeds the bearing

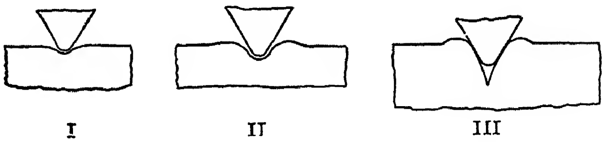


FIG. 55.—Cutting.

strength of the material, it is permanently pushed back. In a plastic non-porous material, some of the substance is forced up by the pressure, as shown in Fig. 55, II. In a porous body such as wood, there is an increase in density adjacent to the cutting surface. The wheel of a wagon cutting into soft earth illustrates both cases. If the earth is wet clay, it is pushed up

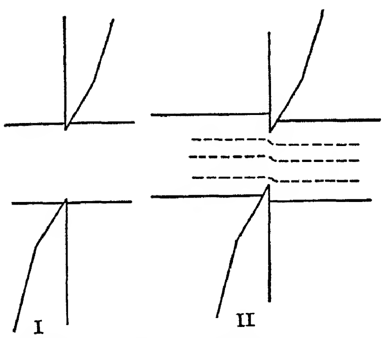


FIG. 56.—Cutting with shears.

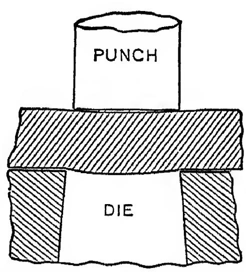


FIG. 57.—Punching a plate.

at the sides of the tire. If it is dry loam, it is compressed under the tire.

When a cutting tool has penetrated a little distance, it acts as a wedge and exerts a tensile stress upon the material in front of its edge. This is shown in Fig. 55, III.

Figure 56 shows the behavior of a pair of scissors or shears. At the beginning, the cutting is due to the bearing stress on the cutting edges, as shown in Fig. 56, I. As the edges penetrate

into the material, the bearing force is increased at each blade. These forces produce shearing stresses in all portions of the body in the plane of the cutting edges. The corresponding shearing deformations are shown by the dotted lines in Fig. 56, II.

Figure 57 represents the punching of a metal plate. The plate is bent a little at first, which makes the surface of contact a narrow ring at the edge of the punch and die. When the compressive stress on these rings exceeds the bearing strength of the plate, cutting begins. This is followed by shear as in the operation of cutting with scissors.

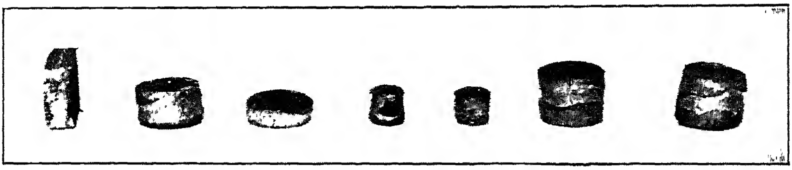


FIG. 58.—Slugs punched from steel plates.

Figure 58 shows some slugs which were punched from steel plates. The ends of the slugs are permanently curved by bending. The sides show the characteristic appearance of the fracture. The two slugs of least diameter were punched from a relatively thick plate. The bearing stress was too great, and the punch failed after making about a dozen holes.

39. Cast Iron.—Table X gives the results of a test of a cast-iron bar, which was about $\frac{3}{4}$ inch square and 10 inches long. The bar was tested as cast without removing the hard skin at the surface. The ends were cut off square and the bar was tested on parallel compression plates without ball-and-socket joints. Two of these plates are shown in Fig. 51. The short cylinder at the right was made of soft steel which was carefully case-hardened. The central portion of the other short cylinder was made of hardened tool steel. To prevent fracture and the danger of flying pieces if fracture should occur, this hard-steel cylinder has been forced down under the testing machine into a rim of soft steel.

These cast-iron test pieces have internal stresses. When loaded the first time, the stress diagram is irregular and begins to curve at very low loads. After one loading the diagram approaches a straight line under loads which greatly exceed the allowable stress.

TABLE X.—COMPRESSION TEST OF CAST IRON

Bar 0.792 inch square; area, 0.627 square inch. Gage length, 8 inches. Bar previously loaded to 51,000 pounds per square inch. Released to 100 pounds per square inch and held at this stress for 10 minutes before beginning test.

Total load, pounds	Unit stress, pounds per square inch	Extensometer reading	Deformation		
			Divisions	Inches	Unit
63	100	3,294	0	0	0
1,254	2,000	3,260	34	0.00068	0.000085
2,508	4,000	3,204	90	180	225
3,762	6,000	3,142	152	304	380
5,016	8,000	3,092	202	404	505
6,270	10,000	3,035	259	0.00518	0.000647
7,524	12,000	2,981	313	626	782
8,778	14,000	2,922	372	744	930
10,032	16,000	2,866	428	856	0.001070
11,286	18,000	2,806	488	976	1220
12,540	20,000	2,759	535	0.01070	0.001337
13,794	22,000	2,700	594	1188	1485
15,048	24,000	2,635	659	1318	1648
17,556	28,000	2,522	772	1544	1930
18,810	30,000	2,460	834	1668	2085
20,064	32,000	2,408	886	0.01772	0.002215
21,318	34,000	2,350	944	1888	2360
22,572	36,000	2,297	997	1994	2492
23,826	38,000	2,236	1,058	2116	2645
25,080	40,000	2,180	1,114	2228	2785
26,334	42,000	2,110	1,184	0.02368	0.002960
27,588	44,000	2,050	1,244	2488	3110
28,842	46,000	1,985	1,309	2618	3272
30,096	48,000	1,918	1,376	2752	3440
31,450	50,160	1,820	1,474	2948	3685
31,977	51,000	1,780	1,514	0.03028	0.003785

The stress-strain diagram up to 40,000 pounds per square inch is plotted on Fig. 59.

Table XI gives the average of 6 tests of cast iron in tension which were made at the Watertown Arsenal. While the diagram for any single test piece would show considerable deviations from a smooth curve, the average of 6 (as is always true if the experi-

ments are honestly conducted) shows very little deviation. Figure 60 shows the same curve for tension, together with a compression curve made from the average of 12 tests of cast iron from the same heat (see "Tests of Metals," 1885, pages 475–490).

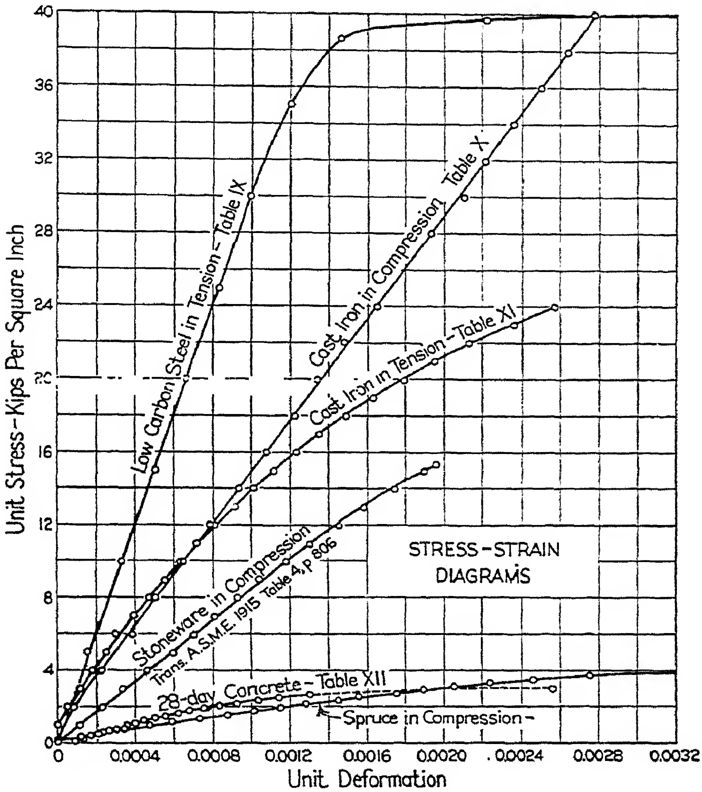


FIG. 59.—Stress-strain diagrams for several materials.

Figure 61 shows on the right the two pieces of a cast-iron bar, originally 2 inches long, which was cut from a bar similar to that of Table X. The dimensions were 0.775 inch by 0.770 inch. The area was 0.597 square inch. The maximum load was 64,780 pounds. The load dropped to 63,550 pounds when the piece broke suddenly. With a unit bearing stress of more than 100,000 pounds per square inch, ordinary steel compression heads would be indented 0.1 inch. It is necessary to use especially prepared heads, such as those of Fig. 51.

Figure 61 shows on the left two pieces of a cast-iron test cylinder, which was originally 1⅛ inch in diameter and 2¼ inches long. This failed under a load of 98,900 pounds.

Both test pieces of Fig. 61 failed by shear. The angle between the plane of shear in each case is about 60 degrees. It will be shown in Art. 192 that the angle of shear in compression is

Table XI.—Tension Test of Cast Iron

From Watertown Arsenal, "Tests of Metals," 1905. Average of six tests, specimens 8014, 8041, 8050, 8051, 8053, and 8063. Diameter, 1.129 inch; area, 1 square inch. Gage length, 10 inches. Bars machined to size, removing all the hard skin.

Load, pounds per square inch	Elongation, inches	
	In gage length	Per inch length
1,000	0	0
2,000	0.00056	0.000056
3,000	0.00115	0.000115
4,000	180	180
5,000	245	245
6,000	0.00318	0.000318
7,000	390	390
8,000	470	470
9,000	550	550
10,000	635	635
11,000	0.00723	0.000723
12,000	815	815
13,000	912	912
14,000	0.01005	0.001005
15,000	1118	1118
16,000	0.01225	0.001225
17,000	1348	1348
18,000	1488	1488
19,000	1633	1633
20,000	1795	1795
21,000	0.01947	0.001947
22,000	2126	2126
23,000	2364	2364
24,000	2570	2570
26,450 Average ultimate load		

45 degrees plus one-half the angle of friction. An angle of 60 degrees does not agree well with this formula. However, the pieces were bent considerably before breaking. The angle between an element of the surface and the plane of fracture is

about 38 degrees. On the assumption that the force is parallel to the element of the surface, instead of vertical, this gives

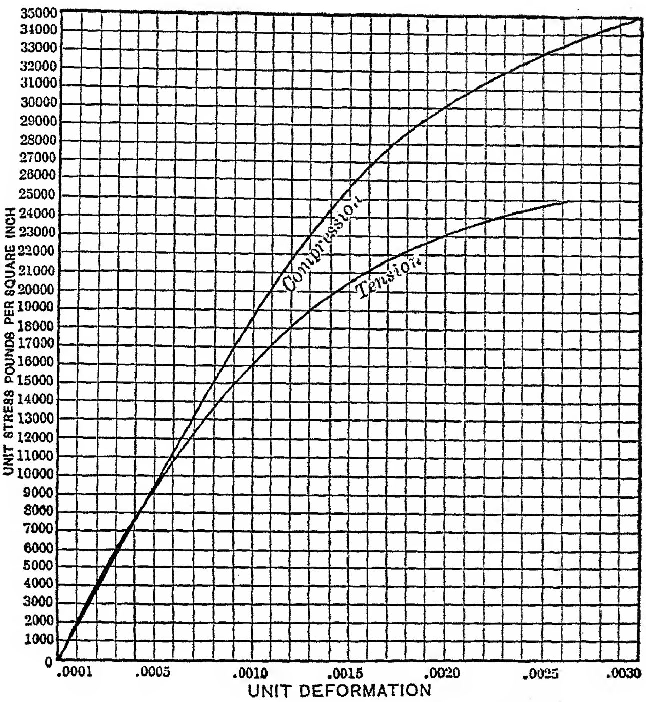


FIG. 60.—Stress-strain diagrams for cast iron.

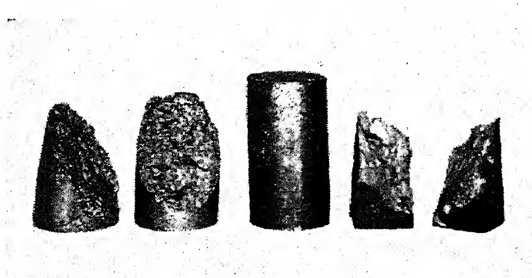


FIG. 61.—Failure of cast iron in compression.

52 degrees as the angle of failure. Pieces which bend little frequently give an angle of about 53 degrees with the horizontal.

Problems

1. Calculate the ultimate compressive stress of the parallelepiped of Fig. 61.
2. Calculate the ultimate compressive stress of the cylinder of Fig. 61.

3. From Table XI find the modulus of elasticity for cast iron in tension by means of the interval 1,000 to 6,000 unit stress.

Ans. $E = 15{,}700{,}000$ lb./in.^{2}

4. From Table X find E for cast iron in compression for the intervals 2,000 lb. per sq. in. to 12,000 lb. per sq. in. and 4,000 lb. per sq. in. to 14,000 lb. per sq. in. Use table of reciprocals.

Ans. $E = 14{,}350{,}000$; $E = 14{,}180{,}000$.

40. Concrete in Compression.—Table XII gives the records of a test of 28-day concrete with water-cement ratio of 1.3 by

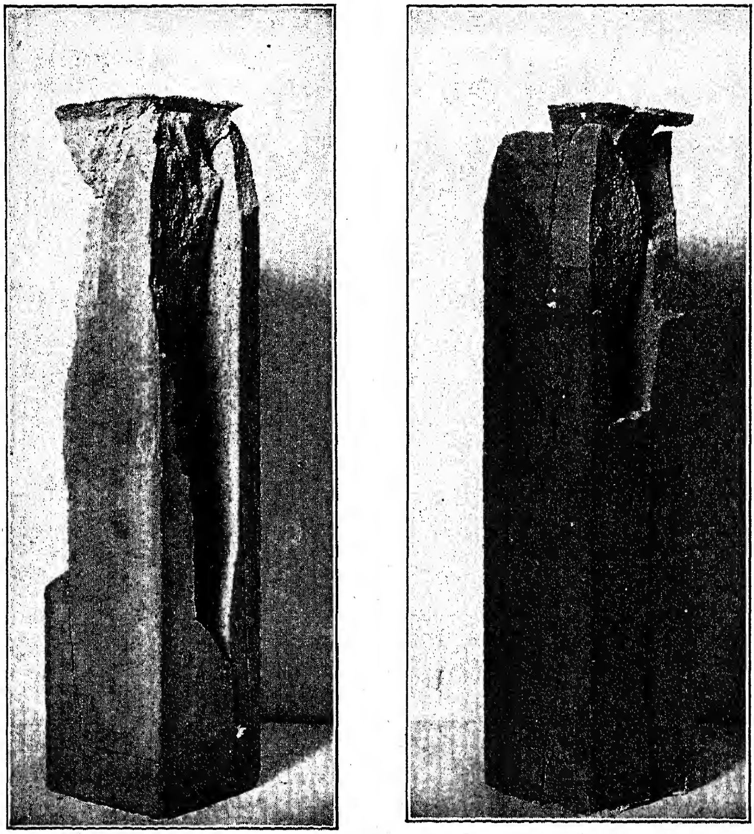


FIG. 62.—Cement in compression.

volume. Three consecutive runs were made. Each of the first two carried the unit stress to 1,272 pounds per square inch, while the last run was continued to failure at 3,015 pounds per square inch. The concrete curve of Fig. 59 was made from the last run.

When steel is stressed beyond the yield point and when timber is stressed considerably beyond the elastic limit, deformation

TABLE XII.—COMPRESSION TEST OF CONCRETE CYLINDERS

Diameter, 6 inches; length, 12 inches. Mix 1:4¼:4 by volume, water cement 1.3 by volume. Compression measured by two Berry strain gages. Cross-head speed, 0.05 inch per minute. Continuous run, gages read without stopping machine. Cylinders made and tested by Profs. J. R. Shank and G. E. Large.

Total load, pounds	Unit stress, pounds per square inch	Unit deformation		
		First run	Second run	Third run
2,000	71	0	0.000079	0.000088
4,000	141	0.000015	88	94
6,000	212	32	0.000108	0.000109
8,000	283	53	120	128
10,000	354	73	139	146
12,000	424	0.000095	0.000160	0.000171
14,000	495	0.000117	179	191
16,000	565	141	202	213
18,000	636	167	226	236
20,000	707	191	244	260
22,000	777	0.000212	0.000267	0.000280
24,000	848	235	286	304
26,000	919	259	312	326
28,000	989	284	335	349
30,000	1,060	310	358	373
32,000	1,131	0.000340	0.000381	0.000403
34,000	1,201	367	403	419
36,000	1,272	398	430	445
38,000	1,342	....	....	470
40,000	1,413	....	....	507
44,000	1,555	....	....	0.000557
48,000	1,696	....	....	619
52,000	1,837	....	....	683
56,000	1,980	....	....	753
60,000	2,120	....	....	835
64,000	2,260	....	....	0.000924
68,000	2,400	....	....	0.001035
72,000	2,544	....	....	1150
76,000	2,685	....	....	1300
80,000	2,825			
84,000	2,965			
85,400	3,015 Failed	....	....	0.002560

continues to increase with no increase of load. This is called *creep* or *flow*. For concrete, creep occurs at very low stresses. If the machine is stopped to take extension readings, the form of the curve depends largely upon the elapsed time. Different observers may get quite different curves. It is best to make all experiments for the determination of stress-strain diagrams with the machine running continuously at a constant speed. This is especially important for concrete. Professors J. R. Shank and G. E. Large, who made this test, kept the movable crosshead of the testing machine running at a constant speed of 0.05 inch per minute and read the two strain gages at intervals of 2,000 pounds total load.

Figure 62 shows the failure of two 4-inch by 4-inch blocks of 1:1 cement mortar. Each of these failed by shearing at the ends to form a pyramid and then splitting lengthwise. The unit compressive stress at failure was 4,000 pounds per square inch. At the right end of Fig. 32 are two pieces of a cylinder of dental plaster. This cylinder was originally $1\frac{1}{4}$ inches in diameter and 2 inches long. The cylinder failed by shearing to form a cone at a load of 1,700 pounds and then splitting. Similar concrete cylinders frequently fail in the same manner.

Table XIII gives the results of two tests of concrete which was twelve months old. Number 1055 was test core taken from a concrete pavement. Longitudinal and transverse deformations were measured by means of instruments which give unit deformation to one-millionth inch per inch. Up to 1,600 pounds per square inch, the readings for the two samples agree closely.

Problems

1. From the third run of Table XII, find E for 28-day concrete for the intervals 6,000 lb. to 16,000 lb., 8,000 lb. to 18,000 lb., and 10,000 lb. to 20,000 lb. total load. *Ans.* $E = 3{,}400{,}000$; 3,270,000; 3,100,000.
2. Solve Problem 1 from the first run and from the second run.
3. From Table XIII find E for 12-month concrete for cylinder 1055 for the intervals of 0 to 800, 100 to 900, and 200 to 1,000 unit stress. *Ans.* $E = 3{,}190{,}000$; 3,080,000; 3,090,000.
4. Solve Problem 3 for cylinder 104. *Ans.* $E = 3{,}260{,}000$; 3,260,000; 3,170,000.
5. Find Poisson's ratio for concrete from cylinder 1055 of Table XIII for loads of 700, 800, and 900 lb. per sq. in. *Ans.* 0.141; 0.139; 0.140.
6. Solve Problem 5 for cylinder 104.
7. Calculate Poisson's ratio for cylinders 104 and 1055 for unit stress of 2,600. Why are these results incorrect?

8. Plot curve for cylinder 1055 of Table XIII to the scale 1 in. = 500 lb. per sq. in. unit stress and 1 in. = 0.0002 unit deformation.

TABLE XIII.—POISSON'S RATIO FOR CONCRETE

Test cylinders, 9 inches long and 4.5 inches diameter. Gage length 4 inches. Tested by Dean A. N. Johnson. *Proc. A.S.T.M.*, 1924, Part II, p. 1024.

Unit stress, pounds per square inch	No. 104; 1:2:3 mix; age 12 months		No. 1055; 1:2:4 mix; age 12 months	
	Unit deformation in 0.000001 inch			
	Vertical	Horizontal	Vertical	Horizontal
50	16	2	15	3
100	33	4	26	5
200	58	7	59	10
300	89	10	91	13
400	121	15	121	17
500	151	18	155	22
600	181	24	185	26
700	213	28	220	31
800	245	32	251	35
900	278	38	286	40
1,000	311	42	318	44
1,200	383	55	385	56
1,400	455	72	454	69
1,600	540	87	525	83
1,800	649	118	600	97
2,000	794	195	689	123
2,200	946	326	790	179
2,400	. . .	. . .	919	208
2,600	Failed at 2,500 lb./in.²		1,038	288
			Failed at 3,500 lb./in.²	

41. Curves for Various Materials.—Figure 59 offers a comparison of several of the most important materials. The first curve is a part of the stress-strain diagram for the soft steel of Table IX. The second curve is part of the diagram for cast iron of Table X. The slope from the vertical is about twice as great as that of the steel. The modulus of elasticity of cast iron is one-half the modulus of steel. The third curve represents cast iron in tension. The next curve, which is plotted from Table IV

(page 806) of *Transactions of The Society of Mechanical Engineers* for 1915, gives the stress-strain diagram for stoneware in compression to 15,500 pounds per square inch where the extensometers were removed. This material is similar to the best hard paving brick. The ultimate compressive strength in the form of a cylinder 16 inches long and 1 inch in diameter was 21,500 pounds per square inch. The diagram is nearly straight. Porcelain for electrical insulators has the same compressive strength and higher modulus of elasticity. The fifth curve is the diagram for

FIG. 63.—Hard brick in compression.

28-day concrete from Table XII. The last curve is the diagram for western spruce such as is used for airplane struts. The data were taken from a test of strut *E*-2, 12.625 inches long, which was made on the 220,000-pound Emery testing machine of the Bureau of Standards.

42. Johnson's Apparent Elastic Limit.—Since it is somewhat difficult to determine the proportional elastic limit accurately, especially in hard steel, where there is a wide range between this stress and the yield point, and in materials (such as cast iron) which have no yield point, the late Prof. J. B. Johnson proposed another point which he called the *apparent eleastic limit.** He defined the apparent elastic limit as "the point on the stress diagram at which the rate of deformation is 50 per cent greater than at the origin." It is that point on the curve at which the

* See JOHNSON'S "Materials of Construction," pages 18–20.

slope of the tangent from the *vertical* is 50 per cent greater than that of the straight-line part of the curve.

This term has not yet come into general use among engineers. In some investigations of the strength of materials, it has been found useful in comparing the results of different tests.*

Figure 64, which is drawn from the concrete test of Table XII, illustrates the graphical determination of Johnson's apparent

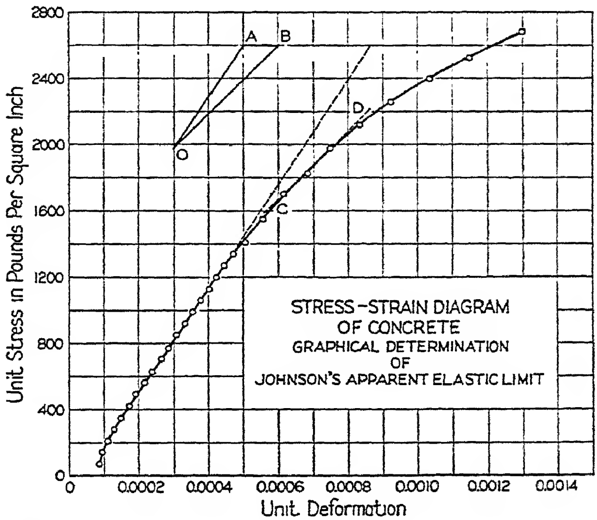


FIG. 64.—Graphic determination of Johnson's elastic limit from Table XII.

elastic limit. Through the point A a line is drawn parallel to the straight-line portion of the stress-strain diagram. The intersection of this line with the vertical line two units to the left of A gives the point O. The point B, with the same ordinate as A, is three units to the right of the vertical line through O. The line $O\,B$ gives the direction of the required slope. The broken line $C\,D$ is tangent to the stress-strain diagram at about 1,820 pounds per square inch.

Johnson's apparent elastic limit may be calculated from the intervals of the table without plotting the stress-strain diagram. For the third run of Table XII, inspection shows that the difference of deformation for an interval of 2,000 pounds total load is fairly constant from 8,000 pounds to 30,000 pounds. Four intervals of 12,000 pounds each, beginning with the interval from 10,000 to 22,000, give an average unit deformation of

* See work of H. F. MOORE in *Bull.* No. 42 and ALBERT J. BECKER in *Bull.* No. 85 of the University of Illinois Engineering Experiment Station.

0.0001335. This is 0.0000445 for an interval of 4,000 pounds. Three-halves of this interval is 0.0006675. From 48,000 to 52,000 pounds the deformation is 0.000064. From 52,000 to 56,000 the deformation is 0.000070. Johnson's apparent elastic limit lies between 50,000 and 54,000 and divides the interval in the ratio of 22.75 to 60.

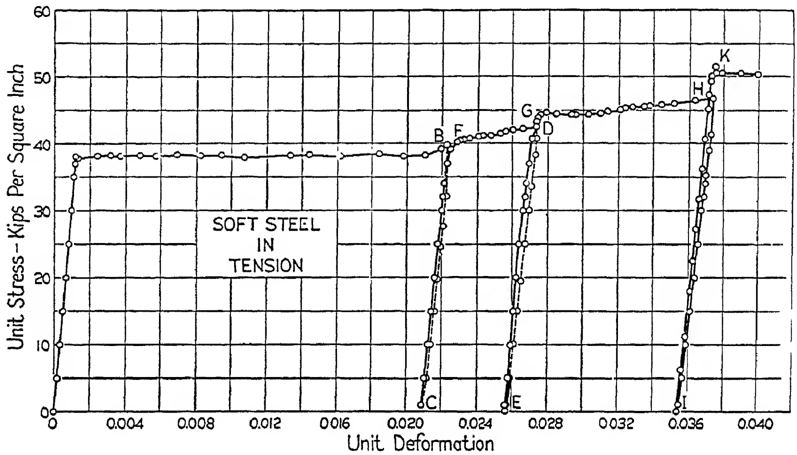


Fig. 65.—Repeated loading of soft steel from Table XIV.

$$\text{Load} = 50{,}000 + \frac{22.75 \times 4{,}000}{60} = 51{,}517 \text{ pounds, which is}$$

1,824 pounds per square inch.

Problems

1. Find Johnson's apparent elastic limit for the hard steel of Appendix C. *Ans.* 70,000 lb./in.^{2}
2. Find Johnson's apparent elastic limit for the longleaf yellow pine of Table IV.

43. Work Hardening.—Figure 65 shows part of the stress-strain diagram of low-carbon steel (carbon, 0.20 per cent). The test piece was taken from the same rod as that of the lower curve of Fig. 34. The plastic stage ended at unit elongation of 0.0199 and unit stress of 38,059 pounds per square inch (Table XIV). The loading was continued to a unit elongation of a little over 2 per cent and unit stress of nearly 40,000 pounds per square inch. The load was then released to 1,000 pounds per square inch. The stress-strain diagram, shown by the broken line, is very steep at first. From about 35,000 pounds per square inch to about

5,000 pounds per square inch, it is practically a straight line. Below 5,000 pounds per square inch, the slope becomes still smaller.

After reaching 1,000 pounds per square inch, loads were applied up to unit stress of 42,660 pounds per square inch. The time down from point *B* of the curve to point *C* was 14 minutes, and the entire time from *B* to *D* was 36 minutes. The ascending curve is practically straight and nearly parallel to the descending curve from 5,000 to 35,000 pounds per square inch. The curves intersect at 37,000 pounds per square inch. The ascending curve deviates from the straight line some distance below the highest stress of the first portion. Beyond *F* the curve is a continuation of the first portion to *B*.

After reaching the stress of 42,660 pounds per square inch at *D*, the load was released to 100 pounds per square inch at *E*. The descending curve is much like *B C*. The bar then stood for 16 hours under a small load, before measurements were made for the curve *E G H I*.

For the third application of the load, after an interval of 16 hours, the upward curve from *E* is a straight line to 40,000 pounds per square inch. It intersects the previous ascending curve *F D* at its highest point *D* and rises several thousand pounds higher to a new yield point at *G*. After passing the upper yield point, the curve drops to a lower yield point and then rises gradually to *H*. It then drops to 100 pounds per square inch at *I*.

After standing 48 hours under a small load, other loads were applied which are not given in Fig. 65 or in Table XIV. One loading was carried up to 43,000 pounds per square inch, which is only a little above *D*, the maximum of the second run. When this load was relieved, there was no additional set. The next loading was carried to 46,800 pounds per square inch, which was nearly to *H*, the maximum of the third loading. After this load was released, there was an additional permanent set of 0.0001 inch in the gage length. The last loading, represented by curve *I H K*, was then applied. The curve continues nearly straight up to 50,000 pounds per square inch. There is a marked upper yield point at *K* at 51,530 pounds per square inch.

When soft steel is stretched at ordinary temperatures to a stress beyond its yield point, it receives a permanent elongation. When it is again loaded, a new yield point is found, which is

above the maximum stress of the previous loading. With low-carbon steel the yield point may be raised a number of

TABLE XIV.—REPEATED TENSION IN SOFT STEEL

Test piece from rod used for Appendix A, lower stress-strain diagram of Fig. 34. Crosshead speed, 1/40 inch per minute.

Unit stress, lb. per sq. in.	Unit deformation in 0.000001 in.	Unit stress, lb. per sq. in.	Unit deformation 0.000001 in.	Unit stress, lb. per sq. in.	Unit deformation 0.000001 in.
3 P.M., June 28		3:59 P.M.		Reversed	
100	0	40,180	22,815	41,422	37,307
1,000	25	41,084	24,042	38,962	37,217
5,000	145	42,032	26,062	35,440	37,072
10,000	320	42,664	27,355	30,000	36,812
15,000	482	4:21 P.M., June 28		25,000	36,587
		Reversed			
20,000	650	38,443	27,295	20,000	36,360
25,000	820	33,409	27,120	15,000	36,120
30,000	970	29,955	26,977	10,000	35,900
35,000	1,130	25,000	26,772	5,000	35,645
36,610	1,185	19,548	26,555	1,000	35,417
				100	35,367
38,262	1,235	15,000	26,317	9:58 A.M., June 29	
37,923	1,325	10,000	26,069	3 P.M., July 1	
38,126	2,430	5,000	25,837	6,772	35,592
38,104	3,712	1,000	25,627	11,287	35,765
38,059	19,860	100	25,570	18,059	36,007
		4:41 P.M., June 28		22,573	36,192
39,278	21,980	8:50 A.M., June 29		27,088	36,357
39,847	22,340	1,000	25,582		
Reversed		5,000	25,707	31,603	36,547
32,054	22,210	10,000	25,895	36,343	36,727
27,765	22,072	15,000	26,067	40,632	36,925
24,605	21,940	20,000	26,265	45,140	37,115
				47,404	37,215
19,814	21,755	25,000	26,462		
15,000	21,535	30,000	26,672	49,278	37,292
10,000	21,330	32,000	26,770	50,113	37,330
5,000	21,105	34,000	26,852	50,790	37,422
1,000	20,885	37,043	26,987	51,530	37,505
3:59 P.M.				50,677	37,530
5,000	21,010	40,750	27,162		
10,000	21,200	43,386	27,322	50,609	37,642
15,000	21,380	44,130	27,477	50,519	37,972
20,000	21,567	44,492	27,640	50,384	38,942
25,000	21,785	44,458	28,492	50,452	39,917
30,000	22,002	44,323	29,122		
32,000	22,080	45,011	32,092		
34,000	22,180	46,117	35,212		
37,170	22,312	46,456	36,312		
39,232	22,460	46,840	37,407		

times. The form of the curves depends upon the elapsed time between loadings and the total unit deformation.

When a ductile material at ordinary temperatures is permanently deformed by the application of force, it is said to be *cold worked.* Cold working raises the elastic limit and yield point of wrought iron and low-carbon steel. It also raises the ultimate strength in terms of the area after cold working. Cold-worked material is said to be *work hardened.* One great advantage in the use of low-carbon steel depends upon this property. When an inaccurate construction or unusual loading throws excessive stress on some part of a structure made of soft steel, some permanent deformation may occur but the structure is not weakened in the least.

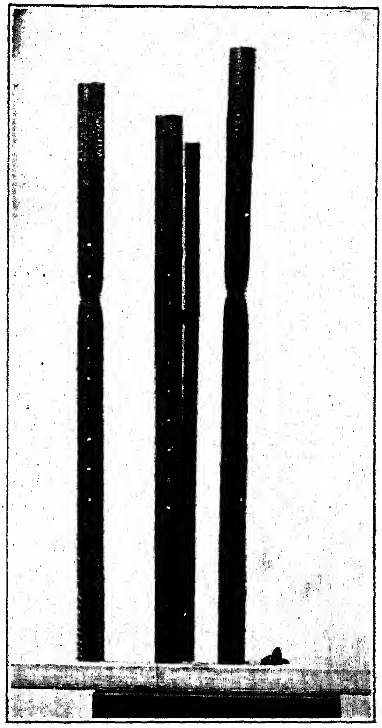


FIG. 66.—Soft steel in tension: left, cold-rolled; right, annealed.

Cold-drawing and cold rolling are two common methods of cold working. Cold-drawing is used in the manufacture of wire. Larger material is cold rolled. Figure 66 shows the effect of cold rolling. The middle rod is a piece of $\frac{7}{8}$-inch cold-rolled shafting. The left one is an exactly similar rod after testing in tension. Its ultimate strength was over 86,000 pounds per square inch, its yield point was about 80,000 pounds per square inch, and its elongation about 10 per cent. On the right is a third rod, originally like the others, which was annealed by heating to redness and slowly cooling to destroy the effect of the previous cold rolling. When tested in tension, its ultimate strength was found to be 60,000 pounds per square inch, its yield point was 40,000 pounds per square inch, and its elongation 22 per cent. It will be seen from these tests that cold rolling raises the yield point to nearly the ultimate strength and that it increases the ultimate strength a considerable amount.

Table XV is the record of the tension test of a piece of soft steel from the same rod as the test piece of Table XIV. Before testing in tension the piece was work-hardened by twisting until it made one complete revolution

TABLE XV.—TENSION TEST OF LOW-CARBON STEEL

Test piece from rod used for Table XIV. Previously tested in torsion. Twisted four revolutions in a length of 10 inches. Diameter after torsion about 0.749 inch. Original area used in calculation.

Total load, pounds	Unit stress, pounds per square inch	Elongation	
		Total, inches	Unit, inches per inch
443	1,000	0	0
2,215	5,000	0.00126	0.000157
4,430	10,000	266	332
6,645	15,000	406	507
8,860	20,000	548	685
11,075	25,000	0.00698	0.000872
13,290	30,000	858	0.001072
15,505	35,000	0.01042	1302
17,720	40,000	1212	1515
18,606	42,000	1298	1622
19,492	44,000	0.01374	0.001717
19,935	45,000	1412	1765
20,378	46,000	1466	1832
21,264	48,000	1552	1940
22,150	50,000	1636	2045
23,036	52,000	0.01730	0.002162
23,922	54,000	1838	2297
24,365	55,000	1894	2367
24,808	56,000	1956	2445
25,694	58,000	2060	2575
26,580	60,000	0.02180	0.002725
27,466	62,000	2308	2885
28,352	64,000	2446	3057
28,795	65,000	2534	3167
29,238	66,000	2640	3300
29,681	67,000	0.02714	0 003392
30,124	68,000	2814	3517
30,567	69,000	2910	3637
31,010	70,000	3014	3767
34,000	76,750	0.05	0.0062
36,000	81,260	0.08	0.010
37,000	83,520	0.10	0.0125
37,400	84,430	0.13	0.0162
37,450	Maximum		
26,000	Broke	0.16	0.02

Diameter of neck, 0.627 inch. Area, 0.309 square inch.
Actual unit stress at rupture, 84,000 pounds per square inch. Broke diagonally.

every 2½ inches and failed by twisting off at one of the grips. The remainder was then pulled in the tension machine. The torsion test reduced the diameter to about 0.749 inch. The original area of 0.443 square inch has been used in calculating the unit stress.

The tension test of this piece was run 20 hours after the completion of the torsion test. The table shows no indication of a yield point. The elongation of the gage length for a stress interval of 2,000 pounds per square inch is 0.00056 at 10,000 pounds per square inch. It is 0.00084 from 48,000 pounds per square inch to 50,000 pounds per square inch and is 0.00200 from 68,000 pounds per square inch to 70,000 pounds per square inch. The final elongation was less than 2 per cent. Most of the fracture took place along a single plane.

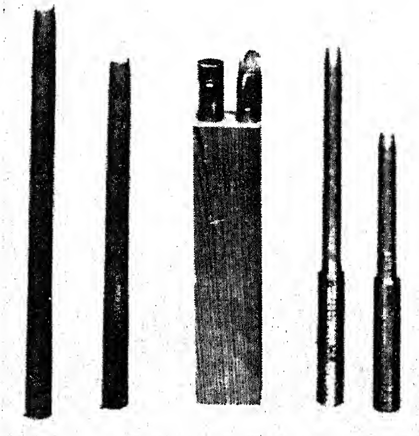


FIG. 67.—Test pieces of Table XII and Table XV.

The short piece at the right on the block of Fig. 67 shows the fracture. After testing, the piece was ground down ${}^1\!/_{16}$ inch to secure a plane surface for a Brinell hardness test. The large depression at the front is the indentation of the sphere of the Brinell apparatus. The other short piece is a part of the original bar which was not subjected to stress. The Brinell indentation on this piece is larger than that on the tested piece. Its Brinell number was 118, while that of the tested piece was 207.

The Rockwell B hardness test was also applied. At the surface of the work-hardened piece this test gave 81, while at the surface of the original material it gave 60. At ${}^1\!/_{16}$ inch below the surface the Rockwell test gave 90 for the work-hardened piece and 69 for the other.

The Brinell hardness is made with a hard-steel ball 10 millimeters in diameter. For hard metals such as steel this ball is pushed down on a plane surface with a force of 3,000 kilograms for 30 seconds. The Brinell hardness number is the quotient obtained by dividing the load P (3,000 kilograms for steel) by the area of the depression.

$$H = \frac{P}{\frac{\pi D}{2}(D - \sqrt{D^2 - d^2})},$$

in which D is the diameter of the ball and d is the diameter of the depression. For instance, if the depression has a diameter of 6 millimeters,

$$H = \frac{3,000}{5\pi\,(10 - 8)} = \frac{3,000}{10\,\pi} = 95.$$

Problems

1. From the first run of Table XIV calculate E for the interval 1,000 lb. per sq. in. to 25,000 lb. per sq. in. unit stress. Solve also for the interval 5,000 lb. per sq. in. to 30,000 lb. per sq. in.
Ans. 30,200,000; 30,300,000.
2. From the second run of Table XIV (beginning at 3:59) calculate E for the interval 5,000 to 25,000 unit stress and for the interval 10,000 to 30,000 unit stress. *Ans.* 25,800,000; 24,900,000.
3. From the third run of Table XIV (beginning at 8:50, June 29) calculate E for the interval 1,000 to 25,000 and the interval 5,000 to 30,000 unit stress. *Ans.* 27,300,000; 25,900,000.

The results of Problem 2 with the measurements taken immediately after the load had been reduced to the minimum are much lower than those of Problem 1. The results of Problem 3 also are lower. However, the area at the beginning of these runs is smaller than the area at the beginning of the first run and the unit stresses are correspondingly greater. Also the initial length has been increased so that the unit deformation is smaller than that given in the table. At the beginning of the second run, the unit deformation was 0.021. The stress intervals of the table should be multiplied by 1.021 and the deformation intervals should be divided by the same figure. The values of E from the table should be multiplied by the square of 1.021, which is 1.042.

$$25{,}800{,}000 \times 1.042 = 26{,}900{,}000.$$
$$24{,}900{,}000 \times 1.042 = 25{,}900{,}000.$$

At the beginning of the third run the unit deformation was 0.0256. The calculated values of E should be multiplied by 1.0518, which is the square of 1.0256.

$$27{,}300{,}000 \times 1.0518 = 28{,}700{,}000.$$
$$25{,}900{,}000 \times 1.0518 = 27{,}200{,}000.$$

4. From the run of Table XIV which began at 3 P.M., July 1, calculate E for the interval 6,772 to 27,088 and for the interval 11,287 lb. per sq. in. to 31,603 lb. per sq. in. Correct for the initial area and gage length.
Ans. Corrected E = 28,400,000; E = 27,900,000.
5. For the first run on July 1 of the steel of Table XIV some readings (which are not given in the table) were

Unit Stress	Unit Deformation
900	35,367
2,000	35,379
3,000	35,439
25,000	36,234
30,000	36,422
32,000	36,479

Calculate E for three largest intervals using no reading twice. Correct for area and true gage length at beginning of run.
Ans. Corrected E = 29,800,000; 28,800,000; 29,900,000.

The measurements of Problem 5 were made after the piece had been stressed and then stood for 48 hr. under small load. Before the run of Problem 4, the load was carried above the maximum of two days before. When soft steel is stressed beyond its previous yield point, considerable time must elapse before the modulus of elasticity reaches its maximum value.

6. A bar of cold-rolled steel, tested Sept. 19, 1931, had an average diameter of 0.7485 in. When the load changed from 0 lb. to 13,200 lb., the elongation of an 8-in. gage length was 0.0080 in. Find E. Find the

TABLE XVI.—SPECIFICATIONS FOR SOME METALS ADOPTED BY THE AMERICAN SOCIETY FOR TESTING MATERIALS, STANDARDS, 1934

Material		Yield point, lb. per sq. in.	Tensile strength, lb. per sq. in.	Gage length, inches	Elongation, per cent	Reduction of area, per cent
Class B steel castings	Hard	0.45 of tensile strength	80,000	2	17	25
	Medium		70,000	2	20	30
	Soft		60,000	2	24	35
Wrought-iron forged [illegible]	Blooms	0.5 of tensile strength	45,000	4	22	30
	Forgings Classes A and B		45,000	4	24	33
Malleable-iron castings	Grade 32,510	32,500	50,000	2	10	
	35,018	35,000	53,000	2	18	
Gray-iron castings	Regular	......	21,000			
	Higher strength	......	30,000			
Refined wrought-iron bars	....	25,000	48,000	8	22	
Naval brass rods for structural purposes	Diameter 1 in. or less	31,000	62,000	2	25	
	Over 1 in. to 2.5 in.	30,000	60,000	2	30	
	Over 2.5 in. to 3.5 in.	25,000	56,000	2	35	
	Over 3.5 in.	22,000	54,000	2	40	
Hard-drawn copper wire	Diameter, inches					
	0.460		49,000	10	3.75	
	0.229		59,000	10	1.79	
	0.114		64,300	60	1.02	
	0.057		66,400	60	0.89	
	0.040		67,000	60	0.85	
Soft-copper wire	0.460 to 0.290		36,000	10	35	
	0.289 to 0.103		37,000	10	30	
	0.102 to 0.021		38,500	10	25	
	0.020 to 0.003		40,000	10	20	

TABLE XVII.—APPROXIMATE COMPOSITION OF ALUMINUM ALLOYS MOST FREQUENTLY USED STRUCTURALLY

Alloy	Per cent of other elements added to aluminum				Weight, lb. per cu. ft.
	Copper	Manganese	Magnesium	Silicon	
Wrought:					
3S.....	...	1.25	...	...	171
17S.....	4.0	0.5	0.5	...	174
25S.....	4.5	0.8	...	0.8	174
51S.....	...	...	0.6	1.0	168
Cast:					
12......	8.0	...	...	...	178
43......	...	...	...	5.0	166
195-4....	4.0	...	...	...	174

total work in gage length. Find the work per unit volume by two different methods. *Ans.* $E = 30{,}000{,}000$ lb./in.^{2} $U = ?$

7. When the load of Problem 6 changed from 2,200 lb. to 15,400 lb., the elongation of the gage length changed from 0.00134 in. to 0.00940 in. Find E. Find the work for this change of length. Find the work per unit volume by two methods.
8. When the load of Problem 6 increased from 15,400 lb. to 24,400 lb., the elongation of the gage length increased to 0.01500 in. Find E. Is the last load beyond the proportional elastic limit?
9. The ultimate load of the cold-rolled steel of Problem 6 was 42,950 lb. Find the ultimate strength.
10. The load at rupture for the steel of Problem 6 was 24,100 lb. The average diameter of the neck was 0.7341 in. The elongation of the 8-in. gage length was 0.645 in., and the elongation of the upper 4 in., which did not include the neck, was 0.215 in. Find the actual stress at the rupture, the actual stress in the upper 4-in. portion, the percentage of elongation, and the percentage of reduction of area.

44. Some Specifications.—Table XVI gives some specifications for metals which have been adopted by the American Society for Testing Materials. Table VIII of Art. 33 gives specifications for rolled structural steel. Table XVIII gives the properties of light aluminum alloys and Table XVII gives the approximate composition of these alloys. Tables XVII and XVIII are from the "Structural Aluminum Handbook," 1930, of The Aluminum Company of America. In the first column of Table XVIII, the last letter designates the heat treatment. Annealed 3S is 3SO; fully hardened 3S is 3SH. When the stronger alloys are heat treated and artificially aged, they are designated by T, as

TABLE XVIII.—TYPICAL MECHANICAL PROPERTIES OF ALUMINUM ALLOYS

Alloy and temper	Tension, lb. per sq. in.		Elongation in 2 inches, per cent	Compression, lb. per sq. in.		Ultimate shearing strength, lb. per sq. in.
	Yield point	Ultimate strength		Yield point	Ultimate strength	
Wrought						
3SO	5,000	16,000	40	5,000	16,000	11,000
3S½H	18,000	21,000	20	18,000	21,000	14,000
3SH	25,000	29,000	10	25,000	29,000	16,000
17ST	35,000	58,000	20	35,000	58,000	35,000
25SW	25,000	48,000	18	25,000	48,000	30,000
25ST	35,000	58,000	20	35,000	58,000	35,000
51SW	20,000	35,000	24	20,000	35,000	24,000
51ST	35,000	48,000	14	35,000	48,000	30,000
Cast						
12	14,000	22,000	2	16,000	38,000	20,000
43	9,000	19,000	4	9,000	25,000	15,000
195–4	16,000	31,000	8	27,000	43,000	27,000

25ST; when heat treated alone, they are designated by W, as 25SW.

Yield point in Table XVIII is the "commercial yield point" for materials which do not have a true yield point. This is a unit deformation of 0.002, which is one part in 500, or two-tenths of 1 per cent. According to this definition, the yield point of the S.A.E.-1095 steel of Appendix C, and upper curve of Fig. 34 has a yield point of about 59,000 pounds per square inch.

Problems

1. A 1-in. wrought-iron bar tested in tension failed under a load of 37,200 lb. The elongation in 8 in. was 1.80 in. Does this wrought iron meet the minimum requirements of the A.S.T.M. Specifications? The load at the yield point must exceed what value?
2. A ¾-in. wrought-iron rod reaches the yield point at a load of 10,600 lb. The ultimate load is 21,200 lb., and the elongation in 8 in. is 1.78 in. Does this meet the minimum specifications of the A.S.T.M.?
3. A cast-steel test piece 0.505 in. in diameter fails under a load of 13,300 pounds. The elongation of 2 in. is 0.46 in. The diameter at the neck at rupture is 0.408 in. Does this piece meet the minimum requirements of the A.S.T.M. Specifications for soft cast steel?
4. A cast-steel test piece 0.505 in. in diameter fails under a load of 15,200 lb. The elongation in 2 in. is 0.37 in., and the diameter of the neck is 0.421 in. Does this piece meet the minimum requirements of the A.S.T.M. Specifications for medium cast steel?

5. Find the safe load in tension on hard-drawn copper wire, 0.114 in. in diameter with a factor of safety of 5.
6. Find the size of a structural brass rod to carry a load of 30,000 lb. with a factor of safety of 5.
7. An aluminum bar 2 in. wide and $\frac{1}{4}$ in. thick is stretched 0.22 in. in a length of 2 in. and fails at 15,000 lb. The elongation of the gage length at 12,500 lb. is 0.0038 in. Does this meet the minimum specifications for aluminum 3SH?
8. A $\frac{1}{2}$-in. round rod of 3SH aluminum alloy hangs vertically and supports a load of 1,800 lb. at the end. What may be the length of the rod if the factor of safety is 2.5? How much will the 1,800 lb. stretch the rod? (Use handbook when necessary.)
9. Solve Problem 8 for a hard-drawn copper wire 0.229 in. in diameter if the load at the bottom is 800 lb.

*

CHAPTER IV

RIVETED JOINTS

45. Kind of Stress.—Riveted joints afford an excellent illustration of tension, compression, and shear, and of the manner of transmission of stress. Figure 68 represents a pair of plates, each of breadth b and thickness t, transmitting a pull P in the direction of their length. The plates are united by means of a pin C, which fits tightly in a hole in the lower plate and passes through a hole in the upper plate. The portion of the upper plate to the left of the pin is in tension. The intensity of this tensile

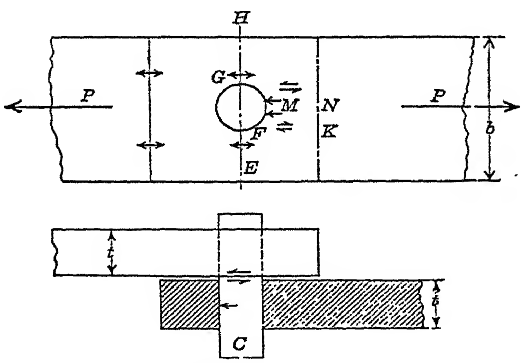


FIG. 68.—Stress at a bolted joint.

stress is found by dividing the pull P by the area of the *gross section* $b\,t$. At the section $E\,H$ in the plane of the axis of the hole, the stress is still tension. The area of this *net section* is smaller than that of the gross sections to the left and the unit stress is greater. If the hole is in the middle of the section and in the line of the pull, half of the total stress is transmitted by the section $E\,F$ and half by the section $G\,H$. The stress which passes $E\,F$ as tension passes $F\,K$ as shear. The intensity of this shearing stress in the plate may be calculated by dividing the pull $\frac{P}{2}$ by the section of length $F\,K$ and the thickness t. At M, the surface of contact of the pin and plate, the stress is compression.

The force is transmitted as shearing stress from the portion of the pin in the upper plate to the portion in the lower plate. Finally, the force is transmitted as compression from the lower half of the pin to the portion of the lower plate on the left.

The stress may be regarded as flowing like an electric current, as is illustrated in Fig. 69. The circuit is completed through the bodies by which the force is applied.

If the pin in Fig. 69 is slightly smaller than the hole, all the bearing pressure is applied to a narrow strip of the plate at M.

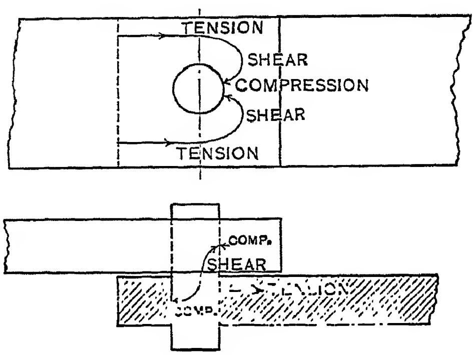


FIG. 69.—Flow of stress.

The entire shearing stress is then transmitted across two planes which are very close together and extend from M to the edge of the plate at N. In calculating the shearing stress in the space between a rivet hole and the edge of the plate, it is customary to consider the minimum distance instead of the distance $F K$.

Problems

1. The plates of Fig. 69 are 2¾ in. wide and ⅜ in. thick. The bolt is 1 in. in diameter and exactly fills the hole in each plate. The total pull is 9,240 lb. Find the unit shearing stress in the bolt, and the unit tensile stress in the net section and in the gross section of each plate.
   Ans. s_s = 11,760 lb./in.²; s_t = 8,960 lb./in.² in gross section; s_t = 14,080 lb./in.² in net section.
2. Solve Problem 1 if the plates are 2½ in. wide. What is the ratio of the tensile stress in the gross section to the stress in the gross section of Problem 1? Solve without writing. What is the ratio of the stress in the net section of Problem 1 to that of the gross section of Problem 2?
3. Solve Problem 1 if the plates are 5⁄16 in. thick.
4. Solve Problem 1 if the lower plate is 3 in. wide and ¼ in. thick.

46. Bearing or Compressive Stress.—In calculating the unit bearing or compressive stress at the surface of contact of the pin

and plate, it is customary, among engineers, to regard the bearing area as the product of the thickness of the plate multiplied by the diameter of the pin. If d is the diameter of the pin and t is the thickness of the plate, the *bearing area* is $t\,d$. The bearing area is the projection of that portion of the pin which is inside the plate upon any plane parallel to the axis of the pin.

Figure 70 shows a rectangular bar of thickness d, which is placed across the edge of a plate of thickness t. If the bar crosses

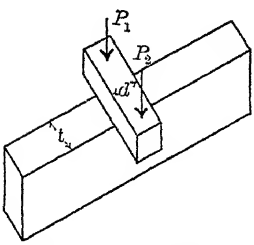


FIG. 70.—Bearing.

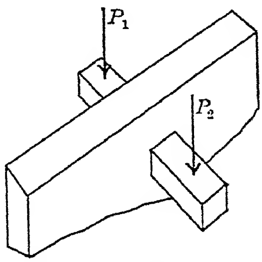


FIG. 71.—Bearing.

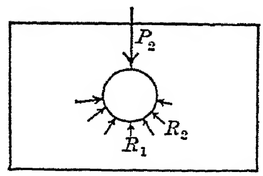


FIG. 72.—Bearing.

the plate at right angles, it is evident that the bearing area is $t\,d$. If, as in Fig. 71, the bar passes through a hole in the plate, the bearing area is the same; and, if the forces P_1 and P_2 are balanced with respect to the center of the plate, the bearing stress is uniform over the entire area. If the forces are not balanced, the area and the *average* bearing stress remain the same, but the maximum bearing stress is greater. If there is force on the bar on one side of the plate only, the maximum bearing stress is much greater than the average.

Figure 72 shows a round bolt or pin passing through a plate. The actual area of contact is the lower half of the surface of the cylinder of length t and diameter d.

The reactions R_1, R_2, etc., are not all vertical but are nearly normal to the surface of contact. If, as in the case of liquid pressure, these reactions were exactly normal and of equal intensity, the resultant of their vertical components would be the same as if that unit pressure were exerted on the horizontal projection of this cylindrical surface.

Problems

1. What is the compressive stress between the bolt and each plate for Problem 1 of Art. 45? *Ans.* $s_c = 24{,}640$ lb./in.^{2}
2. Two ½-in. plates, each 3 in. wide, are united by a single 1-in. rivet. The unit shearing stress in the rivet is 10,000 lb. per sq. in. Find the

unit bearing stress between the rivet and the plates. Find the unit tensile stress in the net sections and in the gross sections of the plates.
Ans. s_c = 15,708 lb./in.²; s_t = 7,854 lb./in.² in net section.

3. Two steel plates, each approximately 4 in. wide and ¾ in. thick, were united by three ⅞-in. rivets in a single row *lengthwise* the plates (Watertown Arsenal, 1911, p. 129). When tested in tension, the first slipping occurred at a load of 45,200 lb., and the joint failed by shearing the rivets at 89,200 lb. Find the unit tensile stress in the net section of the plates when slipping occurred and the unit bearing stress between rivets and plates.
Ans. s_t = 19,285 lb./in.² at net section at rivet farthest from end of the plate; s_c = 22,960 lb./in.² at any rivet.

4. In Problem 3, what was the ultimate shearing strength of the rivets? What was the maximum tensile stress at any net section? What was the maximum compressive stress between rivets and plates?
Ans. s_s = 49,448 lb./in.²; s_t = 38,062 lb./in.²; s_c = 45,300 lb./in.²

5. A joint similar to that of Problem 3 had 1⅛-in. rivets. First slipping occurred at 48,900 lb., and the joint failed by fracture of one plate at the outer rivet hole under a load of 128,500 lb. Find the ultimate tensile *strength* of the material. Find the maximum unit shearing stress, and the maximum unit bearing stress.
Ans. Tensile strength = 59,590 lb./in.²; s_s = ?; s_c = ?

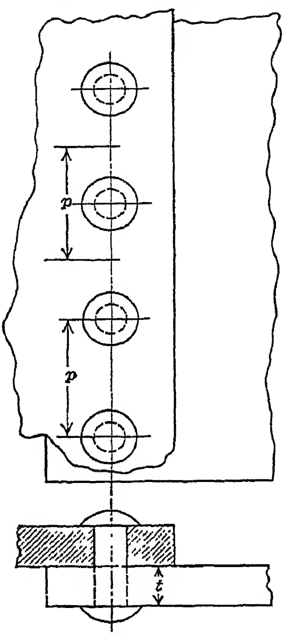


FIG. 73.—Single-riveted lap joint.

47. Lap Joint with Single Row of Rivets. Figure 73 shows a *lap joint* with a single row of rivets. In any riveted joint the distance p from center to center of adjacent rivets in a row is called the *pitch*. In solving problems, it is often convenient to consider a single strip of width equal to the pitch. The problem of a lap joint with a single row of rivets then becomes the same as that of Art. 45. This strip may extend from center to center of adjoining rivets, as is shown between the two lower rivets of Fig. 73. The tension is transmitted by the net section between the rivets, and the shear is equally divided between the upper half of the lower rivet and the lower half of the second rivet. The unit strip may be taken to include one rivet, as is shown at the third rivet from the bottom of Fig. 73. The shear is then carried by the single rivet, while the tension is divided.

In the problems, unless otherwise stated, the rivet will be considered as entirely filling the rivet hole. In practice, when rivet holes are punched and not reamed, it is customary to make some allowance for the material near the hole which is weakened by overstrain. This allowance will not be made in the problems which follow. It is assumed that all rivet holes are reamed or drilled, and that the finished rivets exactly fit.

When the width of the plate is given, it is generally best to consider the entire plate as the unit.

Problems

1. Two $\frac{1}{2}$-in. plates, each 14 in. wide, are united by five 1-in. rivets in a single row to form a lap joint. The joint transmits a pull of 47,124 lb. Find the unit tensile stress in the gross section and in the net section. Find the unit shearing stress in the rivets and the unit bearing stress between rivets and plates.

   Ans. $s_t = 47{,}124 \div 7 = 6{,}732$ lb./in.^{2}; $s_t = 47{,}124 \div (14 - 5)\frac{1}{2} = 94{,}248 \div 9 = 10{,}472$ lb./in.^{2}; $s_c = 47{,}124 \div (5 \times 1 \times \frac{1}{2}) = 94{,}248 \div 5 = 18{,}850$ lb./in.^{2}; $s_s = 47{,}124 \div (5 \times 0.7854) = 12{,}000$ lb./in.^{2}
2. Two $\frac{3}{8}$-in. boiler plates are united by a single row of $\frac{7}{8}$-in. rivets to form a lap joint. The pitch is $3\frac{1}{8}$ in. The unit stress in the gross section is 6,400 lb. per sq. in. Find the total tension in a unit strip of width equal to the pitch. Find unit tensile stress in net section, unit shearing stress, and unit bearing stress.

   Ans. $P = 6{,}400 \times \frac{3}{8} \times \frac{25}{8} = 7{,}500$ lb.; $s_t = 7{,}500 \div (\frac{25}{8} - \frac{7}{8})\frac{3}{8} = 7{,}500 \times \frac{32}{27} = 8{,}889$ lb./in.^{2}; $s_s = 7{,}500 \div 0.6013 = 12{,}470$ lb./in.^{2}; $s_c = 7{,}500 \times \frac{64}{21} = 160{,}000 \div 7 = 22{,}857$ lb./in.^{2}
3. Solve Problem 2 if the pitch is 3 in. and the unit shearing stress in the rivets is 8,000 lb. per sq. in. *Ans.* $s_c = 14{,}660$ lb./in.^{2}
4. For a lap joint with a single row of $\frac{7}{8}$-in. rivets, what must be the thickness of the plates if the bearing stress is to be twice the shearing stress? *Ans.* $t = 0.3436$ in.

48. Butt Joint.—A butt joint is made when the two principal plates are in the same plane and are united by one or two additional plates which are called *butt straps*. A butt joint with a single butt strap is equivalent to a pair of lap joints placed tandem.

Figure 74 shows a butt joint with double butt straps. Since the rivets are in double shear, the total shear transmitted by each rivet is twice as great as in a lap joint.

Example

Two $\frac{1}{2}$-in. plates are united to form a butt joint by two $\frac{5}{16}$-in. butt straps. There is one row of $\frac{7}{8}$-in. rivets on each side. If the allowable unit tensile stress in the plate is 10,000 lb. per sq. in., and the allowable unit shearing stress in the rivets is 8,000 lb. per sq. in., what should be the pitch?

The area of one rivet is 0.6013 sq. in., and each rivet is in double shear. The net cross section which carries the tension equal to the shear in one rivet is $\frac{1}{2}(p - \frac{7}{8})$, so that

$$\tfrac{1}{2}(p - \tfrac{7}{8})\,10{,}000 = 2 \times 0.6013 \times 8{,}000,$$
$$p - \tfrac{7}{8} = 1.924 \text{ in.},$$
$$p = 2.80 \text{ in.}$$

Problems

1. Two $\frac{1}{2}$-in. plates are united by two $\frac{3}{8}$-in. butt straps to form a butt joint with one row of $\frac{7}{8}$-in. rivets on each side. The pitch is $2\frac{3}{4}$ in. Find the unit bearing stress between rivets and plates, the unit tensile

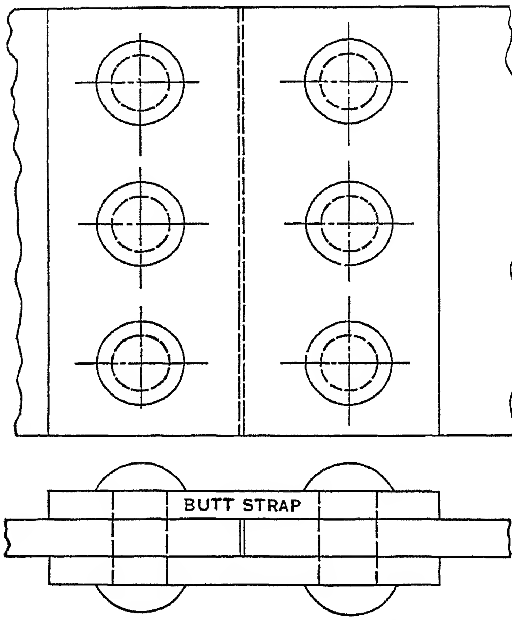


FIG. 74.—Single-riveted butt joint.

stress in the net section, and the unit tensile stress in the gross section when the unit shearing stress in the rivets is 6,000 lb. per sq. in.

Ans. $s_c = 16{,}490$ lb./in.^{2}; $s_t = 7{,}696$ lb./in.^{2}; $s_t = 5{,}347$ lb./in.^{2}

2. Two $\frac{5}{8}$-in. plates are united by two $\frac{3}{8}$-in. butt straps. There is one row of 1-in. rivets on each side. The pitch is $2\frac{3}{4}$ in. The unit tensile stress in the gross section is 6,379 lb. per sq. in. Find the unit bearing stress between the rivets and the plates and between the rivets and the butt straps. Find the unit shearing stress and the unit tensile stress in the net sections.

Ans. $s_c = 17{,}542$ lb./in.^{2} and 14,618 lb./in.^{2}; $s_t = 10{,}025$ lb./in.^{2}

3. What is the unit bearing stress between rivets and plates, and what is the unit tensile stress in the gross section for the example above?

Ans. $s_c = 21{,}990$ lb./in.^{2}; $s_t = ?$

Figure 75 represents a set of tests made at the Watertown Arsenal, to study the behavior of riveted joints. A plate of width b and thickness t was planed down for a portion of its length to some convenient width and united to a pair of plates, thus forming one-half of a butt joint. Wrought-iron rivets were used of nominal diameter $\frac{1}{16}$ inch less than the diameter of the holes. In calculating it was assumed that the finished rivets entirely filled the rivet holes.

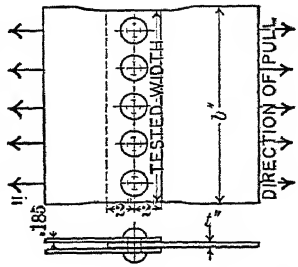


FIG. 75.—Half of butt joint.

Problems

4. In test piece 1353 (Watertown Arsenal, 1885, p. 867), the breadth b was 14.90 in.; the tested width, 14.39 in.; the actual thickness of the plate, 0.248 in. There were five rivets in 1-in. drilled holes. The joint failed by tension along the line of the rivet holes under a pull of 156,440 lb. The calculated results as published are

Areas	Square Inches
Gross sectional area of plate	3.569
Net sectional area of plate	2.329
Bearing surface of rivets	1.240
Shearing area of rivets	7.854

Maximum Stress on Joint	Pounds per Square Inch
Tension in gross section of plate	43,830
Tension in net section of plate	67,170
Compression on bearing surface of rivets	126,160
Shearing in rivets	19,920

Verify these results.

5. In test piece 1355 the results were

Tested width of plate	15 in.
Actual thickness	0.251 in.
Ultimate load	167,200 lb.

There were five rivets in 1-in. holes. "Fractured two outside sections of plate at edge along line of riveting; the two middle sections sheared in front of the rivets."

Compute all unit stresses as in Problem 4.

Figure 76 is a copy of a photograph of this plate after failure. It shows failure by tension across the net section and shear in front of the rivets. It also shows elongation of the rivet holes due to bearing pressure on the plate, combined with shear.

In order to compare the strength of the material in the net section of a riveted joint with the ordinary tension tests, two

strips were sheared from each sheet of steel, one lengthwise, the other crosswise the sheet. These were planed to a width of 1.5 inches and tested in the usual way.

From the sheet used in No. 1353 two test pieces were taken. These gave as ultimate tensile strengths:

	Pounds per Square Inch
No. 1213, lengthwise	59,180
1224, crosswise	60,840

Four test strips were taken from the sheets used for No. 1355:

	Pounds per Square Inch
No. 1214, lengthwise	58,680
1220, lengthwise	62,300
1225, crosswise	61,230
1226, crosswise	60,890

The ultimate strength of these test pieces was considerably smaller than the unit stresses in the net sections of the riveted

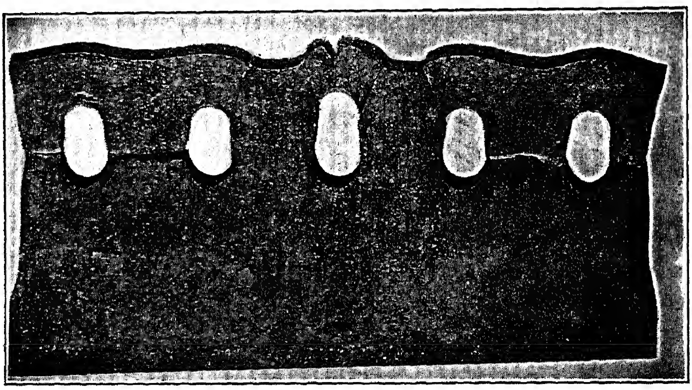


FIG. 76.—Failure of riveted plate.

plates which failed in tension. These tension tests show no definite difference between the strength of the strips taken lengthwise of the plate and those taken crosswise of the plate. This is explained by the fact that the plates were rolled in both directions. When rolled metal is worked in one direction only, its tensile strength is greater in that direction.

Problems

6. In a test piece similar to Fig. 76 (Watertown Arsenal, 1886, p. 1401), the following data are given: tested width, 13.11 in.; thickness, 0.630 in.;

five rivets in 1-in. drilled holes; failed by shearing the rivets under a pull of 295,500 lb.; rivet holes elongated 0.31 in., 0.32 in., 0.26 in., 0.25 in., 0.24 in.

Calculate the unit stresses.

		Pounds per Square Inch
Ans.	Tensile stress in net section	57,840
	Bearing stress	93,810
	Shearing stress on the rivets	37,620

7. In Problem 6, the butt straps were 0.384 in. thick. Find the unit tensile stress in the net section.

Figure 77 is a copy of a photograph of a rivet which failed by shear as in Problem 6 (Watertown Arsenal, "Tests of Metals," 1886, page 1567).

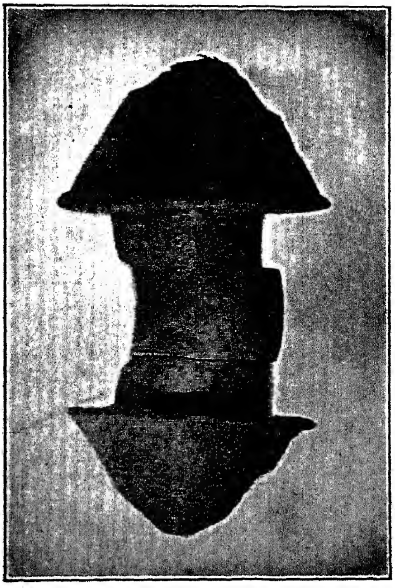


FIG. 77.—Failure of rivet.

49. Rivets in More Than One Row.—Rivets are frequently arranged in two or more rows. The rivets in the second row may be placed directly behind the rivets in the first row, or they may be arranged zigzag as shown in Fig. 78. Two adjoining rows of zigzag rivets must not be placed too close together or the plate will fail along the diagonal line joining the rivets of the two rows. The Boiler Code of the American Society of Mechanical Engineers specifies that the minimum distance between rows of rivets (called *back pitch*) shall be twice the diameter of the rivet holes, when the pitch is not more than four times the diameter of the rivet holes, and the rivets are arranged as in Fig. 78.*

In computing problems of two or more rows of rivets it is customary to regard the unit shearing stress the same in all rivets.

Where narrow plates are united by several rows of rivets, it is best to take the entire width of the plate as the unit.

* See *Report* of Boiler Code Committee, American Society of Mechanical Engineers, 1918 ed., p. 44, for full specifications.

Example

Two ¾-in. by 12-in. plates are united to form a lap joint by means of ten 1-in. rivets arranged as shown in Fig. 79. The joint transmits a pull

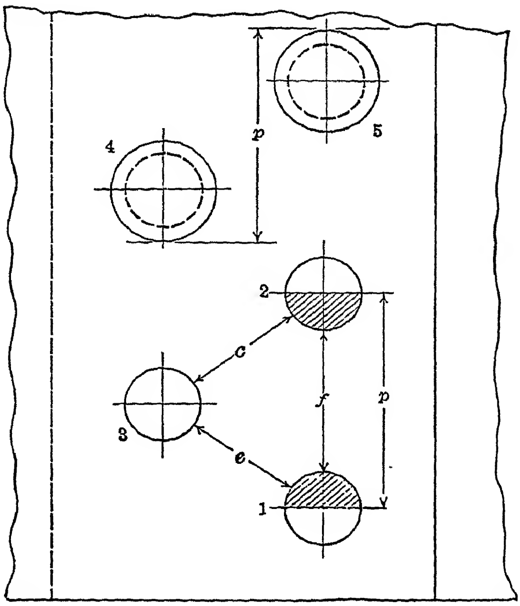


FIG. 78.—Double-riveted lap joint.

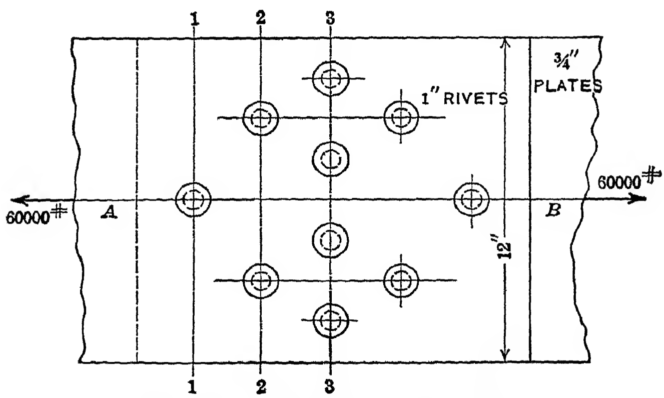


FIG. 79.—Multiple-riveted lap joint.

of 60,000 lb. Find the unit shearing stress in the rivets and the unit tensile stress in plate A at sections 1–1, 2–2, and 3–3.

$$s_s = \frac{60,000}{10 \times 0.7854} = 7,639 \text{ lb. per sq. in.}$$

At section 1–1 the net area is $(12 - 1)\frac{3}{4}$ and

$$s_t = \frac{60{,}000}{11 \times \frac{3}{4}} = 7{,}273 \text{ lb. per sq. in.}$$

At section 2–2 the net width is 10 in.; but since one-tenth of the total pull has been transmitted by rivet 1 from plate A to plate B, the total tension transmitted through this net section is only 54,000 lb., and

$$s_t = \frac{54{,}000}{10 \times \frac{3}{4}} = 7{,}200 \text{ lb. per sq. in.}$$

At section 3–3 the net width is 8 in., but three-tenths of the total pull has been transmitted to plate B through the rivets in sections 1–1 and 2–2 so that the total tension in net section 3–3 is only 42,000 lb.

$$s_t = \frac{42{,}000}{6} = 7{,}000 \text{ lb. per sq. in.}$$

Problem

1. Solve the foregoing example if the plates are 10 in. wide instead of 12 in.

$$\textit{Ans.} \begin{cases} \text{At 1–1, } s_t = 8{,}889 \text{ lb./in.}^2 \\ \text{At 2–2, } s_t = 9{,}000 \text{ lb./in.}^2 \\ \text{At 3–3, } s_t = 9{,}333 \text{ lb./in.}^2 \end{cases}$$

Notice that in Problem 1 the greatest tensile stress is at section 3–3, while in the example the greatest stress is at section 1–1.

In wide plates, such as are used in boilers, it is not convenient to consider the entire width, but it is better to divide the width up into a number of equal units, each of which includes a group of rivets. Generally the width of such a unit is the pitch in the row having the fewest rivets.

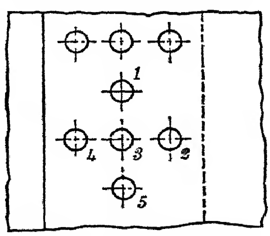


FIG. 80.—Rivets in three rows.

In Fig. 78, in which the pitch is the same in both rows, the width of the unit strip is equal to the pitch. The unit strip may be regarded as extending from center to center of adjacent rivets in each row, as from the center of rivet 1 to the center of rivet 2 in the right row of Fig. 78; or the strip may be regarded as embracing all of one rivet in each row, as rivet 4 in the left row and rivet 5 in the right row of Fig. 78. In either case the equivalent of one rivet in each row is included in the unit strip.

Figure 80 shows a lap joint with twice as many rivets in the middle row as in either of the others. The unit strip is taken as

equal to the pitch in the outer rows (called the *long pitch*). The unit strip may extend from the center of rivet 1 to the center of rivet 5. It includes the area of four rivets, *viz.*, the front half of rivet 1, the rear half of rivet 5, and the whole of rivets 2, 3, and 4. The strip may include all of rivet 5 and none of rivet 1. Whatever distribution is considered, the unit strip has the same width and includes two rivets in the middle row and one rivet in each of the outer rows.

Problems

2. Two ⅜-in. plates are united by ¾-in. rivets to form a lap joint. The rivets are in three rows as in Fig. 80, with the long pitch 5 in. and the short pitch 2½ in. The unit stress in the rivets is 9,000 lb. per sq. in. Find the unit tensile stress in the gross section of each plate. Find the unit tensile stress in the net sections at each row of rivets. Find the unit compressive stress between rivets and plates.

		Pounds per Square Inch
Ans.	s_t in gross section	8,483
	s_t right upper and left lower	9,979
	s_t at middle row	9,088
	s_t left upper and right lower	2,495
	s_c	14,138

3. A butt joint is formed of two ½-in. plates united by two ⅜-in. butt straps. There are two rows of ⅞-in. rivets on each side. The pitch of the inner rows is 3 in. and of the outer rows is 6 in. The unit stress in the gross section of the plate is 8,000 lb. per sq. in. Find the unit tensile stress in the net section of the ½-in. plates at each row of rivets. Find the unit stress in the net section at the inner rows of the butt straps. Find the unit shearing stress in the rivets and the unit bearing stress between the rivets and the ½-in. plates.

		Pounds per Square Inch
Ans.	s_t in ½-in. plates at outer rows	9,366
	s_t in ½-in. plates at inner rows	7,529
	s_t in butt straps at inner rows	7,529
	s_s in all rivets	6,652
	s_c between rivets and ½-in. plates	18,286

4. In Problem 3, what would be the maximum tensile stress in the butt straps if they were only ¼ in. thick? What should be the minimum thickness of each butt strap in order that maximum tensile stress at the inner rows should be equal to the maximum tensile stress in the ½-in. plates?

Ans. $s_t = 11{,}294$ lb./in.²; $t = 0.301$ in.

5. Solve Problem 3 with the butt straps of unequal width as shown in Fig. 81.

		Pounds per Square Inch
Ans.	s_t in ½-in. plates at outer rows	9,366
	s_t in ½-in. plates at inner rows	9,035
	s_s in all rivets	7,983
	s_t in lower butt strap at inner rows	9,035
	s_c in ½-in. plates at inner rows	21,940
	s_c in either butt strap	14,630

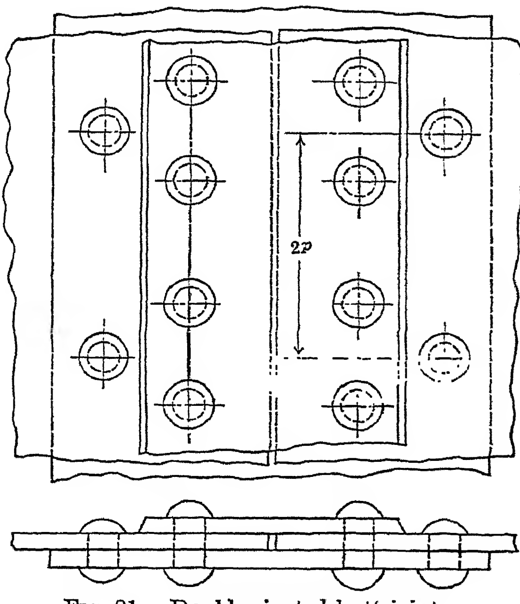


FIG. 81.—Double-riveted butt joint.

50. Efficiency of a Riveted Joint.—The ratio of the strength of a riveted joint to the strength of one of the plates which it unites is called the *efficiency* of the joint. The efficiency may also be defined as the ratio of the unit stress in the gross section, when the joint is stressed to its allowed limit, to the allowable unit stress in the plates. If the joint is so designed as to make it at least as strong in shear and compression as it is in tension at the net section, the efficiency becomes the ratio of the net section to the gross section.

The calculations for efficiency may be based upon either the ultimate strength or the allowable unit stress. The Boiler Code of The American Society of Mechanical Engineers specifies:

	Pounds per Square Inch
Tensile strength of steel plate	55,000
Crushing strength of steel plate	95,000
Shearing strength of steel rivets	44,000
Shearing strength of iron rivets	38,000

The code further specifies that the strength of a rivet in double shear is twice its strength in single shear.

With a factor of safety of 5, which is the one generally used in boiler design, the allowable unit stresses become:

	Pounds per Square Inch
s_t for steel	11,000
s_c for steel	19,000
s_s for steel	8,800
s_s for iron	7,600

To find the efficiency of a riveted joint, the strength of the unit strip is calculated in tension at the net sections, in shear of the rivets, in compression between rivets and plates, and in all combinations of these by which failure may occur. The smallest strength thus obtained is then divided by the strength of the gross section of the unit strip to get the efficiency. When relatively narrow plates are connected by riveting, the strength of the entire plates may be used instead of the unit strip. Either ultimate strength or allowable stress may be used in these calculations.

In the calculation of stress in the preceding articles, the total load on the unit strip is given (or calculated from one of the unit stresses) and the unit stresses are computed from this load. It is advisable to reverse this process in the calculation of efficiency. All the ultimate or allowable stresses are given, and the total strength or the total allowable force on the unit strip is calculated from each of these stresses. For an ideal design, the total strength of the unit strip would be the same for every possible kind of failure. Since the sizes of commercial rivets differ by $\frac{1}{8}$ inch, this ideal condition cannot be attained.

In these calculations it is assumed that the rivet holes are reamed and that each rivet fills the hole. Where rivet holes are punched, it is customary to make allowance for damaged material in the plate and reduce the net section accordingly.

Example

Two $\frac{3}{8}$-in. by 12-in. plates are united to form a lap joint by eight $\frac{3}{4}$-in. rivets in two equal rows across the plates. Find the efficiency of the joint, using the entire width of the plates as unit strips, and the ultimate strengths of the A.S.M.E. Boiler Code.

		Pounds
Tensile strength of gross section	$12 \times \frac{3}{8} \times 55{,}000 =$	247,500
Tensile strength of net section..	$9 \times \frac{3}{8} \times 55{,}000 =$	185,625
Shearing strength of one rivet..	$0.4418 \times 44{,}000 =$	19,440
Shearing strength of eight rivets	$=$	155,520
Bearing strength of one rivet...	$\frac{9}{32} \times 95{,}000 =$	26,720

Since the bearing strength of each rivet is greater than the shearing strength, it is not necessary to calculate the bearing strength of eight rivets. Joint is weakest in shear..

$$\text{Efficiency} = \frac{15{,}552}{24{,}750} = 0.628 = 62.8 \text{ per cent.}$$

Problems

(*Use ultimate strengths of the A.S.M.E. Boiler Code in the first five problems.*)

1. Solve the foregoing example for $\frac{7}{8}$-in. rivets.
   Ans. Joint weakest in tension at net section. Efficiency = 70.8 per cent.
2. After calculating all the strengths in Problem 1 to find that the joint is weakest in tension, compute the efficiency from the ratio of the net section to the gross section.
3. Two $\frac{11}{32}$-in. plates are united by two rows of $\frac{7}{8}$-in. rivets to form a lap joint. Find the efficiency if the pitch is $3\frac{3}{4}$ in.

		Pounds
Ans.	Strength of gross section	70,900
	Strength of net section........	
	Shearing strength of two rivets....	
	Bearing strength of two rivets...	

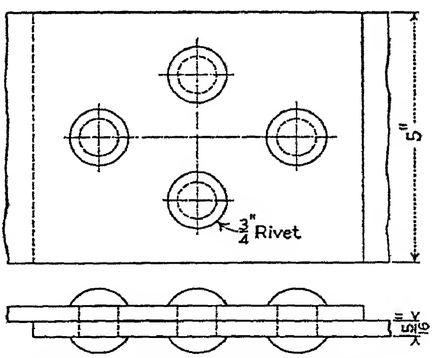


FIG. 82.—Triple-riveted lap joint.

Efficiency = 52,914 ÷ 70,900 = 0.746 = 74.6 per cent.

4. Solve Problem 3 for a pitch of $3\frac{1}{2}$ in.
   Ans. Efficiency = 49,629 ÷ 66,172 = 0.75 = 75 per cent; efficiency = $\frac{21}{8} \div \frac{28}{8} = \frac{3}{4} = 0.75$.
5. For what pitch in Problem 3 would the tensile strength of the net section exactly equal the shearing strength?

(For the next three problems use allowable stresses frequently specified for steel structures: $s_c = 24{,}000$, $s_t = 18{,}000$, $s_s = 12{,}000$.)

6. Two $\frac{5}{16}$-in. by 5-in. plates are united by four $\frac{3}{4}$-in. rivets to form a lap joint. The rivets are arranged in diamond form as shown in Fig. 82. Find the efficiency of the joint.

		Pounds
Shear on single rivet	$0.4418 \times 12{,}000 =$	5,302
Bearing on single rivet	$\frac{15}{64} \times 24{,}000 =$	5,625
Shearing strength of four rivets		21,208
Tension net section outer rivets	$\frac{85}{64} \times 18{,}000 =$	23,906
Tension gross section	$\frac{25}{16} \times 18{,}000 =$	28,125
Tension net section at two rivets	$\frac{35}{32} \times 18{,}000 =$	19,687
Add shearing strength of outer rivet		5,302
Shear outer rivet and tear middle net section		24,989

Ans. Efficiency = 75.4 per cent.

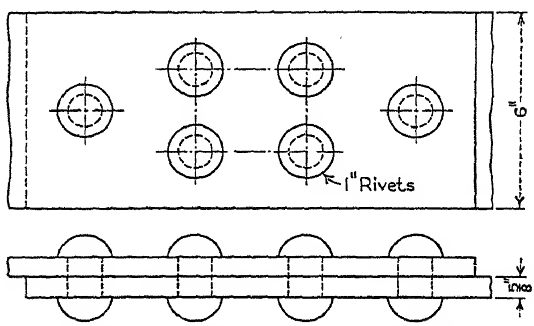


FIG. 83.—Quadruple-riveted lap joint.

7. Two $\frac{5}{8}$-in. by 6-in. plates are united by six 1-in. rivets to form a lap joint as shown in Fig. 83. Find the efficiency.

$$\textit{Ans.}\ \text{Efficiency} = \frac{45{,}000 + 9{,}425}{67{,}500} = 0.806 = 80.6 \text{ per cent.}$$

8. Solve Problem 7 for $\frac{1}{2}$-in. plates with all other data unchanged.

	Pounds
Allowable load net section outer rivet	45,000
Net section at two rivets plus shear outer rivet	45,425

Ans. Efficiency = 83.3 per cent.

(Use A.S.M.E. ultimate strengths for the next problems.)

9. Two $\frac{3}{4}$-in. plates are united to form a butt joint with two rows of 1-in. rivets on each side. The outer (long) pitch is 6 in. and the inner (short) pitch is 3 in. Find the efficiency.

	Pounds	
One rivet, double shear..................	69,115	
Shear three rivets......................	207,345	
Bearing, rivet on ¾-in. plate.............	71,250	
Tensile strength of gross section...........		247,500
Tensile strength net section, outer row.....		206,250
Net section, inner row....................	165,000	
One rivet, double shear..................	69,115	
Total, tension inner row plus shear outer...		234,115

Ans. Efficiency = 83.3 per cent.

10. Find the minimum thickness of the butt straps of Problem 9.

$$\textit{Ans.}\ \frac{5 \times \frac{3}{4}}{4} = \frac{15}{16}\ \text{in. total.}\quad \text{Use two}\ \frac{1}{2}\text{-in. straps.}$$

11. Two ¾-in. plates are united to form a butt joint with two rows of 1-in. rivets on each side. The long pitch is 6 in. and the short pitch is 3 in. The butt straps are ½ in. thick. The lower strap (Fig. 81) embraces two rows of rivets on each side of the joint, and the upper narrow strap embraces one row of rivets. Find the strength and efficiency of the joint.

		Pounds
Shearing strength all rivets................		172,790
Tensile strength net section, outer rows.....		206,250
Bearing one rivet and ¾-in. plate..........	71,250	
Bearing two rivets inner row and single shear outer row............................		177,058
Tension, net section inner row.............	165,000	

Efficiency = 172,790 ÷ 247,500 = ?

12. Two ¾-in. plates are united to form a butt joint with three rows of 1-in. rivets on each side. The pitch of the two outer rows is 10 in. and the pitch of the four inner rows is 5 in. The ½-in. butt straps are of equal width. Find the strength and efficiency of the joint.

Ans. Efficiency = 83.8 per cent.

13. Solve Problem 12 if the two outer rows on each side have a pitch of 7½ in. and the inner row on each side has a pitch of 3¾ in.

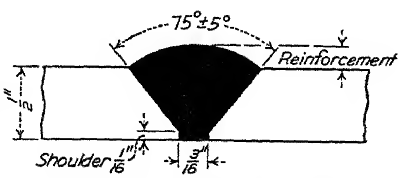


FIG. 84.—Vee-welded butt joint.

51. Welding.—Figure 84 shows two plates connected by a single *Vee* arc-welded butt joint, which is reinforced 20 per cent of the plate thickness. A gas-welded joint is similar with a slightly larger angle of the Vee and smaller clearance at the bottom. A weld of this kind is generally used in tension. Wide plates welded in this way are cut perpendicular to the weld, milled to a width of 1.5 inches for a length of 9 inches,

and loaded to failure on a tension machine as one *qualification test for welders.*

Figure 85 shows a fillet weld, which forms a right-angled triangle. The smallest distance from the right angle to the

FIG. 85.—Fillet welds.

hypotenuse is called the *throat.* When the sides are equal, the throat is the length of either side multiplied by 0.7071.

The American Welding Society's "Code for Fusion Welding and Gas Cutting in Building Construction" specifies the following permissible unit stresses in kips per square inch:

Shear	11.3
Tension	13.0
Compression	15.0

"provided the welder has been qualified and provision has been made for bending stresses that may be introduced due to eccentricity." These permissible stresses give a factor of safety

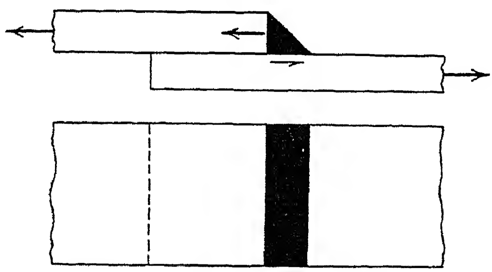


FIG. 86.—Fillet end welds.

4 with ordinary welding rod, and still higher factor with some of the special rods now in use.

For a $\frac{1}{4}$-inch fillet weld with equal sides, the throat is 0.1767 inch, and the area of the throat for 1 inch of weld is 0.1767 square inch. The allowable stress of 11.3 kips per square inch of throat gives 1.997 kips as the strength of 1-inch length of $\frac{1}{4}$-inch fillet weld. It is customary to allow for fillet welds:

$\frac{1}{4}$-inch weld	2 kips per inch of length
$\frac{3}{8}$-inch weld	3 kips per inch of length
$\frac{1}{2}$-inch weld	4 kips per inch of length

Figure 86 shows an "end weld." The lower plate exerts shearing force on the lower surface of the fillet. The upper plate

exerts tension. The stresses in the weld are combined shear and tension.

Figure 87 shows a "side weld." In this figure, the side of the fillet is smaller than the thickness of the plate. Both plates exert shear on the material of a side weld. The allowable

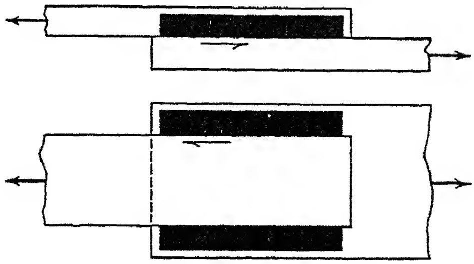


Fig. 87.—Fillet side welds.

shearing stresses for shear at the throat are used for both side welds and end welds.

52. Effective Length of a Weld.—An arc weld does not have its full section at the ends. For testing purposes, the Structural Steel Welding Committee of the American Bureau of Welding (*Report*, page 45) states:

> Examination of arc-welded specimens showed that fillet welds generally attain full section and fusion within ⅛ inch of the beginning of the weld, and end in a crater whose length is about ¼ inch. It is the opinion of the committee, in line with general experience, that the efficiency of an unfilled crater may be set at 50 per cent, or the equivalent of ⅛ inch of length. The effective length of arc fillets has, therefore, been taken as the gross length reported by the laboratory less ⅛ inch at each end, *i.e.*, less ¼ inch per weld.
>
> In the case of arc butt welds the effective length has been taken as identical with their gross length since in these specimens the craters were filled.
>
> In the case of all forms of gas welds the effective length has been taken as identical with the gross length for the reason that, preliminary to depositing the weld, the base metal is heated so that perfect fusion is secured at the start of the weld, and at the end of the weld the gas process leaves no crater.

The joint at the right of Fig. 88 consists of two ⅜-inch by 2-inch plates joined by ¼-inch by 3-inch butt straps. The craters appear at the right ends of the end welds which connect these plates. Figure 89 shows the same joint after it had failed in tension under a load of 43,300 pounds. Failure occurred

in one $\frac{3}{8}$-inch by 2-inch plate with no signs of weakening at the welds. The ultimate strength of the plate was $43{,}300 \div \frac{3}{4} = 57{,}700$ pounds per square inch. If the effective length of each weld is taken as 2 inches, the full width of the plates, 4 linear inches of $\frac{1}{4}$-inch fillet weld resists a pull of 43,300 pounds without failure. While the altitude of the fillet is $\frac{1}{4}$ inch, the thickness of the butt strap, the base is about $\frac{3}{8}$ inch, and the minimum length of the throat is about 0.21 inch, which multiplied by 11.3 gives 2.37 kips per inch as the allowable load. If a factor of safety of 4 is assumed, the joint should have

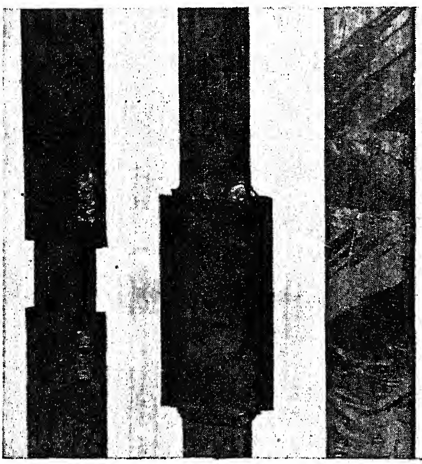


Fig. 88.—Welded joints.

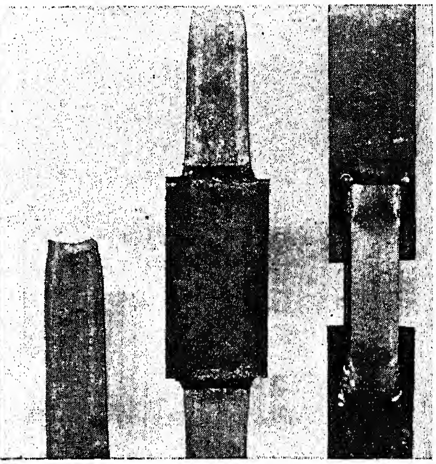


Fig. 89.—Tested welds.

failed under a pull of 37,500 pounds. The actual factor of safety was greater than 4.5.

The joint at the left of Fig. 88 consists of two $\frac{3}{8}$-inch by $2\frac{1}{2}$-inch plates joined by two $\frac{1}{4}$-inch by $1\frac{1}{2}$-inch butt straps. The craters appear at the top of the upper side welds and the bottom of the lower side welds. This joint was not tested, since calculations and previous similar experiments showed that the butt straps would fail in tension. The area of the two straps is $\frac{3}{4}$ square inch. If the ultimate strength of the material is 60,000 pounds per square inch, the ultimate load would be 45,000 pounds. The four side welds at each end have a combined length of 6 inches. At 2,000 pounds per linear inch and a safety factor of 4, the ultimate strength of the weld would be 48,000 pounds.

The side-weld joint of Fig. 89 was the same as that of Fig. 88 except that the welds were shorter. The average for the lower

joint was about 1 inch, while the lower left one was a little less than an inch. The joint failed by shearing the lower left weld and by shearing the plate at the lower right weld. The deformation of the butt strap at the lower end shows the great shearing deformation. The darker color of the butt strap at the ends, except at the edges, indicates smaller stress at the surface here than at the middle of the length.

At a load of 26,800 pounds the elongation of an 8-inch gage length, which included the 6-inch butt straps and the welds, was 0.09 inch. At 32,400 pounds, the elongation was 0.28 inch. Failure took place at 33,500 pounds. This is 8,375 pounds per linear inch for a $\frac{1}{4}$-inch fillet weld in shear.

Problems

1. Two $\frac{3}{8}$-in. by 5-in. plates are united by two $\frac{1}{4}$-in. by 4-in. plates to form a butt joint. The allowable tensile stress in the plates and straps is 16,000 lb. per sq. in. There are 3 in. effective end weld at the ends of each strap. How much side weld is required at each end of each strap?
Ans. 4.5 in. effective; 2.25 in. effective at each side.
2. Two $\frac{1}{2}$-in. by 12-in. plates are united to form a butt joint by three 3-in. by $\frac{3}{8}$-in. butt straps on each side. There are $\frac{3}{8}$-in. fillet welds across the ends of each strap. The allowable tensile stress in the plates and straps is 18,000 lb. per sq. in. How much effective side weld is required at the end of each strap to make the total weld 20 per cent stronger than the plates? *Ans.* 4.2 in. of $\frac{3}{8}$-in. fillet.

53. Circumferential Stress in Hollow Cylinders.—In a hollow vessel inclosing a liquid or gas under pressure, the pressure of the fluid develops stresses in the walls of the vessel. The pressure of a fluid at any point is normal to the surface. The resultant pressure on any portion of a curved surface in any *given direction* is equal to the pressure on a plane surface perpendicular to the given direction and equal in area to the projection of the curved surface upon its plane. Figure 90 represents a portion of the surface of a cylinder of diameter D, and of length l perpendicular to the plane of the paper. If P is the pressure on this surface in pounds per square inch, the total pressure on the semicircular surface to the right of the plane AB is $\frac{P \pi D l}{2}$. The resultant pressure on this surface in the direction normal to $A B$ is $P D l$, since $D l$ is the area of the projection of the curved surface upon the vertical plane. There is an equal pressure in the opposite direction upon the curved surface to the left of $A B$.

These two equal and opposite forces are resisted by the circumferential tensile stresses in the sections at A and B. If t is the thickness of the wall of the cylinder, the area in tension is $2\,t\,l$, and

$$2\,t\,l\,s_t = P\,D\,l, \tag{1}$$

$$s_t = \frac{P\,D}{2\,t}. \qquad \text{Formula VII}$$

Problems

1. A boiler shell 30 in. in diameter is subjected to a steam pressure of 300 lb. per sq. in. The allowable tensile stress is 9,000 lb. per sq. in. in the gross section. Required: the thickness of the plates. *Ans.* $t = \frac{1}{2}$ in

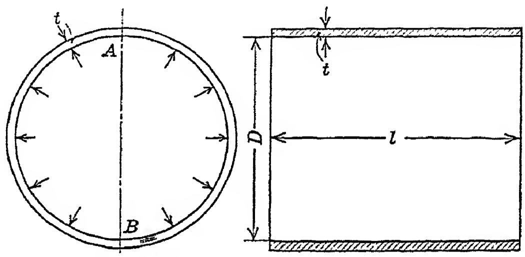


FIG. 90.—Hollow cylinder with internal pressure.

2. A boiler shell is 5 ft. in diameter and $\frac{5}{8}$ in. thick. The efficiency of the longitudinal riveted joint is 80 per cent. If the allowable tensile stress is 11,000 lb. per sq. in., what is the maximum allowable steam pressure? *Ans.* 183 lb./in.^{2}
3. A $\frac{3}{4}$-in. boiler plate has a longitudinal welded butt joint which is assumed to have an efficiency of 80 per cent. A $\frac{3}{8}$-in. butt strap covers the joint on the outside and is attached by $\frac{3}{8}$-in. fillet end welds. Taking 11,000 lb. per sq. in. as the allowable tensile stress in the plate and using the structural value of the allowable strength of fillet weld, how much stronger is the joint than the plate? *Ans.* $1\frac{3}{11}$.
4. If the boiler in Problem 3 is 4 ft. in diameter, find the allowable pressure.

54. Longitudinal Stress in a Hollow Cylinder.—The force exerted by pressure of a liquid or gas in any *given direction* upon a surface is equal to product of the pressure per unit area multiplied by the projection of surface upon a plane perpendicular to the given direction. To find the total pressure exerted upon the head of a cylinder in the *direction of the length of the cylinder,* the area of cross section of the cylinder is multiplied by the pressure per unit area. The pressure in the required direction is the same, no matter what may be the form of the cylinder head.

Problems

1. A cylinder 4 ft. inside diameter is subjected to an internal pressure of air at 240 lb. per sq. in. What is the total force on the head?

Ans. 434,290 lb.

2. How many ¾-in. bolts with nuts would be required to hold the head of the cylinder of Problem 1?
3. If the head of the cylinder of Problem 1 is held on by a fillet weld, what size fillet should be used?

If D is the internal diameter of a cylinder, the area of cross section is $A = \frac{\pi D^2}{4}$ and total pressure longitudinally is $\frac{P \pi D^2}{4}$. The cross section of the cylinder walls which resist longitudinal tension is approximately equal to the inner circumference multiplied by the thickness.

$$s_t \pi D t = \frac{P \pi D^2}{4}, \tag{1}$$

$$s_t = \frac{P D}{4 t}. \tag{2}$$

Comparison of Equation (2) with Formula VII shows that longitudinal (or axial) unit stress in a hollow cylinder is one-half as great as the circumferential unit stress.

Equations (1) and (2) apply also to hollow spheres subjected to internal pressure.

Since the longitudinal tensile stress is only one-half as great as the circumferential unit stress, it is necessary to have the efficiency of the circumferential joints which transmit longitudinal tension only a little greater than one-half the efficiency of the longitudinal joints which resist circumferential stress.

Problems

4. A hollow cylinder, 50 in. in diameter, is made of ½-in. plates. The internal pressure is 240 lb. per sq. in. Find the total longitudinal force on one end.
5. In Problem 4, what is the approximate area of steel which resists the axial pull? What is the axial unit stress?

Ans. $A = 78.54$ sq. in.; $s_t = ?$

6. What is the circumferential unit stress in the cylinder of Problem 5?
7. A spherical tank, 6 ft. 8 in. inside diameter and 7 ft. 0 in. outside diameter, is used to transport helium gas at a pressure of 150 atm. What is the approximate average unit stress?
8. A cylindrical tank, 6 ft. 8 in. inside diameter and 7 ft. outside diameter, is used to transport helium gas. If the ends of this cylinder are hemispheres, what must be its length in order to carry 1.2 times as much gas as the sphere of Problem 7 with the same unit stress?

CHAPTER V

TORSION

55. Torque.—A shaft or rod subjected to a pair of equal opposite couples which are in parallel planes at right angles to its length is in *torsion* between these planes. Figure 91 shows a horizontal shaft which is supported by two bearings and carries two pulleys. A rope is wound a part of the way around each pulley and fastened to it. Each rope supports a load. The load P on the smaller pulley and part of the reactions of the bearings form a counterclockwise couple. The load W on the larger pulley and a part of the reactions form a clockwise couple. If there is no friction at the bearings, these opposite couples are equal, provided the shaft is stationary or moving in either direction with constant speed. The moment of either couple is the twisting moment or torque in the portion of the shaft between the two pulleys. Torque, which is represented in algebraic equations by T, is expressed in foot-pounds or inch-pounds. In order to distinguish torque and bending moment from work, some writers use *pound-feet* and *pound-inches* for the first two and reserve foot-pounds and inch-pounds to mean work or energy. This distinction, however, is not generally made.

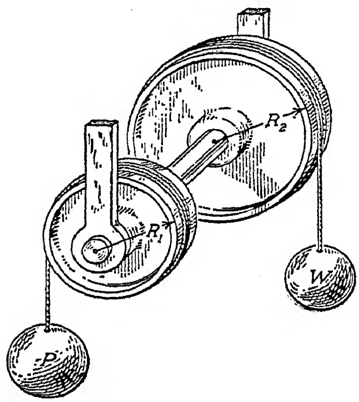


FIG. 91.

Problems

1. In Fig. 91, the diameter of the smaller pulley is 40 in. The load P, of 600 lb., is hung on a $\frac{3}{4}$-in. rope. Find the torque.
   Ans. $T = 12{,}225$ in.-lb.
2. The load W of Fig. 91 is 400 lb. and hangs on a 1-in. rope. What is the diameter of the larger pulley if this load balances the load of Problem 1?
3. Viewed from either end, what is the direction of the torque in the shaft of Fig. 91 between the pulleys?

4. A horizontal shaft carries a crank, which is 32 in. long from axis of shaft to axis of crank pin. Find the torque when a vertical load of 500 lb. is hung on the crank pin and the crank makes an angle of 35° with the horizontal.
5. A 4-ft. pulley is placed on the shaft of Problem 4. A load of 400 lb. is hung on a 1-in. rope which is wound round this pulley. With the 500-lb. load on the crank pin, what angle will the crank make with the horizontal when the shaft is in equilibrium?
6. A shaft carries a 5-ft. pulley. A belt over this pulley exerts a pull of 1,800 lb. on one side and a pull of 400 lb. on the other. Find the torque if the thickness of the belt is neglected.

56. Deformation and Stress at Surface of Shaft.—Figure 92 represents a shaft fixed at the lower end. $D\,B$ and $E\,F$ are lines

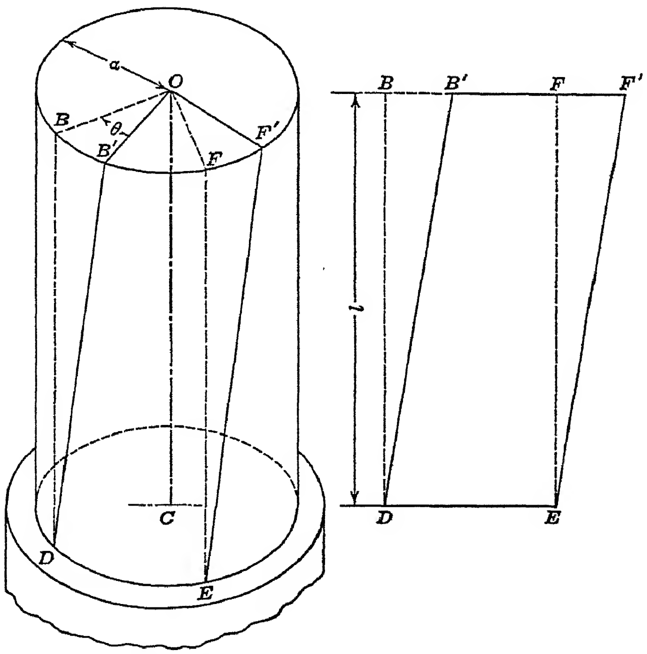


FIG. 92.—Portion of shaft in torsion.

in its surface parallel to the axis $C\,O$. If the cylindrical surface between the lines $D\,B$ and $E\,F$ is developed, it forms the plane rectangle $D\,B\,F\,E$. If a torque is applied to the shaft, twisting it in a counterclockwise direction, the point B is displaced to B' and the point F is displaced to F'. The developed surface $D\,B\,F\,E$ suffers a shearing deformation and becomes the parallelogram $D\,B'F'E$. Every point on the surface at the upper end is displaced the distance $B\,B'$. If a is the radius of the cylinder

and θ (in radians) is the angle through which the top is turned with reference to the base, the displacement $B\,B'$ is equal to $a\,\theta$. If l is the length of the shaft, the unit shearing deformation is given by

$$\delta_s = \frac{a\,\theta}{l}, \tag{1}$$

and the unit shearing stress in the outer fibers is given by

$$S_s = \frac{E_s a\,\theta}{l}. \tag{2}$$

Problems

1. A 4-in. solid shaft is twisted 3° in a length of 20 ft. What is the unit shearing deformation?
Ans. $\delta_s = 2 \times \frac{3\,\pi}{180} \div 240 = \frac{\pi}{7{,}200} = 0.000436.$
2. A ¾-in. shaft is twisted 1°43′ in a length of 10 in. Find the unit shearing deformation. (Use Macmillan's Logarithmic Tables which give degrees, minutes, and seconds in radians on p. 91.) *Ans.* 0.11235.
3. If the modulus of rigidity is 11,200,000, in Problem 1, find the unit shearing stress at the surface. *Ans.* S_s = 4,883 lb./in.^{2}
4. If the unit stress in Problem 2 is 12,135 lb. per sq. in., what is the modulus of elasticity in shear?
5. A shaft 0.902 in. in diameter is twisted 1° 37′ in a length of 10 in. when the shearing stress changed from 5,570 lb. per sq. in. to 19,496 lb. per sq. in. Find E_s. *Ans.* E_s = 10,945,000 lb./in.^{2}
6. A test piece of low-carbon steel, S.A.E.-1020, was 0.899 in. in diameter. The gage length was 5 in. Some readings were

Torque, Inch-pounds	Twist, Radians
0	0
200	0.00138
400	259
600	396
2,000	0.01350
2,200	1482
2,400	1602
2,600	1725

Find the unit deformation at the surface for four intervals of 2,000 in.-lb. torque.
Ans. 200 to 2,200, δ_s = 0.0012082.

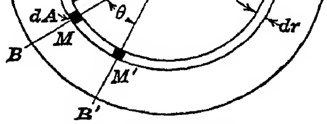


FIG. 93.—Shear displacement of torsion.

57. Relation of Torque to Angle of Twist.—Figure 93 shows the upper end of the shaft of Fig. 92. An element of area dA is at

the position M at a distance r from the axis. When the shaft is twisted and the top is turned through an angle of θ *radians*, this area is moved to M'. Its displacement is $r\,\theta$ and the unit shearing displacement is given by

$$\delta_s = \frac{r\,\theta}{l}. \tag{1}$$

The unit shearing stress on dA is given by

$$s_s = \frac{E_s r\,\theta}{l}. \tag{2}$$

The part of the shaft between the radius r and the radius $r + dr$ is a hollow cylinder of thickness dr, which may be developed into a rectangular solid of width $2\,\pi\,r$ and thickness dr. The area of cross section of this hollow cylinder is $2\,\pi\,r\,dr$. The shearing force required to deform this cylinder is the product of its cross section multiplied by the unit shearing stress.

$$\text{Shearing force} = 2\,\pi\,r\,dr \times \frac{E_s r\,\theta}{l} = \frac{2\,\pi\,E_s\theta}{l} r^2 dr. \tag{3}$$

The moment of this shearing force with respect to the axis of the cylinder is the product of the force by the distance r.

$$\text{Resisting moment} = \frac{2\,\pi\,E_s\theta}{l} r^3 dr. \tag{4}$$

The entire shaft may be regarded as made up of a series of concentric hollow cylinders of thickness dr, and the total resisting moment, which is equivalent to the external torque, is the integral of Equation (4) between the limits $r = 0$ and $r = a$.

$$T = \frac{2\,\pi\,E_s\theta}{l} \int r^3 dr = \frac{\pi\,E_s\theta}{2\,l} \Big[r^4 \Big]_0^a;$$

$$T = \frac{E_s\theta}{l} \frac{\pi\,a^4}{2}. \tag{5}$$

The expression $\frac{\pi\,a^4}{2}$ is the polar moment of inertia of the circle of radius a and is usually represented by J. Equation (5) becomes

$$T = \frac{E_s\theta\,J}{l}, \tag{6}$$

from which

$$\theta = \frac{T\,l}{E_s J}. \qquad \text{Formula VIII}$$

This theory applies rigidly to circular shafts only. In Fig. 93, the straight line $O\ M\ B$ remains straight when the shaft is twisted, provided the sections are circular. When the sections are not circular, a straight line from the center to the surface does not remain straight when torque is applied. The unit stress is not, therefore, proportional to the distance from the axis, and the equations above are not valid.

Problems

1. A 3-in. solid shaft is twisted 2° in a length of 10 ft. Find the torque if $E_s = 11{,}400{,}000$. *Ans.* $T = 26{,}370$ in.-lb.
2. A steel rod 0.901 in. in diameter was twisted 1° 37′ in a gage length of 10 in. when the torque changed from 1,000 in.-lb. to 3,000 in.-lb. Find the modulus of rigidity. Use table of areas of circles and table of squares in the calculation of J. *Ans.* $E_s = 10{,}960{,}000$.
3. The steel rod of Problem 2 was twisted 1° 36′ in a gage length of 10 in. when the torque changed from 1,200 in.-lb. to 3,200 in.-lb. Find E_s from these readings.
4. In Problem 2, find the unit stress for a torque of 2,000 in.-lb. from the modulus of rigidity and the unit deformation.
5. A steel shaft for which the modulus of rigidity is 11,200,000 is twisted 2° 12′ in a length of 10 ft. The shaft is hollow with inside diameter 4 in. and outside diameter 6 in. Find the torque. What would be the torque if the shaft were solid?
6. Find the modulus of rigidity of the steel of Problem 6 of Art. 56 by Formula VIII for intervals 200 in.-lb. to 2,200 in.-lb., and 400 in.-lb to 2,400 in.-lb. torque. *Ans.* $E_s = 11{,}640{,}000$ from 200 to 2,200.

58. The Relation of Torque to Shearing Stress.—From Equation (6) Art 57,

$$T = \frac{E_s \theta\, J}{l}; \tag{1}$$

and from Equation (2) of Art. 57,

$$S_s = \frac{E_s a\, \theta}{l}; \tag{2}$$

from which

$$\frac{E_s \theta}{l} = \frac{S_s}{a}. \tag{3}$$

Substituting in Equation (1):

$$T = \frac{S_s J}{a}, \tag{4}$$

$$S_s = \frac{T\, a}{J}. \qquad \text{Formula IX}$$

Formula IX may be derived by another method, which is not based on the formulas of the preceding articles. Figure 93 shows that the element dA is displaced a distance $r\,\theta$ when the shaft is twisted. The displacement is proportional to r. The unit displacement, and, consequently, the unit stress, is proportional to r, which is the distance of the element dA from the axis of the shaft.

If s_1 is the unit shearing stress at unit distance from the axis, the unit shearing stress at a distance r is s_1r. The shearing stress on an area dA is $s_1r\,dA$ and the resisting moment is s_1r^2dA. The total moment is given by

$$T = s_1\int r^2dA; \tag{5}$$

Since $\int r^2dA$ is the polar moment of inertia, which is represented by J,

$$T = s_1J. \tag{6}$$

Since s_1 is the unit stress at unit distance from the axis, the unit stress at a distance r from the axis is s_1r, and the unit stress at the surface at a distance a from the axis is given by

$$S_s = s_1a; \tag{7}$$

from which

$$s_1 = \frac{S_s}{a}. \tag{8}$$

Substituting in Equation (6):

$$T = s_1J = \frac{S_sJ}{a}. \qquad \text{Formula IX}$$

Formula IX may be written

$$T = S_s\frac{\pi\,a^3}{2};$$

$$S_s = T \div \frac{\pi\,a^3}{2} = T \div \frac{\pi\,d^3}{16}.$$

Problems

1. A 3-in. solid shaft is twisted by a force of 1,200 lb. applied by an arm 4 ft. in length, measured from the axis of the shaft. Find the unit shearing stress at the surface.

$$\textit{Ans. } S_s = \frac{57{,}600 \times 2}{\pi\left(\frac{3}{2}\right)^3} = 10{,}865 \text{ lb./in.}^2$$

2. A 6-in. solid shaft exerts a torque of 36,000 ft.-lb. Find the unit shearing stress at the surface. Solve by the formula and check by proportion from the answer of Problem 1.
3. What is the unit stress in a shaft 0.901 in. in diameter caused by a torque of 1,000 in.-lb.?
4. Solve Problem 2 if the shaft is hollow with inside diameter of 4 in. Find stress at inner and at outer surface.

Ans. S_s at outer surface = 12,693 lb./in.²
s_s at inner surface = 8,462 lb./in.²

5. What is the ratio of the maximum stress in Problem 2 to the maximum stress in Problem 4? *Ans.* 65 : 81.
6. Solve Problem 2 if the shaft is hollow with inside diameter 4 in. and the outside diameter such that the volume is the same as that of a solid 6-in. shaft.
7. The inside diameter of a hollow shaft is 4 in. Find the outside diameter in order that the unit stress in the outer fibers shall be the same as the unit stress in the outer fibers of a solid 6-in. shaft. What is the percentage of saving of material?
8. If a is the outside radius of a hollow shaft and b is the inside radius, what is the expression for the unit stress at each surface?
9. In Table XIX calculate the unit stress in the outer fibers for a torque of 400 in.-lb. and a torque of 2,400 in.-lb. Calculate the unit deformation for each torque. From these solve for the modulus of rigidity by dividing change of unit stress by change of unit deformation. Solve also by Formula VIII of Art. 57.

59. Torsion Failure.—Table XIX gives part of the data from a test of a hollow cylinder of steel, S.A.E.-1020, having 0.17 per cent carbon. The first column gives the torque in inch-pounds. The second column gives the angle in radians. Up to 3,020 inch-pounds torque, the angle was measured by Brown and Sharpe micrometers on 10-inch arms. The reading divided by 10 gave the angle. Two arms at 180 degrees were used to correct for possible eccentricity. This arrangement by which distances were measured from a sphere to parallel planes gives the sine of the angle. The largest angle measured was 0.08 radian for which the sine differs from the arc by less than 1 part in 800. For the larger angles the arms were reset to zero.

The fourth column gives the unit stress in the outer fibers calculated by Formula IX. Up to 2,600 inch-pounds torque, these are correct. Beyond 2,640 inch-pounds the stress does not vary as the unit deformation, and Formula IX does not hold. It is customary, however, to use this formula up to the breaking load. The value of the ultimate strength calculated in this way gives a comparison of two similar materials. For brittle materials which have no yield point the formula is fairly good.

TABLE XIX.—TORSION TEST OF LOW-CARBON STEEL, S.A.E.-1020

Outside diameter, 0.900 inch; inside diameter, 0.500 inch. Gage length, 5 inches. Angles measured by two Brown and Sharpe micrometers on 10-inch arms. Large angles measured with graduated circle.

Torque, inch-pounds	Angle, radians	Unit deformation	Unit stress, $\frac{Ta}{J}$	Torque, inch-pounds	Angle, radians	Unit deformation	Unit stress, Eq. (2)
200	0.0012	0.000108	1,544	3,140	0 4747	0.0427	19,857
400	29	261	3,089	3,290	5358	482	20,806
600	42	378	4,633	3,420	5934	534	21,628
800	56	504	6,177	3,530	6475	582	22,324
1,000	72	648	7,721	3,630	7243	651	22,956
1,200	86	774	9,266	3,770	7976	718	23,841
1,400	0.0101	909	10,810	3,950	9425	848	24,980
1,600	116	0.001044	12,355	4,110	1.0280	925	25,922
1,800	130	1170	13,899	4,220	1.1694	0.1052	26,687
2,000	146	1314	15,444	4,340	1.2409	.1117	27,446
2,200	160	1440	16,988	4,380	1.3526	.1217	27,699
2,400	175	1575	18,532	4,580	1.4608	.1315	28,964
2,600	190	1710	20,077	4,720	1.6301	.1467	29,849
	Running balance			4,870	1.8186	.1637	30,798
				4,920	2.0560	.1850	31,114
2,640	270	2502	20,486	5,310	2.6691	.2402	32,442
2,750	469	4225	21,235	5,530	3.0054	.2705	34,972
2,750	557	5013	......	5,860	4.1574	.3742	37,059
2,740	634	5706	......	6,000	4.6286	.4166	37,944
2,740	688	6192	......	6,180	5.5537	.4998	39,082
2,740	787	7083	......	6,360	6.3390	.5705	40,221
2,740	845	7614	......	6,510	7.2990	.6569	41,169
2,770	957	8613	......	6,700	8.8698	.7983	42,370
2,790	0.1119	0.010071	......	6,850	10.5628	.9507	43,319
2,790	1753	15777	......	7,180	12.7270	1.1454	45,406
2,820	2758	24822	......	7,200	13.8091	1.2428	45,532
2,840	3227	29043	......	7,220	14.5596	1.3104	45,658
2,870	3446	31014	......	7,250	15.2228	1.3701	45,849
2,920	3714	33426	......	7,280	15.9549	1.4359	46,039
3,020	0.3998	0.035982	......	7,320	16.2002	1.4580	46,291
				7,330	17.2125	1.5491	46,355
				7,330	17.5965	1.5837	46,355

Figure 94 is plotted from Table XIX. Two divisions of the lower curve represent a unit deformation of 0.001. The yield point is at unit deformation of 0.0017 and torque of less than 2,640 inch-pounds. The outer fibers reach the yield point first, while the stress at fibers beneath the surface is still increasing. When the torque has reached 2,750, the angle of twist is

about twice as great as it was when the outer fibers began to yield. Since the inner radius is five-ninths as great as the outer radius, when the angle of twist reaches 0.0469 radian the unit deformation at the inner surface is greater than it was at the outer surface when yielding began. Consequently, all the material has reached

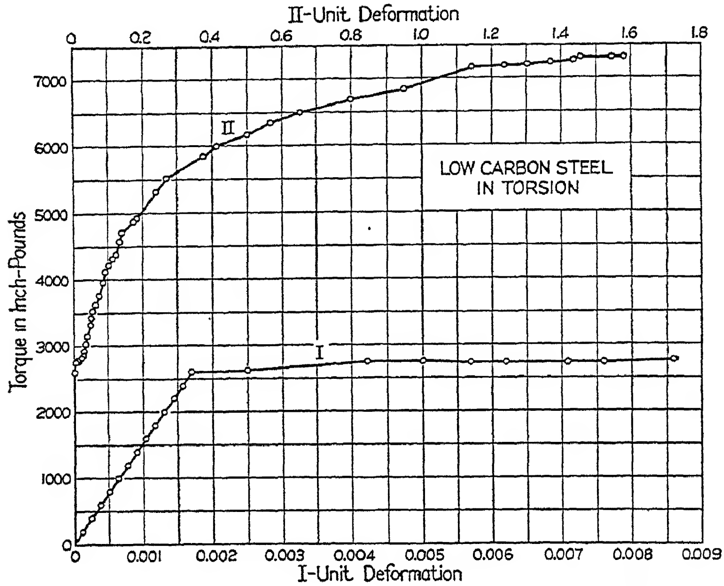


FIG. 94.—Torsion stress-strain diagram from Table XIX.

the yield point at the torque of 2,750, and the stress is the same in all fibers.

If $\bar{s}_s$ is a shearing stress which is constant throughout the section, the total stress on a circular element of radius r and thickness dr (Fig. 95) is $\bar{s}_s\, 2\,\pi\, r\, dr$. The resisting torque of the stress on this element is the total stress multiplied by r.

$$T = 2\,\pi\,\bar{s}_s \int^{a} r^2 dr = \frac{2\,\pi\,\bar{s}_s}{3}(a^3 - b^3); \tag{1}$$

$$\bar{s}_s = \frac{3\,T}{2\,\pi(a^3 - b^3)}. \tag{2}$$

For a solid cylinder, Equation (2) becomes

$$\bar{s}_s = \frac{3\,T}{2\,\pi\,a^3}. \tag{3}$$

A solid cylinder does not differ so greatly from a hollow cylinder. Since the torque varies as the fourth power of the

diameter when the force varies as the distance from the axis, a hollow cylinder with inside radius one-half the outside radius has fifteen-sixteenths as great torque as a solid cylinder. If a small portion of a solid cylinder near the axis has not been deformed to the yield point, the error is not great if it is assumed that the stress is constant throughout.

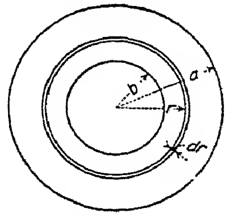


FIG. 95.—Circular element of area.

After the unit deformation passes 0.05, the torque-deformation diagram rises rapidly until the unit deformation becomes about 0.4 and then rises more slowly until rupture. On the steep part of the curve a calculation by Equation (2) may be considerably in error, but it is certain that the stress at a distance of one-tenth the radius from the axis is not less than the stress at yield point. The curve is nearly flat from unit deformation of 1.2 to 1.6. Equation (2) may be used with confidence in the determination of the ultimate shearing strength of soft steel in torsion if the gage length has an inside diameter which is three-fourths as great as the outside diameter. If, however, the material has been strained nearly to failure and then permitted to stand for some time, the stress on reloading may be proportional to the distance from the axis.

Problems

1. Find the unit shearing stress for the hollow cylinder of Table XIX when the torque was 2,750 in.-lb. Compare with Formula VIII. *Ans.* $\bar{s}_s = 17{,}390$ lb./in.^{2}
2. Find the unit stress at failure by Eq. (2) from the test piece of Table XIX.
3. The solid rod of Problem 6 (Art. 56), made of the same rod as that of Table XIX, had a yield point at 3,100 in.-lb. torque. Find the yield-point stress. *Ans.* $s_s = 21{,}730$ lb./in.^{2}
4. The torque on the solid bar of Problem 3 rose to 3,300 in.-lb. and then dropped to 3,260 in.-lb. at unit deformation of 0.036. Calculate the average stress by Eq. (3). Compare with Problem 1. *Ans.* $\bar{s}_s = 17{,}140$ lb./in.^{2}
5. The rod of Problem 3 had a unit shearing deformation of 0.135 when the torque was 5,240. Find the average unit stress by Eq. (3). Compare with Table XIX. *Ans.* $\bar{s}_s = 27{,}450$ lb./in.^{2}
6. The rod of Problem 3 had a unit deformation of 0.463 when the torque was 7,020 in.-lb. Find the unit stress by Eq. (3) and compare with Table XIX for similar deformation. *Ans.* $\bar{s}_s = 36{,}900$ lb./in.^{2}

7. The maximum torque for the solid rod of Problem 3 was 8,170 in.-lb. and the angle of twist was 12.86 radians in the gage length. Find the ultimate shearing strength by Eq. (3) and the unit deformation.

Ans. $\bar{s}_s = 42{,}950$ lb./in.^{2}; $\delta_s = 1.156$.

(The solid rod of Problem 3 stood one day under a torque of 3,500 in.-lb. with an angle of twist of 0.4 radian, which may have modified its subsequent behavior.)

8. Rods from the bar of Table XIX were tested in transverse shear by the apparatus of Fig. 24.

	Rod 1	Rod 2
Diameter	0.750 in.	0.992 in.
Single shear	19,320 lb. 19,200 lb.	33,250 lb. 33,410 lb.
Double shear	38,710 lb. 38,910 lb.	67,370 lb.

Calculate the unit shearing stress from each reading.

9. A bar of medium steel, 0.901 in. in diameter, twisted 2° 47′ in a gage length of 10 inches when the torque changed from 100 in.-lb. to 3,600 in.-lb. Find E_s.

Ans. $E_s = 11{,}160{,}000$.

10. The torque at the yield point of the bar of Problem 9 was 4,000 in.-lb. The ultimate torque was 10,600 in.-lb. with a twist of 1,577° in the gage length. Find the yield-point stress and the maximum unit deformation. *Ans.* $s_s = 27{,}850$ lb./in.^{2}; $\delta_s = 1.24$.

11. Find the shearing strength of the rod of Problem 10 by Formula VIII and by Eq. (3).

Ans. $s_s = 73{,}800$ lb./in.^{2}; $\bar{s}_s = 55{,}350$ lb./in.^{2}

FIG. 96.—Torsion failure of soft steel.

Figure 96 shows the test bar of Problem 9 after fracture. A chalk line was drawn lengthwise along the bar before testing. This line appears as a spiral in the figure.

Figure 97 shows a cast-iron bar which was tested in torsion. Shearing failure took place at right angles with the direction of the resultant tensile stress which is caused by shearing stress (Art. 22). The 45-degree triangle shows how accurately the experiment agrees with the theory. The rupture spiral in this figure is opposite the spiral of the chalk line of Fig. 96.

Problems

12. Sketch an element in the *front surface* of a vertical shaft, with four arrows representing the shear forces acting on this element from the

adjacent material when the applied torque is counterclockwise, viewed from either end. Draw lines on this element to show the direction of the maximum tension and the direction of the line of rupture which would result from this tension. Compare with Fig. 97.

13. The rod of Fig. 97 was 1.24 in. in diameter and failed under a torque of 9,010 in.-lb. Find the unit tensile strength.

Ans. $s_t = s_s = 24{,}070$ lb./in.^{2}

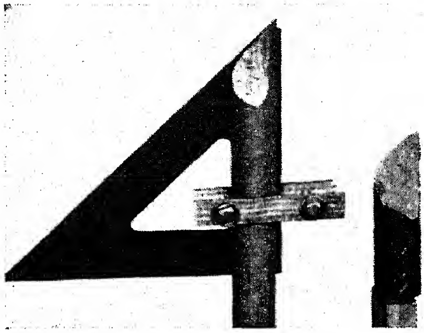


FIG. 97.—Torsion failure of cast-iron.

60. Relation of Torque to Work.—To an arm of length R, measured from the axis of a shaft, a force P is applied which is perpendicular to the plane passing through the axis of the shaft and the point of application of the force. The torque is RP. When the shaft makes one revolution, the point of application of the force moves through a distance $2\pi R$. The work done by the force P is $2\pi RP$. Since PR is the torque,

$$\text{Work} = 2\pi RP = 2\pi T.$$

The work per revolution is 2π times the torque. This relation is true, whether the torque is due to a single force or to a number of forces.

In problems relating to the work done by a rotating body, solve first for the torque. When this is obtained, it may be used in Formulas VIII and IX.

Problems

1. A shaft transmits 600 hp. at 240 r.p.m. Find the work per revolution. Find the torque in foot-pounds.

Ans. 82,500 ft.-lb. of work; 13,130 ft.-lb. of torque (or lb.-ft.).

2. What must be the diameter of the shaft of Problem 1 if the allowable unit shearing stress is 6,000 lb. per sq. in.? *Ans.* $D = 5\frac{1}{8}$ in.

3. How many horsepower may be transmitted by a hollow shaft which is 4 in. inside diameter and 10 in. outside diameter if the allowable shearing stress is 5,000 lb. per sq. in. and the speed is 250 r.p.m.? *Ans.* 3,795 hp.

4. If Ss is the allowable unit shearing stress, N is the number of revolutions per minute, hp is the horsepower, and a is the radius of a solid shaft, show that

$$a^3 = \frac{33{,}000 \times 12 \times hp}{\pi^2 \times N \times S_s} = \frac{40{,}123\ hp}{N \times S_s}.$$

5. If the allowable unit shearing stress is 5,000 lb. per sq. in., show that the diameter of a solid shaft should be approximately

$$d = 4\sqrt[3]{\frac{hp}{N}}.$$

61. Helical Springs.—An interesting illustration of torsion is the elongation or compression of a helical spring, such as is shown in Fig. 98. A helical spring is made by winding a wire or rod on a cylinder (in a single layer, usually). The radius of the coil of the spring is the sum of the radii of the wire and the cylinder about which it is wound. When the spring is to be used in tension, the ends of the wire are turned in to the center, in order that the force may be applied in the line of the axis. Figure 98, II, is a plan of the lower turn. The force P is normal to the plane of the paper. Any portion of the spring $C\ B\ O$ may be considered as a free body. The section at O is perpendicular to the wire. The plane of this section passes through the center C. The force P at C has no bending moment with respect to the section at O. The effect of the force P acting on the arm $C\ B\ O$ is independent of the form of the arm. As far as the stresses at O are concerned, $C\ B\ O$ might be a straight rod from C to O. The effect of the force P on the section at O is a torque $P \times R$. Since O is any point on the helix, the entire wire, except the portion $C\ B$ and a similar portion at the top, is subjected to the torque $P \times R$. In addition to this torsion, there is a constant total shear P. Since the coils are not exactly horizontal, there is another slight correction. Both of these, however, are neglected in ordinary calculations.

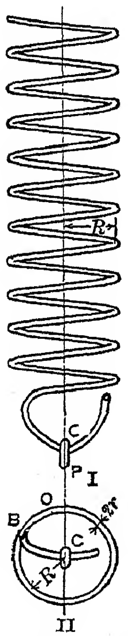


FIG. 98.—Helical spring.

The total elongation of a helical spring is calculated by multiplying the angle of twist in the entire length of the wire by the radius of the coil.

Problems

1. A rod 0.2 in. in diameter is used to make a helical spring of 20 turns. The radius of the coil from the axis to the center of all sections is 1 in. What is the elongation, due to a load of 3 lb., if the modulus of rigidity is 12,000,000 lb. per sq. in.?

$$T = 3 \text{ in.-lb.};\ J = \frac{0.0001\pi}{2};\ \text{length of rod, } 40\pi.$$

$$\theta = \frac{3 \times 40\,\pi \times 2}{12{,}000{,}000 \times 0.0001\,\pi} = 0.2 \text{ radian.}$$

Ans. Elongation $= 0.2 \times 1 = 0.2$ in.

2. What is the unit shearing stress in Problem 1?

Ans. $S_s = \frac{6{,}000}{\pi} = 1{,}910$ lb./in.^{2}

3. If the same rod were used to make a spring of 10 turns, each of 2-in. radius, what would be the elongation due to a load of 3 lb., and what would be the unit shearing stress? *Ans.* 0.8 in., 3,819 lb./in.^{2}

4. At Watertown Arsenal, a steel rod 1.24 in. in diameter and about 241 in. long was formed into a helical spring 7.64 in. *outside* diameter. A load of 5,000 lb. shortened this spring 4.64 in. Find the modulus of shearing elasticity. *Ans.* 11,460,000.

5. In Problem 4 find the unit shearing stress under the load of 5,000 lb.

Ans. 42,740 lb./in.^{2}

6. If R is the radius of the helix, r the radius of the rod, P the load, E_s the modulus of elasticity in shear, and n the number of turns, prove that

$$\text{Elongation} = \frac{4PR^3n}{E_sr^4}.$$

7. If S_s is the allowable unit shearing stress, find the elongation of a spring in terms of S_s, E_s, R, r, and n.

Ans. Elongation $= \frac{2\pi S_sR^2n}{E_sr}$.

8. Find the expression for the elongation of a helical spring in terms of S_s, E_s, R, r, and l, in which l is the length of the rod.

Ans. Elongation $= \frac{S_sRl}{E_sr}$.

62. Resilience in Torsion.—A force P at the end of an arm R twists a shaft of length l through an angle of θ radians. If there is no torque at the beginning, the average force is $\frac{P}{2}$, and the work of twisting is given by

$$\text{Work} = \frac{PR\theta}{2} = \frac{T\theta}{2} = U, \tag{1}$$

$$U = \frac{T^2l}{2E_sJ}. \tag{2}$$

If the shaft is circular and of radius a,

$$U = \frac{J^2S_s^2l}{2a^2E_sJ} = \frac{JS_s^2l}{2a^2E_s}. \tag{3}$$

For a solid circular shaft

$$U = \frac{\pi\, a^2 l\, S_s^2}{4\, E_s} = \frac{S_s^2}{4\, E_s} \times \text{volume}, \qquad (4)$$

and the energy per unit volume is

$$U = \frac{S_s^2}{4\, E_s}. \qquad \text{Formula X}$$

Since the modulus of rigidity of metals is much less than the modulus of elasticity in tension or compression, the energy of torsion is relatively large. When Poisson's ratio is $\frac{1}{4}$, E_s is $\frac{2\,E}{5}$. When this value of E_s is substituted in Formula X, the result is

$$U = \frac{5\, S_s^2}{8\, E}. \qquad (5)$$

The elastic limit in shear is somewhat smaller than the elastic limit in tension or compression. If the limits were the same, a solid cylinder in torsion would store more energy than a block under direct stress. With direct tension or compression, the force must be very large and the displacement must be very small. With torsion, as in a helical spring, the displacement may be large and the force correspondingly small.

Problems

1. In Problem 4 of Art. (61), find the work done by the load of 5,000 lb. in shortening the spring and the work per cubic inch.
   Ans. 11,600 in.-lb.; 39.8 in.-lb./in.^{3}
2. Find the resilience per cubic inch in Problem 1 by means of Formula X and the answers of Problems 5 and 4 of Art. 61.
3. A spring at the Watertown Arsenal was made of 36 lb. of steel rod 1.02 in. in diameter. The outside diameter of the coil was 4.30 in. A load of 11,000 lb. changed the length of this spring from 20.63 in. to 16.67 in. After the load was removed, the spring returned to its original length to within 0.02 in. Find the energy per cubic inch and the energy per pound. *Ans.* 50.4 ft.-lb./lb.
4. In Problem 3 what was the maximum shearing stress due to torsion?
   Ans. 86,580 lb./in.^{2}

*

CHAPTER VI

BEAMS

63. Definition.—A beam is a rigid body subjected to parallel transverse forces. Figure 99 is the front view of a horizontal beam, which is firmly held at the right end and carries a load P at the left end. The wall in which the beam is fixed exerts an upward pressure R_1 and a downward pressure R_2. These two *reactions* and the load P constitute the set of parallel transverse forces. Figure 100 shows a second beam, which is *simply supported* near the ends and carries a concentrated load P between the supports. In addition to the concentrated load P and the reactions R_1 and R_2 in Figs. 99 and 100, the weight of the beam itself furnishes another parallel transverse force. If the material of the beam is of uniform density and all cross sections are alike, the weight is *uniformly distributed.*

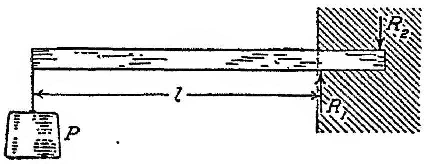


FIG. 99.

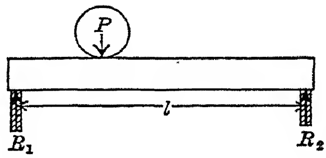


FIG. 100.—Beam supported at ends.

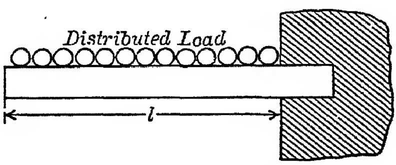


FIG. 101.—Cantilever.

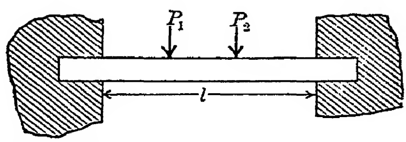


FIG. 102.—Beam fixed at both ends.

64. Kinds of Beams.—Beams may be classified according to the character of the support. Figures 99 and 101 represent beams which are fixed at one end and free at the other. A beam supported in this way is called a *cantilever*. Figure 102 shows a beam *fixed* at both ends. The beam of Fig. 103 is *fixed* at the right end and *supported* at the left end. Figure 104 shows a

simply-supported beam which *overhangs* its supports. A beam with three or more supports, as in Fig. 105, is a *continuous beam.*

Beams may be classified also in accordance with the type of loading. Figures 99, 100, and 103 show a single *concentrated* load on each beam. Figure 101 shows a load which is *uniformly distributed* over the entire length of the beam. The beam of Fig. 105 carries a uniformly distributed load on the left overhang and on one-half of the left *span* and carries another distributed

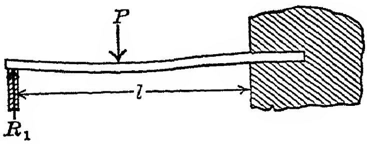


FIG. 103.—Beam fixed at one end and supported at the other.

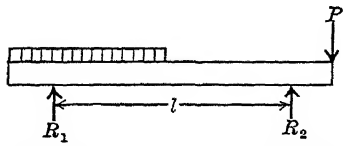


FIG. 104.—Beam overhanging its supports.

load of greater weight per unit length over the right span. The beam of Fig. 104 is loaded uniformly for the left half of its length and has a concentrated load on the right end. Figure 102 shows two concentrated loads, symmetrically placed.

A beam is not necessarily horizontal. A vertical fence post subjected to the horizontal force of the wind or the horizontal reactions of the hinges of a gate is a cantilever beam. A post at the end of a line of wire fence is a vertical beam which is supported horizontally at the top and partially fixed at the bottom and carries a horizontal load at each wire.

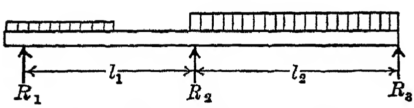


FIG. 105.—Continuous beam.

65. Reactions at Supports.—The calculation of the reactions at the supports of a beam is a problem of the equilibrium of non-concurrent coplanar forces. The general problem of non-concurrent coplanar forces has three unknown quantities and requires three independent equations. When all the forces are parallel, as in most cases of beams, there are only two unknowns, and only two independent equations are required. One of these equations *must* be a *moment* equation; the other *may* be either a *moment* equation or a *resolution* equation. It is best to solve by two moment equations and then check by a resolution equation. In order to eliminate one unknown, the origin of moments for the first equation, at least, should lie on one of the unknown reactions.

Example I

A uniform beam 12 ft. long, weighing 60 lb., is supported at the ends and carries a load of 72 lb. 4 ft. from the left support (Fig. 106). Find the reactions at each support. Remembering that the center of gravity of

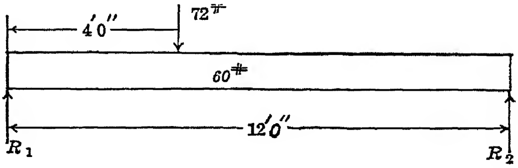


FIG. 106.—Beam supported at ends.

a uniform beam is at the middle of its length, take moments about a horizontal line perpendicular to the beam through the right support.

Load, Pounds		Arm, Feet		Moment, Foot-pounds
60	×	6	=	360
72	×	8	=	576
R_1	×	12	=	936

$$R_1 = 78 \text{ lb.}$$

Taking moments about the left support:

$$60 \times 6 = 360$$
$$72 \times 4 = 288$$
$$R_2 \times 12 = 648$$
$$R_2 = 54 \text{ lb.}$$

Check by vertical resolutions.

Loads	Reactions
60	54
72	78
132	132

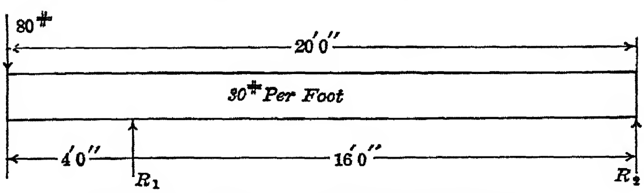


FIG. 107.—Beam overhanging left support.

Example II

A beam 20 ft. long, weighing 30 lb. per foot is supported at the right end and 4 ft. from the left end and carries a load of 80 lb. at the left end (Fig. 107). Find the reactions and check.

The total weight of the beam is $30 \times 20 = 600$ lb.
Taking moments about an axis through the right support:

$$
\begin{array}{rcr}
600 \times 10 & = & 6,000 \\
80 \times 20 & = & 1,600 \\
\hline
R_1 \times 16 & = & 7,600 \\
R_1 & = & 475 \text{ lb.}
\end{array}
$$

Taking moments about an axis through the left support:

$$
\begin{array}{rcr}
600 \times 6 & = & 3,600 \\
80 \times -4 & = & -320 \\
\hline
R_2 \times 16 & = & 3,280 \\
R_2 & = & 205 \text{ lb.}
\end{array}
$$

It will be noticed that in the first part of each example counterclockwise moment is written positively and in the second part clockwise is written positively. This is done for convenience to avoid negative signs as much as possible. It makes no difference which sign is given to a moment expression, provided the same convention is retained throughout one equation. When the moments are not all of the same sign, it is convenient to take as positive the rotation which has the greatest number of terms. The direction of a moment should always be determined by noting which way it tends to rotate about the axis of moments rather than by observing the mathematical sign of the forces and the arms.

In the second example, the moment of the left 4 feet of the beam is counterclockwise about the left support, while that of the remaining 6 feet is clockwise. Some students would write these two portions separately taking 120 pounds with a moment arm of 2 feet and 480 pounds with a moment arm of 8 feet. The method used in the illustrative example, where the whole weight is treated as concentrated at its center of gravity 6 feet from its right support, is shorter. Again, some would write these moments in the form of an equation, the first part of the second example being

$$600 \times 10 + 80 \times 20 = 16R_1.$$

This is sometimes convenient when there are factors which can be cancelled, but generally it is better to arrange the work as shown in the example. Where there are a large number of terms, several of which are negative, it is advisable to put the

positive moments in one column and the negative moments in another.

Problems

(*Always make a sketch of the beam showing all dimensions and loads in the solution of problems and examples. If there is a sketch in the book, make your own sketch. Failure to make sketches and failure to work examples through frequently cause failure in the term's work.*)

1. A simply supported beam is 12 ft. long between supports and carries a load of 720 lb. 5 ft. from the left support. Find the reaction at each support caused by this load. Solve by two moments and check by a vertical resolution. *Ans.* $R_1 = 420$ lb.; $R_2 = 300$ lb.
2. A horizontal beam, 18 ft. long, is supported at the right end and 3 ft. from the left end. It carries a load of 240 lb. per ft., including its own weight. Find the reactions and check. *Ans.* $R_1 = 2{,}592$ lb.; $R_2 = 1{,}728$ lb.
3. A horizontal beam, 20 ft. long, is supported at the right end and 3 ft. from the left end. It carries a load of 340 lb. per ft., including its own weight, a load of 1,020 lb. at the left end, and a load of 180 lb. 3.4 ft. from the right end. Find the reactions and check. *Ans.* $R_1 = 5{,}236$; $R_2 = 2{,}764$.
4. A horizontal beam, 24 ft. long, is supported 4 ft. from the left end and 2 ft. from the right end. It carries a uniformly distributed load, including its own weight, of 360 lb. per ft. A concentrated load of 1,200 lb. is 1 ft. from the left end, and a load of 2,160 lb. is 7 ft. from the right end. Find the reactions and check. *Ans.* $R_1 = 6{,}800$; $R_2 = 5{,}200$.
5. In Problem 4, the concentrated load of 2,160 lb. at 7 ft. from the right end is replaced by a load which makes the end reactions equal. Find this load and check.
6. A beam 4 ft. long, weighing 60 lb., with its center of gravity at the middle, is hinged at the lower right corner (Fig. 108) and held horizontal by a horizontal pull 8 in. above the hinge. Find this horizontal pull (H), the horizontal component of the hinge reaction (C), and the vertical component of the hinge reaction (V). *Ans.* H, 180 lb.; C, 180 lb.; V, 60 lb.
7. The beam of Fig. 108 is hinged at the lower right corner and supported by a rope which is attached to a point 8 in. above the hinge. This rope makes an angle of 10° above the horizontal toward the right. Find the tension in the rope and the components of the hinge reaction. Solve also graphically.

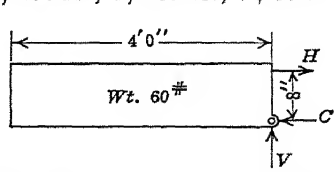


FIG. 108.—Beam supported by horizontal couple.

8. What should be the direction of the rope of Problem 7 in order that the vertical reaction at the hinge may be zero? Solve geometrically.

66. Shear in Beams.—Figure 109 represents a cantilever which is fixed at the right end and carries a concentrated load P

near the left end. A section $E F G$ across this beam separates the left portion as a free body. Figure 113 shows the front elevation of this cantilever. The load P is at a distance a from the left end, and the section $E F G$ is at a distance x from the left end. The weight per unit length is w. When the portion of the beam to the left of the section $E F G$ is considered as the free body in equilibrium, the *external* forces are the load P and the

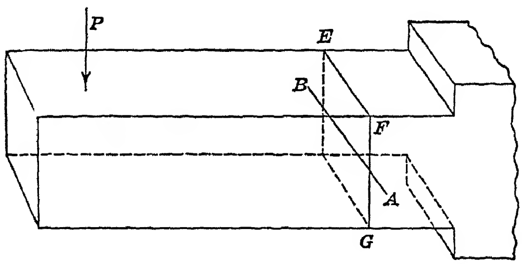


Fig. 109.—Section of cantilever.

weight of the portion, which is $w x$. This portion of the beam is kept in equilibrium by the *internal* forces which the portion on the right of the section $E F G$ exerts across the section.

Figure 110 shows the beam actually cut in two at the section $F G$. A cylinder, with its axis horizontal, perpendicular to the length of the beam, separates the two portions near the bottom,

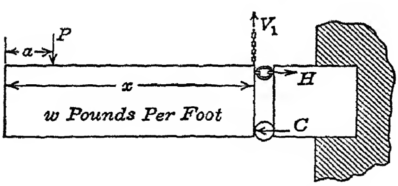


Fig. 110.—Cantilever shear and tension.

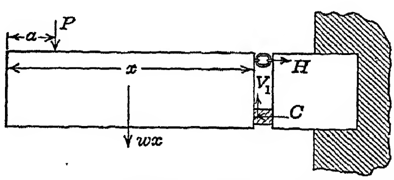


Fig. 111.—Cantilever shear resisted by friction.

and a short horizontal chain connects them near the top. A vertical chain is attached to the right end of the left portion.

The tension in each chain and the compression in the cylinder are calculated as a problem of the equilibrium of non-concurrent coplanar forces. By a vertical resolution, the tension in the vertical chain is shown to be equal to the sum of the two vertical loads. This may easily be verified if the chain is supported by a spring balance. The horizontal resolution shows that the pull H of the horizontal chain is equal to the horizontal push C of the cylinder.

In Fig. 111, the cylinder is replaced by a rectangular block. If the coefficient of friction is sufficiently large, the friction will exert a vertical force equal to the weight of the portion, and the vertical chain may be removed. This vertical force is transmitted across the rectangular block as vertical shear.

In Fig. 112, the portions of Fig. 110 or Fig. 111 are supposed to be glued together. All of the glue is in vertical shear. The upper half is in tension and the lower half is in compression.

Figure 113 represents a similar beam which has not been cut. The material in it at any section, between two imaginary parallel

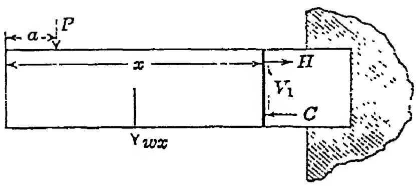


FIG. 112.—Resisting shear and moment at glued section.

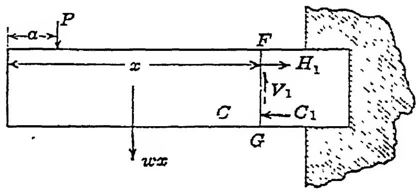


FIG. 113.—Shear and moment at section.

planes very close together, is under the same stresses as the glue of Fig. 112.

The vertical shear, V_1 of Fig. 113, is called the *resisting shear*. The resultant of all the forces parallel to the section which act on the portion of the beam on either side of the section is called the *external shear*. In a horizontal beam the external shear (for a section at right angles to the beam) is vertical and is called the *total vertical shear*. In formulas total vertical shear is represented by V. The resisting shear on one side of any section is equal and opposite to the external shear acting on the portion of the beam on the other side of the section. In Fig. 112, the external shear on the portion of the beam to the left of the section is $P + wx$ acting downward and is equal to the resisting shear with which the portion to the right of the section acts on the glue. Since the entire beam is in equilibrium under the action of the external forces, the external shear on the portion to the right of the section must be equal and opposite to the shear on the left portion. In like manner, the portion to the left of the section exerts a shear equal and opposite to V_1 upon the portion to the right.

The *magnitude* of the vertical shear may be determined from the vertical resolution of all the external forces which act on either the left or the right portion of the beam. The *sign* of the

shear is regarded as positive when the resultant of all the vertical forces which act on the portion to the left of the section is upward. In Figs. 109 to 113, inclusive, the forces P and $w\,x$ are downward. The vertical shear, therefore, is negative. In Figs. 112 and 113, the single-barbed arrow to the right of the section is upward. This arrow represents the *resisting shear* across the section $F\,G$ with which the portion to the right of the section opposes the external shear on the portion which is taken as the free body.

Example

A uniform horizontal beam, 10 ft. long, weighing 12 lb. per ft., is supported at the ends and carries a load of 30 lb. 3 ft. from the left end (Fig. 114).

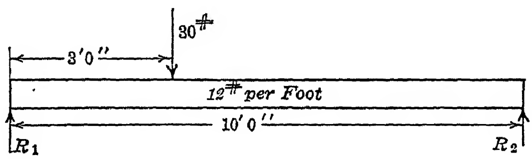


FIG. 114.—Beam supported at ends.

Find the total vertical shear at a section 2 ft. from the left end and at a section 4 ft. from the left end.

The reactions are 81 lb. at the left support and 69 lb. at the right support. With the left portion used as the free body in equilibrium, at 2 ft. from the left end,

$$V_2 = 81 - 2 \times 12 = 57 \text{ lb.}$$

At 4 ft. from the left end,

$$V_4 = 81 - 4 \times 12 - 30 = 3 \text{ lb.}$$

Just before the load of 30 lb. is reached, the shear is

$$V_{3-} = V_2 - 12 = 57 - 12 = 45 \text{ lb.}$$

Just after the load is passed, the shear is

$$V_{3+} = 45 - 30 = 15 \text{ lb.}$$

Problems

1. Check the example by using the right portion of the beam as the free body. The numbers should be the same as in the example with the signs opposite.
2. In Problem 2 of the preceding article, find the shear at 2 ft., 4 ft., 6 ft., and 10 ft. from the left end. Check.
3. In Problem 4 of Art. 65, find the shear at 6 ft., 12 ft., and 18 ft. from the left end.

67. Bending Moment and Resisting Moment.—In Fig. 110, the negative vertical shear is resisted by the tension in the vertical

chain. In Fig. 111, the friction of the rectangular block provides the resisting force. In a horizontal beam subjected to vertical forces, the vertical shearing stress of the material resists the external vertical shear. In so far as vertical linear displacement is concerned, these forces produce equilibrium for the portion of the beam that is regarded as the free body. However, if the cylinder of Fig. 110 should be removed or crushed, the free portion of the beam would *rotate* around the intersection of the chains. The vertical loads and reaction produce a moment with respect to any horizontal axis in the section. For equilibrium,

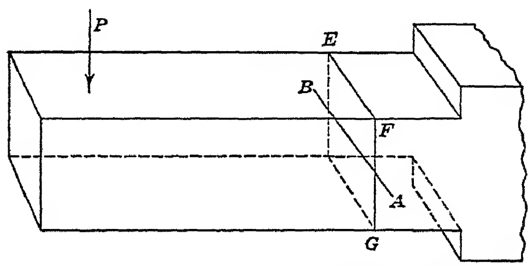


FIG. 115.—Moment of cantilever section.

this *external* moment must be balanced by the *resisting* moment at the section. If moments are calculated with respect to the line of contact of the cylinder of Fig. 110 with the free portion of the beam, the moment of the vertical chain is zero. The force P causes a counterclockwise moment $P \times (x - a)$ and the distributed load causes a counterclockwise moment $w\,x \times \frac{x}{2}$. The clockwise moment of the horizontal tension H must balance the resultant moment of these loads. Expressed as couples, the resultant of P and $w\,x$ downward forms a counterclockwise couple with the equal, upward force V exerted by the vertical chain. This couple is balanced by the clockwise couple formed by the equal, opposite horizontal forces H and C. Since these are couples, the moment of either is the same with respect to any axis perpendicular to their common vertical plane. It is *convenient* to calculate the moment about any axis in the plane of the section, provided this axis is perpendicular to the plane of the external forces.

Since the resisting moment and the external moment are equal, it is customary to speak of the *moment* at the section. Since an

element of volume between two transverse parallel planes, which are infinitely close together, is in equilibrium, the moment of the portion on one side of the section formed by this element must be equal and opposite the moment of the portion on the other side. Moment at any section of a beam may be calculated by taking the moment of all the reactions and loads to the left of the section *or* the moment of all the reactions and loads to the right of the section.

Bending moment is considered positive when the forces to the left of the section tend to cause clockwise rotation about the section. The moment at every section of Figs. 109 to 113, inclusive, is *negative*. If any of these beams were partly cut off, the left portion would turn counterclockwise. The moment at every point in the two cantilevers of Fig. 116 is negative. If the second of these were cut off at F, the portion to the *right* of F would turn *clockwise*, which means that the portion to the *left* of F tends to turn *counterclockwise* and would turn this way if the moment of the load P were removed by cutting the beam at F. A horizontal beam bent by a *negative* moment has its center of *curvature downward.* A *downward* force on a beam *produces negative moment.*

The moment at every section of Fig. 114 is positive. The left reaction tends to turn the left portion in a clockwise direction about any section. It is true that the downward load of 30 pounds tends to turn every section at more than 3 feet from the left end in a counterclockwise direction. However, the bending moment of the left reaction is numerically greater than the moment of the 30 pounds at every section of the beam. At the right end these moments are numerically equal and the resultant moment is zero. By beginning at the right end and using the right portion of the beam, it is seen that the forces on this portion tend to turn it in a counterclockwise direction to an observer at the front of the beam. *Counterclockwise* rotation of the *right portion* or *clockwise rotation* of the left portion means positive moment. When the right portion of the beam is considered, it is convenient to imagine one's self on the opposite side of the beam and facing forward. Clockwise then becomes counterclockwise, and right becomes left.

The moment at any section of a horizontal beam is the same in magnitude and sign, whether calculated from the left portion or the right portion or viewed from the front or the rear. The shear cal-

culated from the left end has the *same magnitude* as the shear calculated from the right end, but the *sign* is *opposite.*

A horizontal beam bent by a positive moment has its *center* of *curvature upward.* An *upward* force *produces positive* moment. The simply-supported beam of Fig. 116 has positive moment throughout its length by any one of the following reasons:

I. Evidently the beam is concave upwards.

II. The left reaction bends the portion of the beam to the left of the concentrated load *clockwise.* The right reaction does the same when viewed from the rear of the beam.

III. Since the left reaction is an upward force, it causes positive moment, at least to the first downward load. Viewed from the rear of the beam, the right reaction has the same effect.

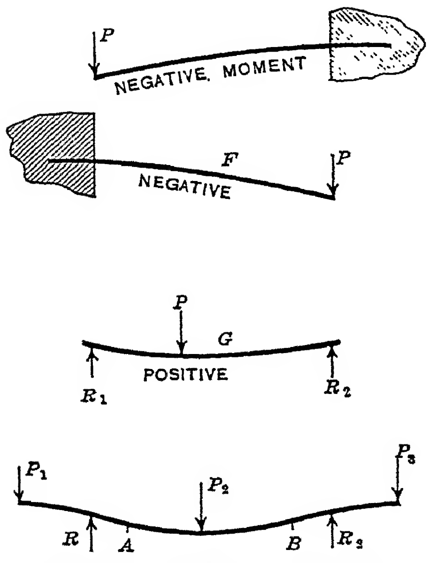


FIG. 116.—Positive and negative moment.

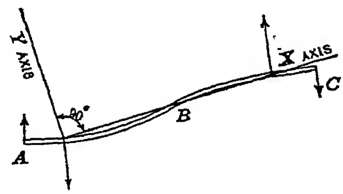


FIG. 117.—Inclined beam.

The lowest beam of Fig. 116 has negative moment outside the supports. The positive moment caused by the left reaction is added to the maximum negative moment over the left support. At A the sum of the positive and negative moments becomes zero. The positive moment increases from A to the load P_2 and then decreases, on account of the negative moment caused by P_2, to zero moment at B.

The sections at A and B, at which the moment changes sign, are called *points of inflection* or *points of contraflexure.*

When a beam is not horizontal, as in Fig. 117, the X axis is taken parallel to the direction of its length and the Y axis at right angles to the direction of the X axis in the counterclockwise direction from that axis. The moment is positive when the ordinate of the center of curvature is positive, and negative when

the ordinate of the center of curvature is negative. In Fig. 117, the moment is positive from A to the point of contraflexure at B and is negative from B to the right end.

Bending moment is represented by M in formulas and equations.

Example

A beam 16 ft. long is supported at the right end and 4 ft. from the left end. It carries a uniformly distributed load of 60 lb. per ft. A load of 240 lb. is

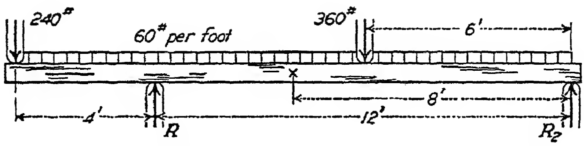


FIG. 118.—Beam for calculating reactions.

on the left end and a load of 360 lb. is 6 ft. from the right end. Find the moment and shear caused by these loads at intervals of 4 ft.

First make a sketch with dimensions and loads.

Total distributed load $= W = wl = 60 \times 16 = 960$ lb.

For external reactions, this load may be regarded as concentrated at the middle of the beam. Its position may be represented by a cross on the sketch. If this force is represented by an arrow, there is danger of using it incorrectly as a concentrated load in the calculation of internal moment or shear.

Taking moments about the right support:

Force	Arm	Moment
240	16	3,840
960	8	7,680
360	6	2,160
1,560	$12\,R_1$	$=13,680$ ft.-lb.
	R_1	$=$ 1,140 lb.

Taking moments around the left support:

240	−4	−960
960	4	3,840
360	6	2,160
	$12\,R_2$	$=$ 5,040
	R_2	$=$ 420; 420 + 1,140 = 1,560, check.

After the reactions have been calculated and checked, they should be written on the sketch. Calculated numbers that are not part of the original data should be designated by a parenthesis or preceded by an equality sign. Figure 119 shows the sketch as it appears after the reactions are written.

At 4 ft. from the left end:

	Force		Arm	Moment	
	− 240		4	− 960	
	− 240		2	− 480	
V_{4-} =	− 480		M_4 =	− 1,440	

At 8 ft. from the left end:

	− 240		8	− 1,920	
	− 480		4	− 1,920	
		1,140	4		4,560
V_8 =		420	M_8 =		720

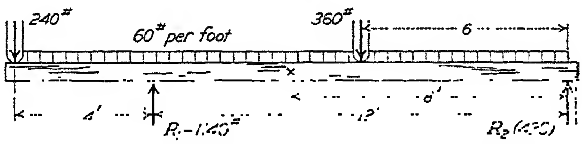


FIG. 119.—Distributed and concentrated loads.

At 12 ft. from the left end:

	− 240		12	− 2,880	
	− 720		6	− 4,320	
		1,140	8		9,120
	− 360		2	− 720	

$V_{12} = -1,320 + 1,140 = -180 \quad M_{12} = -7,920 + 9,120 = 1,200$

At 16 ft. from the left end:

	− 240		16	− 3,840	
	− 960		8	− 7,680	
		1,140	12		13,680
	− 360		6	− 2,160	

$V_{16-} = -1,560 + 1,140 = -420 \quad M_{16} = -13,680 + 13,680 = 0$

The expression V_{4-} means the shear infinitely close to the left of the reaction at 4 ft. The shear infinitely close to the right of this reaction is designated by V_{4+}. This shear is $-480 + 1,140 = 660$.

Problems

1. In the example above, find the shear and moment at 6 ft., 9 ft., 10 ft., and 14 ft. from the left end.
   Ans. $V_6 = 540$, $M_6 = -240$; $V_9 = 360$, $M_9 = 1,110$; $V_{10-} = 300$, $V_{10+} = -60$, $M_{10} = 1,440$.
2. A beam 12 ft. long, weighing 40 lb. per ft., is supported at the ends. Find the moment and shear at 4 ft. from the left end and at the middle.
   Ans. $M_4 = 640$ ft.-lb., $V_4 = 80$ lb.; $M_6 = 720$ ft.-lb., $V_6 = ?$

3. A beam of length l is simply-supported at the ends and carries a load P at the middle. Find the moment at the middle, at one-third the length from the left end, and at two-thirds the length from the left end.

Ans. At the middle, $M = \frac{Pl}{4}$.

4. A beam of length l is simply-supported at the ends and carries a load P at the middle. Find the expression for the moment at a distance x from the left end when x is not greater than one-half the length. Find the expression for the moment when x is greater than one-half the length.

Ans. $M = \frac{Px}{2}$; $M = \frac{Px}{2} - P\left(x - \frac{l}{2}\right) = P\frac{l - x}{2}$.

5. A simply-supported beam of length l carries a uniformly distributed load of w per unit length. Find the moment at the middle, at one-third the length from the left end, and at two-thirds the length from the left end.

Ans. Moment at the middle $= \frac{wl^2}{8}$.

6. In Problem 5, find the expression for the moment at a distance x from the left end. Also find the shear.

Ans. $M_x = \frac{wlx}{2} - \frac{wx^2}{2} = \frac{wx}{2}(l - x)$; $V_x = \frac{wl}{2} - wx$.

7. Solve Problem 5 by the formula of Problem 6. Solve also for the moment at one-fourth and at three-fourths the length from the left end.

68. Shear Diagrams.—It is often convenient to represent the total shear at all sections of a beam by means of a diagram.

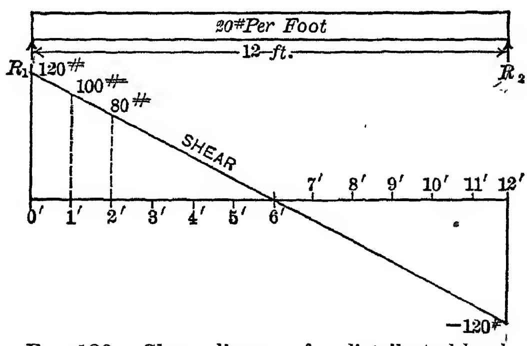


FIG. 120.—Shear diagram for distributed load.

Figure 120 is the *shear diagram* for a uniform horizontal beam, which is 12 feet long, weighs 20 pounds per foot, and is supported at the ends. Each end reaction is 120 pounds. At any section of the beam, the shear is the algebraic sum of the left reaction upward and the weight of the portion of the beam between the left end and the section acting downward. Infinitely near the left support, the weight of the portion of beam to the left is negligible. The shear, therefore, is the left reaction of 120 pounds. Infinitely

near the right support, the shear is the reaction of 120 pounds minus the weight of practically all the beam, which is 240 pounds. The shear is, therefore, minus 120 pounds. A 1 foot from the left end, the shear is

$$V_1 = 120 - 20 = 100 \text{ pounds.}$$

At x feet from the left end,

$$V_x = 120 - 20\,x. \tag{1}$$

Equation (1) is the equation of a straight line. To construct this diagram it is necessary only to find the shear at each end and draw the straight line which connects the extremities of these shear ordinates.

Figure 121 is the shear diagram for a beam which is 10 feet long, weighs 60 pounds per foot, is simply-supported at the ends, and carries a load of 200 pounds 3 feet from the left end. By moments about the right support, the left reaction R_1 is found to be 440 pounds. By moments about the left support, the right reaction is found to be 360 pounds. The sum of these reactions is 800 pounds, which checks the total load.

The shear is 440 pounds infinitely near the left support. It drops 180 pounds in the first 3 feet and is 260 pounds infinitely close to the left of the load of 200 pounds. Under this load, the shear diagram drops vertically 200 pounds. The shear infinitely close to the right of the 200-pound load is 60 pounds. Beyond the concentrated load, the shear drops at the rate of 60 pounds per foot for the remaining 7 feet. It is minus 360 pounds infinitely close to the left side of the right support. The right reaction of 360 pounds raises the diagram to the initial line. The diagram crosses the initial line, or zero ordinate, 1 foot to the right of the concentrated load, which is 4 feet from the left support.

The shear diagram of Fig. 121 is a vertical straight line at each support and at the concentrated load, and the discussion refers to points infinitely near the supports or the load. This method of treatment assumes that the loads and reactions act on mathematical lines. In reality, the *surface* of contact is a band of some width extending across the beam, and the actual shear diagram is something like that represented by the broken curved lines.

In Fig. 121, the shear decreases 60 pounds for every foot. With the origin of coordinates at the left end, the equation of shear for the first 3 feet is

$$V = 440 - 60\,x. \tag{2}$$

For the remainder of the beam,

$$V = 440 - 200 - 60\,x = 240 - 60\,x. \tag{3}$$

If the origin of coördinates is taken at the concentrated load, the equation of the shear for the beam to the right of that load is

$$V = 60 - 60\,x. \tag{4}$$

It is frequently desirable to locate the point at which the shear is zero. This may be done by means of the equation of the shear diagram. If Equation (2) were used, the result would not be

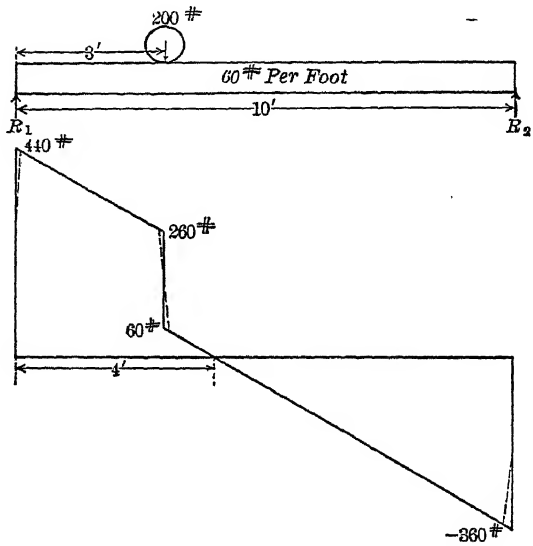


FIG. 121.—Concentrated and distributed loads.

the point desired, since this equation applies only to the part of the beam to the left of the load. Equation (3) gives the point 4 feet from the left end and Equation (4) gives it 1 foot from the load of 200 pounds.

Generally, it is better not to think of the equation of the line. At 3 feet the shear drops to 60 pounds. It continues to drop at the rate of 60 pounds per foot. In what distance will it become zero?

Problems

1. Construct the shear diagram for the example of Art. 66 (Fig. 114) to the scale of 1 in. equals 2 ft. of length and 1 in. equals 20 lb. of shear. Find the point on the beam at which the shear is zero by an equation and compare with the graphical solution.

2. Construct the shear diagram for the example of Art. 67 (Fig. 118) to the scale of 1 in. equals 4 ft. of length and 1 in. equals 200 lb. of shear. Find the position of zero shear algebraically and graphically.
3. A simply-supported beam 10 ft. long carries a load of 400 lb. at the middle. Draw the shear diagram to any convenient scale. The weight of the beam, which is not given, is neglected.
4. Solve Problem 3 if the load is 4 ft. from the left end.
5. A beam of length l is supported at the ends and carries a load $\frac{P}{2}$ at one-third the length from the left end and an equal load at one-third the length from the right end. Construct the shear diagram to any convenient scale.
6. A beam 16 ft. long, supported at 4 ft. from the left end and 2 ft. from the right end, carries 120 lb. on the left end and 240 lb. on the right end. Construct the shear diagram to the scale of 1 in. equals 4 ft. of length and 1 in. equals a shear of 100 lb.

Shear diagrams are usually made up of straight lines. These lines are horizontal from one load to the other when the loads are concentrated and the weight of the beam is neglected. With

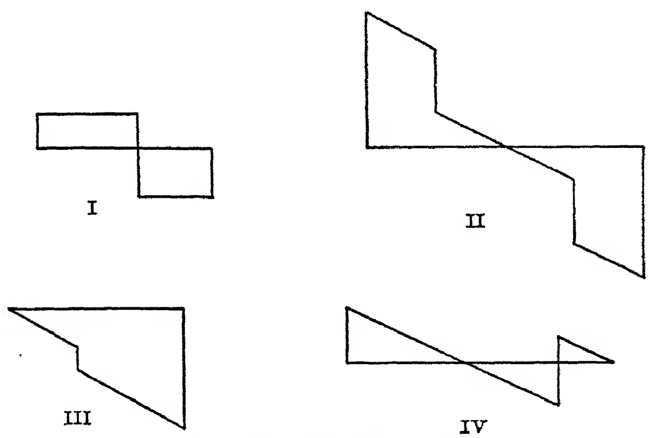


FIG. 122.—Shear diagrams.

uniformly distributed loads, the lines slope downward from left to right. (With distributed loads pushing up, as in the bottom of a boat subjected to water pressure, the lines slope upward.) Where loads are distributed not uniformly, as in the case of the water pressure on a vertical dam, the shear diagram is curved.

The student should become sufficiently familiar with the simpler shear diagrams to be able to recognize the character of the loading at a glance.

Problem

7. Describe the loading and the character of support which give each of the shear diagrams of Fig. 122.

69. Moment Diagrams.—Moment diagrams are constructed in the same way as shear diagrams. The abscissas represent horizontal distances in the beam, and the ordinates represent the external moments. In this book, positive moment is drawn upward. Some writers, however, prefer to draw the moments opposite to the method here used.

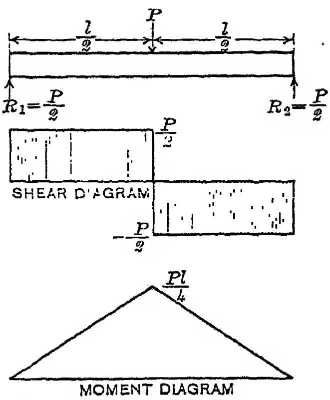


FIG. 123.—Singly concentrated load.

Since shear diagrams usually consist of straight lines, they are easy to construct. Moment diagrams are curved, except when all the loads are concentrated.

Figure 123 shows the shear and moment diagram for a beam supported at the ends and carrying a load P at the middle. The weight of the beam is neglected. The end reactions are $\frac{P}{2}$. The moment at any section at a distance x from the left end is $\frac{Px}{2}$, provided x is not greater than one-half of the length. Under the load the moment is $\frac{Pl}{4}$. The moment diagram for the left half of the beam is a straight line through the points (0, 0) $\left(\frac{l}{2}, \frac{Pl}{4}\right)$. Beyond the concentrated load, the moment caused by the reaction at the left end is diminished by the moment caused by the load at the middle. At a distance x from the left end, when x is greater than $\frac{l}{2}$,

$$\text{Moment} = \frac{Px}{2} - P\left(x - \frac{l}{2}\right) = \frac{Pl}{2} - \frac{Px}{2} = \frac{P}{2}(l - x). \quad (1)$$

This also is a straight line. The last of the expressions for the moment may be obtained directly by using the portion to the

right of the section as the free body. The right reaction is $\frac{P}{2}$ and its moment arm is $l - x$.

Figure 124 gives the shear and moment diagrams for a beam which is supported at the ends and carries a uniformly distributed load over the entire span. For the general case of a beam of length l with a load of w per unit length, the end reactions are $\frac{w\,l}{2}\cdot$ At a distance x from the left end, the moment of the end

L

S

$\int x^2 dL$

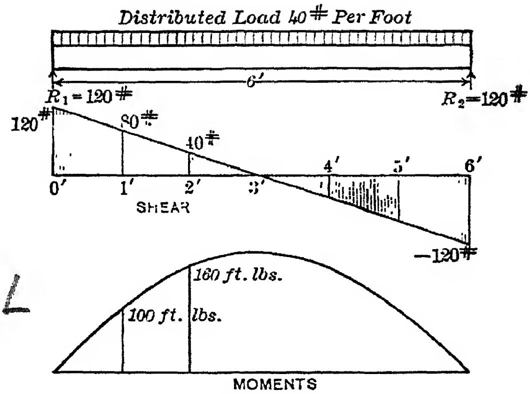


FIG. 124.—Uniformly distributed load.

reaction is $\frac{w\,l\,x}{2}\cdot$ The uniformly distributed load over the length x is $-w\,x$. The moment arm of this distributed load with respect to the section at a distance x from the end is $\frac{x}{2}$, and the moment of the distributed load about this section is $-\frac{w\,x^2}{2}\cdot$ Moment at any section is given by

$$M = \frac{w\,l\,x}{2} - \frac{w\,x^2}{2} = \frac{w\,x}{2}(l - x). \tag{2}$$

Since x and $l - x$ appear in the first degree in Equation (2), it is evident that the curve is symmetrical with respect to a vertical line through the middle; the line $x = \frac{l}{2}\cdot$

Example

A beam 12 ft. long weighs 20 lb. per ft. It is supported at the left end and 2 ft. from the right end and carries a load of 100 lb. 2 ft. from the left

end and a load of 80 lb. at the right end. Construct the shear and moment diagrams to the scale of 1 in. horizontal equals 2 ft. of length, and 1 in. vertical equals 100 lb. shear and 100 ft.-lb. moment.

Figure 125 shows the curves for this example. Under the load of 100 pounds, the shear diagram drops from 120 pounds to

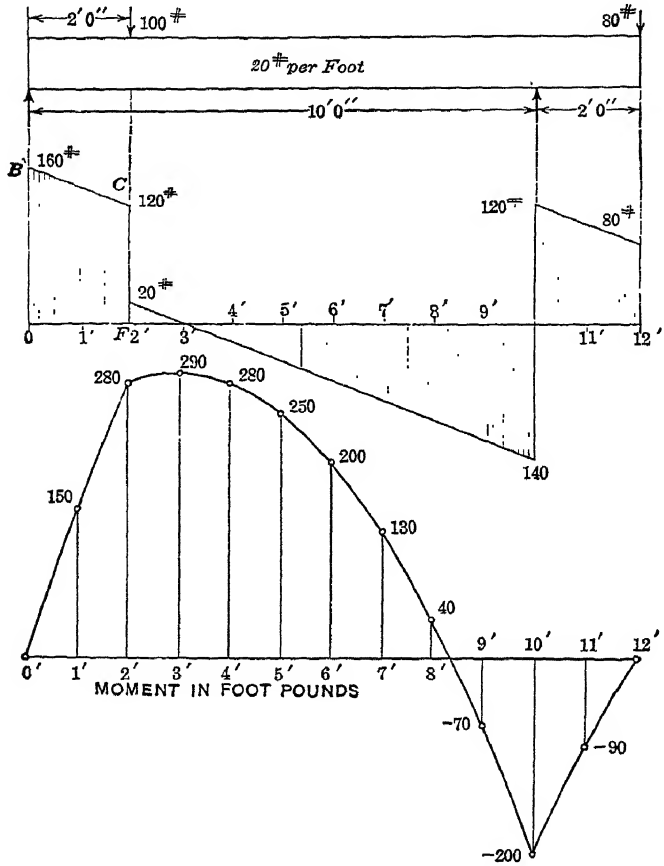


FIG. 125.—Shear and moment diagrams.

20 pounds and the moment diagram has an abrupt change of curvature. At the right support, the shear diagram rises vertically and the moment diagram has a still more abrupt change of curvature and direction. The shear diagram crosses the X axis 3 feet from the left end, at which point the moment is a maximum. The shear again crosses the X axis at the right support, where the moment is a minimum.

Problems

(Construct all diagrams with shear directly below the sketch of the beam, and moment directly below shear.)

1. Construct the shear and moment diagrams for a simply-supported beam of length l which carries a load P at six-tenths the length from the left end. Neglect the weight of the beam.
2. A simply-supported beam is 10 ft. long and carries a uniformly distributed load of 200 lb. per ft. over 6 ft. adjacent to the left support, and no load over the remainder. Construct shear and moment diagrams to the scale of 1 in. equals 2 ft. of length, 1 in. equals 500 lb. of shear, and 1 in. equals 500 ft.-lb. Calculate the shear at only three points. Calculate the moment at intervals of 1 ft. for the first 6 ft.
3. A beam 12 ft. long is supported at the right end and 2 ft. from the left end. It carries a uniformly distributed load of 20 lb. per ft., a load of 80 lb. on the left end, and a load of 100 lb. 2 ft. from the right end. Construct the shear and moment diagrams to the scale of 1 in. equals 2 ft. of length and 1 in. equals 100 lb. of shear and 100 ft.-lb. of moment.
4. Construct the moment diagram for Problem 5 of Art. 68.
5. A uniformly loaded beam, 14 ft. long, is supported at the left end and 4 ft. from the right end. Draw the shear and moment diagram.
6. A uniformly loaded beam of length $2\,l$ is supported at the ends and at the middle. It is known that the reaction at each end is $\frac{3\,w\,l}{8}$. Draw the shear and moment diagrams.

70. The General Moment Equation.—In the examples which have been given, the origin of coördinates has been taken at the left end of the beam. But it is often desirable to be able to write the moment equation with any point as the origin. Figure 126 represents a beam of indefinite extent with the origin

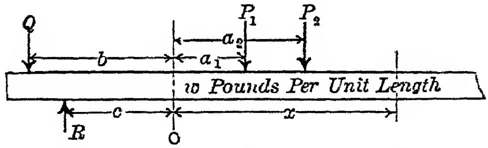


FIG. 126.—General case of loading.

of coördinates on a vertical line through O. To the right of the origin, at distance a_1, a_2, etc., there are concentrated loads P_1, P_2, etc. There is also a uniformly distributed load of w per unit length. There may be any number of vertical loads and reactions to the left of the origin, but all the vertical loads may be replaced by their resultant Q at some definite distance b from the origin, and all the vertical reactions by a single reaction R

at a distance c from the origin. Writing the moment with respect to a section at a distance x from the origin:

$$M = R(c + x) - Q(b + x) - P_1(x - a_1) - P_2(x - a_2) - \frac{w x^2}{2}; \quad (1)$$

$$M = Rc - Qb + (R - Q)x - P_1(x - a_1) - P_2(x - a_2) - \frac{w x^2}{2}. \quad (2)$$

$R c - Q b$ is the moment at the origin, which may be represented by M_0, and $R - Q$ is the shear at the origin, which may be represented by V_0.

$$M = M_0 + V_0 x - P_1(x - a_1) - P_2(x - a_2) - \frac{w x^2}{2}; \quad (3)$$

$$M = M_0 + V_0 x - \Sigma P(x - a) - \frac{w x^2}{2}, \qquad \text{Formula XI}$$

in which $\Sigma P(x - a)$ represents the sum of the moments of all the concentrated loads between the origin and the section considered. When any point on a beam is taken as the origin of coördinates, the moment at any section at a distance x to the right of the origin is the moment at the origin, plus the shear at the origin multiplied by the distance of the section from the origin, plus the moment with respect to the section of each load and reaction between the origin and the section.

Example I

For the beam of Fig. 119 (example of Art. 67), write the moment equation for the portion between the left support and the 360-lb. load from the general moment equation with the origin of coördinates infinitely close to the right of the left support.

$$V_0 = -240 - 240 + 1{,}140 = 660.$$
$$M_0 = -1{,}440.$$

At any distance x from the origin, toward the right,

$$M = -1{,}440 + 660\,x - 30\,x^2. \quad (1)$$

If x is greater than 6 ft., the 360-lb. load enters the equation and

$$M = -1{,}440 + 660\,x - 30\,x^2 - 360(x - 6). \quad (2)$$

Equation (1) is limited to the region between the left support and the 360-lb. load. Equation (2) is valid from the 360-lb. load to the right support.

Problem

1. Using Eq. (1) or Eq. (2), calculate the moment at 5 ft., 6 ft., 8 ft., 9 ft., 11 ft., and 12 ft. from the left end of the beam of Fig. 119.

The general moment equation is especially useful in the solution of problems of beams with more than two supports (indeterminate beams). The equations may be written with M_0 and V_0 unknown quantities, the values of which are calculated later from the deflection, slope, moment, or shear. Deflection and slope, for which this method is most valuable, have not yet been studied. The examples here given, therefore, are limited to moment and shear.

Example II

A cantilever of length l is fixed at the left end and carries a uniformly distributed load of w per unit length. Write the general moment equation with the origin of coördinates infinitely close to the wall (Fig. 127).

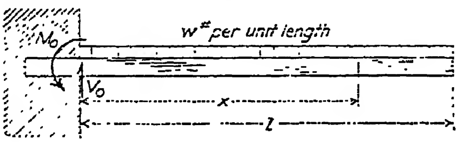


FIG. 127.—Uniformly loaded cantilever.

The shear equation is

$$V = V_0 - wx.$$

At $x = l$, $V = 0$;

$$0 = V_0 - wl;$$
$$V_0 = wl.$$
$$M = M_0 + V_0x - \frac{wx^2}{2}.$$

At $x = l$, $M = 0$. Substituting in the general moment equation for V_0 and x:

$$0 = M_0 + wl^2 - \frac{wl^2}{2};$$
$$M_0 = -\frac{wl^2}{2}.$$

Example III

Write the general moment equation for the beam of Problem 3 (Art. 69) with the origin of coördinates close to the right of the left support.
For x greater than 8 ft. and not greater than 10 ft.,

$$M = M_0 + V_0x - \frac{wx^2}{2} - 100\,(x - 8).$$

At the right support $M = 0$ and $x = 10$.
Over the left support $M_0 = -80 \times 2 - 40 \times 1 = -200$;

$$0 = -200 + 10\,V_0 - 1{,}000 - 200;$$
$$V_0 = 140.$$

To find the reaction R_1,

$$-80 - 40 + R_1 = 140;$$
$$R_1 = 260.$$

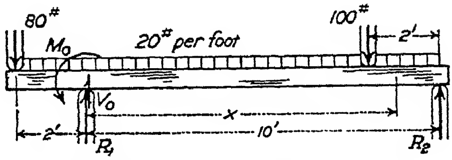


FIG. 128.—Beam for Example III.

Problems

2. Calculate the moment for Example III by the general moment equation for $x = 2$, 3, 5, 6, 8, and 9.
3. In Example III find the shear close to right support, starting with V_0. From the shear close to the support find the right reaction.

71. Relation of Moment and Shear.—Differentiate the general moment equation with respect to x.

$$M = M_0 + V_0x - P_1(x - a_1) - P_2(x - a_2) - \frac{w\,x^2}{2}. \qquad (1)$$

$$\frac{dM}{dx} = V_0 - P_1 - P_2 - w\,x. \qquad (2)$$

The right member of Equation (2) is recognized as the shear at a distance x from the origin.

$$\frac{dM}{dx} = V. \qquad \text{Formula XII}$$

The derivative, with respect to the length, of the moment equation of a beam gives the shear in the beam.

In Fig. 125, there is an abrupt change in the slope of the moment curve at the concentrated load and at the second support. At the concentrated load, the shear changes from 120 pounds to 20 pounds and there is an equivalent relative change in the slope of the tangent to the moment curve. The shear at this point may be said to have any value between 120 pounds and 20 pounds. The derivative of the moment is not *single valued* and Formula XII does not hold. It does hold, however, infinitely close to this point on either side.

In reality, no load can be concentrated at a point or on a line extending across the beam. A so-called *concentrated load* is actually distributed over an area. If this distribution were known, the shear at any point would have a single value and Formula XII would be found to be valid at all sections.

Since

$$V = \frac{dM}{dx},$$

$$\int V\,dx = \int dM;$$

$$\int V\,dx = M_2 - M_1. \qquad \text{Formula XIII}$$

The integral of $V\,dx$ between any two values of x gives the difference of the moments at the corresponding points.

Figure 129 is part of the shear diagram for a beam which is supported at the left end, weighs w pounds per foot, and carries a

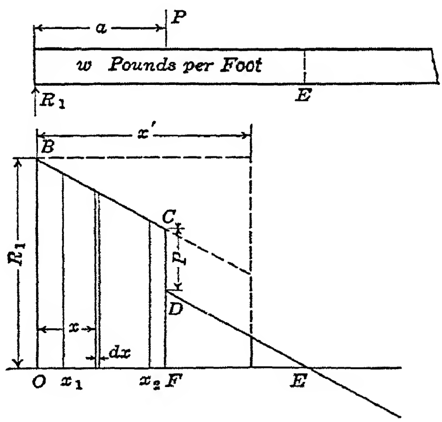


FIG. 129.—Relation of area of shear diagram to moment.

load P at a distance a from the left end. The element of width dx at a distance x from the left of the diagram extends from the X axis to the line $B\,C$. The area of this element is $V\,dx$. The integral of $V\,dx$ between the limits x_1 and x_2 represents the area which is bounded by the shear diagram, the X axis, and the ordinates x_1 and x_2. Since $\int V\,dx = M_2 - M_1$, *the area of the shear diagram between two points is the difference between the moments at these points.* When the shear is negative, the area is below the X axis and, therefore, is negative.

At the ends of a beam, if all the loads and reactions are perpendicular to the length, the moment is zero. It follows, therefore, that the moment at any section is the entire area of the shear diagram from the end of the beam of the section.

Example

In Fig. 125, find the moment at 2 ft., 3 ft., 4 ft., and 7 ft. from the left end by means of the area of the shear diagram.

At 2 ft., the moment is the area of the trapezoid the base of which is 2 ft., and the altitude is 160 units on one side and 120 units on the other side.

$$M_2 = 0 + \frac{160 + 120}{2} \times 2 = 280 \text{ ft.-lb.}$$

At 3 ft., the moment is the moment at 2 ft. plus the area of the triangle 20 units high and 1 ft. wide.

$$M_3 = 280 + 10 = 290 \text{ ft.-lb.}$$

At 4 ft., the negative triangle is subtracted from the moment at 3 ft.

$$M_4 = 290 - 10 = 280 \text{ ft.-lb.}$$

At 7 ft., the negative triangle has a base of 4 ft. and an altitude of 80 units

$$M_7 = 290 - \frac{80 \times 4}{2} = 130 \text{ ft.-lb.}$$

Problems

1. In Fig. 125, find the moment at 5 ft., 7 ft., 8 ft., and 12 ft. from the left end, assuming that the moment at 3 ft. is known.
2. A cantilever of length l is fixed at the right end and carries a load P at the left end. Find the moment at the middle and at the right end by means of the area of the shear diagram.
3. A cantilever of length l is fixed at the right end and carries a uniformly distributed load of w per unit length. Find the moment at the middle and at the right end by means of the shear diagram. Check by definition of moment.
4. From the shear diagram of Problem 2 of Art. 69, calculate the moment at 3 ft., 6 ft., and 9 ft. from the left end. Check by definition of moment.
5. A simply-supported beam of length l carries a load P at six-tenths the length from the left end. Draw the shear diagram. Find the moment at the middle and under the load. Check by definition of moment from the sketch of the beam without reference to the shear diagram.

72. The Dangerous Section.—A section in a beam where the moment has a maximum numerical value is called a *dangerous section*. The mathematical condition for a maximum or minimum value of M is that the derivative with respect to the length shall be zero. But since $\frac{dM}{dx}$ is the shear, this means that there is a dangerous section at every point where the shear becomes zero.

In Fig. 125, the shear diagram crosses the X axis at 3 feet from the left end. This is one dangerous section.

The shear may pass through zero when the moment equation does not fulfill the mathematical condition that the slope of the tangent to the curve is zero. At the right support in Fig. 125, the slope of the moment curve changes abruptly from negative to positive. The negative moment at this point has the maximum numerical value. This is evident from the shear diagram. The shear changes from negative to positive at the support. A positive area must, therefore, be added to the negative moment after the support is passed.

When the loading of a beam is given, always find the dangerous sections by means of a sketch of the shear diagram. If the dangerous section does not come at a support or under a concentrated load, its exact position may be found algebraically.

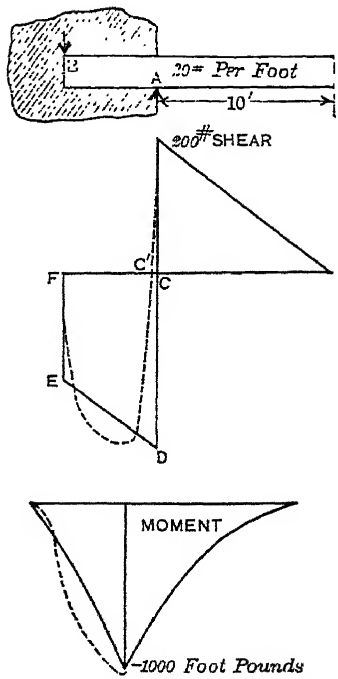


FIG. 130.—Cantilever fixed at left end.

Problems

1. A simply-supported beam of length l carries a uniformly distributed load over six-tenths of the length adjacent to the left support, and no load over the remainder. Find the moment at the dangerous section. *Ans.* $M = 0.0882\ wl^2$.
2. A beam 16 ft. long, weighing 120 lb. per ft., is supported 3 ft. from the left end and 1 ft. from the right end. It carries 300 lb. on the left end, 660 lb. on the right end, and 1,080 lb. 5 ft. from the left end. Draw the shear diagram and locate each dangerous section. Find the moment at each dangerous section algebraically and check by the area of the shear diagram. Write the equation of moments for the portion between the left support and the load of 1,080 lb. and solve for the position of zero moment. The equation has two roots. Which one should be taken? Why? Write the equation and find the other position of zero moment.
3. A beam 12 ft. long, weighing 60 lb. per foot, is supported at the ends and carries a load of 240 lb. 3 ft. from the left end and a load of 720 lb. 2 ft. from the right end. Find the moment at the dangerous section.
4. Find the moment at the dangerous section for a cantilever 10 ft. long, which carries a distributed load of 20 lb. per ft., including its own weight.

Figure 130 shows the moment and shear diagrams for a cantilever. If there are no horizontal forces, the area of the shear diagram inside the wall must equal the area outside. The form of the diagram inside the wall is not known. If all the downward forces were concentrated at B, and all the upward force at A, the shear diagram would be the figure $C\ D\ E\ F$. Since the pressures are distributed, the shear diagram is that shown by the dotted lines, and the dangerous section is at C', a little back of the face of the wall.

The actual moment diagram inside the wall is something like that shown by the dotted line.

Miscellaneous Problems

1. A cantilever of length l is fixed at the right end and carries a load of w per unit length over six-tenths of the length adjacent to the free end. Draw the shear diagram. Calculate the moment at each two-tenths of the length from the definition of moment. From the end of load to the fixed end the moment diagram is a straight line. Where does this line intersect the line $M = 0$? Why?

2. A beam of length l is supported at the ends and carries a load w per unit length over six-tenths of the length adjacent to the left end. Draw the shear diagram. Find the moment at the dangerous section and at the middle.

3. A simply-supported beam, 20 ft. long, carries a distributed load of 1,200 lb. per ft., a load of 3,000 lb. 4 ft. from the left support, and a load of 6,000 lb. 6 ft. from the right support. Find the reactions and check. Draw the shear diagram. Find the dangerous section. Calculate the moment at the dangerous section by definition. Check by area of shear diagram. Find the moment 4 ft. from the left end by shear trapezoid. Find the moment 6 ft. from right end by area between that section and the dangerous section.

4. A beam 24 ft. long is supported 4 ft. from each end. It carries a distributed load of 500 lb. per ft., including its own weight, and a load of 2,000 lb. on the right end. Find the dangerous section. Calculate the moment at the dangerous section two ways. Write the general moment equation with the origin of coördinates just to the right of the left support. Find the moment at the right support, at the dangerous section, and at the right end by this equation modified to suit each particular case.

5. A beam 20 ft. long is supported 5 ft. from the left end and 3 ft. from the right end. It carries 360 lb. on the left end and 600 lb. on the right end. Draw the shear diagram. Draw the moment diagram. Locate the dangerous section.

Ans. Dangerous section anywhere between supports.

6. A beam 20 ft. long is supported at the ends and carries 320 lb. 7 ft. from the left end, and 560 lb. 4 ft. from the right end. Draw the shear diagram and the moment diagram. Find the dangerous section.

7. A beam of length l is supported at the ends and carries a load P at a distance a from the left support and a load Q at a distance b from the right support. What must be the relation of the loads and distances in order that the diagram of the moments caused by these loads may be horizontal from one load to the other?

   Ans. $P \times a = Q \times b$. Each reaction and the adjacent load form a couple.

8. A beam 20 ft. long is supported at the ends. It carries 480 lb. 5 ft. from the left support and 600 lb. 8 ft. from the right support. Draw the shear diagram and calculate the moment at the dangerous section.

*

CHAPTER VII

STRESSES IN BEAMS

73. Distribution of Stress.—At any section of a bent beam, there is tension across the part adjacent to the convex surface and compression across the part adjacent to the concave surface, and there is usually shear parallel to the section. The method of finding the total vertical shear has been given in Chapter VI. The method of determining the unit shearing stress will be given later in Chapter X. The problem of the total tension and compression and of the unit tensile and compressive stresses will now be considered.

If the external forces have no components parallel to the length of the beam, the resultant compressive stress across any section is equal to the resultant tensile stress, and these two forces form a couple, the moment of which is equal to the product of either force multiplied by the distance between them. This moment is equal and opposite to the bending moment.

To calculate these forces (H and C of Figs. 108 to 113) it is only necessary to know the bending moment and the distance between the forces. This distance is easily measured in Figs. 108 and 110. In Fig. 111, the compressive stress is distributed over the small block, and its law of distribution must be known in order to locate its resultant.

In Fig. 113, the tensile stress is distributed over the entire upper portion and the compressive stress is distributed over the entire lower portion. In order to find the moment arm of the couple, it is necessary, therefore, to know the law of distribution of these stresses.

The fibers on the convex side of a bent beam are elongated and those on the concave side are shortened. Between these there is a surface in which the fibers suffer no deformation in the direction of the length of the beam. This surface is called the *neutral surface* of the beam. The intersection of the neutral surface with any transverse section of the beam is called the *neutral axis* of that section.

It is customary to assume that the unit stress at any section varies directly as the distance from the neutral axis. The reasons for this assumption, and the conditions under which it is valid, will be given in Art. 78. Figure 131 represents graphically the variation of unit stress in a beam, the upper part of which is in tension.

Figure 131, I, shows the forces from left to right and also the forces from right to left. Figure 131, II, shows only the forces with which the portion of the beam to the right of the section acts on the portion to the left. It will be noticed that both sets of forces tend to turn the left portion clockwise about the neutral axis at O. Figure 131, III, shows a convenient method of draw-

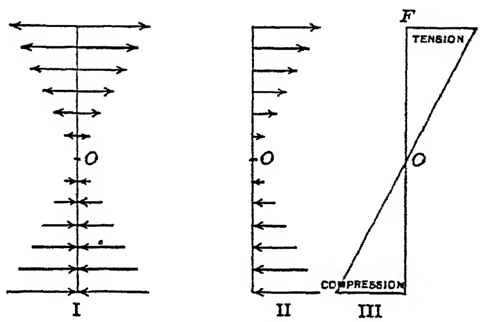


FIG. 131.—Stress variation in a beam.

ing the diagram to show the magnitude of the unit stress at any distance from the neutral axis.

Since the unit stress varies as the distance from the neutral axis, it may be represented by two wedges cut from the beam by two planes which pass through the neutral axis. One of these planes should be normal to the length of the beam and, therefore, represent the section considered, and the other may make any convenient angle. Figure 131, III, may be considered as representing two such planes. The volume of each wedge may be regarded as giving the total stress across its corresponding part of the section, and the distance between the center of gravity of the two wedges, measured parallel to the section, gives the moment arm of these total stresses.

74. Fiber Stress in a Beam of Rectangular Section.—Figure 132 shows the wedges representing the stress distribution of a rectangular beam section of breadth b and depth d. Since the total tension H is equal to the total compression C, the two

wedges which represent the total tension and compression must have equal volume. Since the slope and width of the wedges are the same and their volumes are equal, their heights must be equal, and the neutral axis $B\ B'$ is at a distance $\frac{d}{2}$ from the top or bottom of the section. If S is the unit tensile stress in the outer

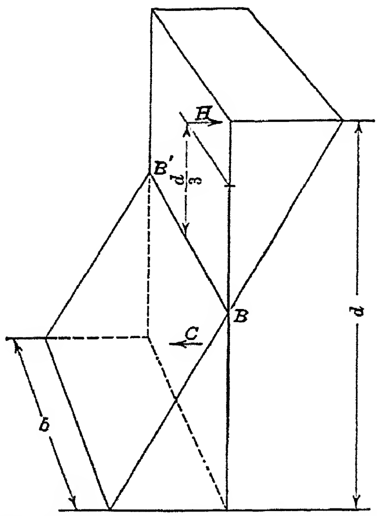


FIG. 132.—Solids representing stress in rectangular prism.

fibers at the top of the beam, $\frac{S}{2}$ is the average tensile stress over the upper half of the section. The total tension is the average stress multiplied by the area above the neutral axis.

$$H = \frac{S}{2} \times \frac{b\,d}{2} = \frac{S\,b\,d}{4}.$$

Problems

1. A beam of rectangular section is 4 in. wide and 12 in. deep. The unit stress in the outer fibers at the convex surface is 1,000 lb. per sq. in. What is the total tension? *Ans.* $500 \times 4 \times 6 = 12{,}000$ lb.
2. A beam of rectangular section is 6 in. wide and 10 in. high. The total tension is 4,800 lb. What is the average tensile stress? What is the maximum tensile stress? *Ans.* Maximum stress = 320 lb./in.^{2}
3. Figure 133 represents a T section with sides parallel. The neutral axis is 3 in. from the top of the flange. The unit stress at the top is given as 240 lb. per sq. in. What is the unit tensile stress at the bottom of the flange? What is the unit compressive stress at the bottom of the stem? What is the unit compressive stress 1 in. from the bottom? *Ans.* $s_t = 80$ lb./in.^{2}; $S_c = 400$ lb./in.^{2}; $s_c = 320$ lb./in.^{2}

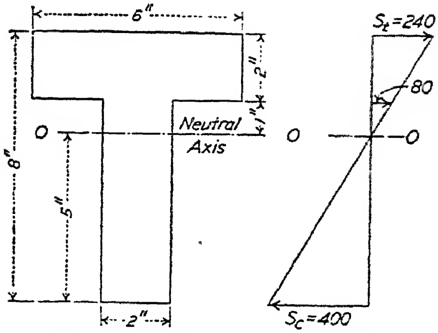


FIG. 133.—Determination of neutral axis.

4. Calculate the total tensile stress in the beam of Problem 3. Calculate the total compressive stress in the lower 5 in. of the stem.

Tension in flange.	$\frac{240 + 80}{2} \times 12 =$	1,920 lb.
Tension in upper inch of stem . . .	$\frac{80 + 0}{2} \times 2 =$	80 lb.
	Total tension $=$	2,000 lb.
Compression in stem.	$\frac{400 + 0}{2} \times 10 =$	2,000 lb.

5. The flange of a T-beam is 5 in. wide and 2 in. high. The thickness of the stem is 1 in. and the net height is 10 in. The neutral axis is 4 in. from the top of the flange or 8 in. from the bottom of the stem. The tensile stress at the bottom of the stem is 1,200 lb. per sq. in. What is the compressive stress at the top of the flange? What is the average compressive stress in the flange? What is the total tension in the lower 8 in. of the stem? What is the total compression in the remainder of the section? *Ans.* Total tension = 4,800 lb.
6. A hollow box girder is 10 in. wide and 12 in. high outside and is 6 in. wide and 8 in. high inside. The maximum unit stress in the top and bottom fibers is 1,200 lb. per sq. in. Find the total tension in one-half. *Ans.* 26,400 lb.

The line of application of the resultant tension H of Fig. 132 passes through the center of gravity of the wedge. Since the center of gravity of a triangle and, consequently, of a triangular wedge is two-thirds the height from the vertex, the distance of H from the neutral axis is $\frac{2}{3} \times \frac{d}{2} = \frac{d}{3}$. In like manner the total compression C is located at a distance $\frac{d}{3}$ below the neutral axis. The total moment arm of the couple made up of the forces H and C is $\frac{2d}{3}$.

Problems

7. What is the moment about the neutral axis of the forces of Problem 1?

Ans. $12{,}000 \times 4 + 12{,}000 \times 4 = 96{,}000$ in.-lb.

8. What is the moment about the neutral axis of the tensile stress of Fig. 133? Construct a three-dimensional sketch of the stress distribution solid.

The middle 2 in. of the stress-distribution solid is a triangular wedge. The remainder makes two trapezoidal wedges. Regard the whole tension part as a triangular wedge 6 in. wide and 3 in. high. Subtract from the moment of this wedge the moment of two triangular wedges each 2 in. wide and 1 in. high.

Total tension of triangular wedge 6 in. wide and 3 in. high is

$$\frac{240 + 0}{2} \times 18 = 2{,}160 \text{ lb.}$$

Total tension of two triangular wedges each 2 in. wide and 1 in. high is

$$\frac{80 + 0}{2} \times 4 = 160.$$

Moment $= 2{,}160 \times 2 - 160 \times 2/3 = 4{,}320 - 106.7 = 4{,}213.3$ in.-lb.

9. Find the moment about the neutral axis of the compressive stress of Problem 3. Find the total moment.

Ans. 6,666.7 in.-lb.; 10,880 in.-lb.

10. Find the total moment about the neutral axis of the forces of Problem 6.

Ans. $(36{,}000 \times 4 - 9{,}600 \times 8/3)\,2 = 236{,}800$ in.-lb.

11. Find the total moment about the axis of the forces of Problem 5.

Ans. $M = 25{,}600 + 14{,}400 = 40{,}000$ in.-lb.

The total resisting moment of a rectangular section is

$$M = \frac{S\,b\,d}{4} \times \frac{2\,d}{3} = \frac{S\,b\,d^2}{6}. \qquad \text{Formula XIV.}$$

By means of Formula XIV, the maximum fiber stress in a beam of rectangular section may be calculated when the loads are known, or the load may be calculated for any given allowable stress.

Example

A 4-in. by 6-in. cantilever carries a load of 240 lb. on the free end. Find the unit stress in the top and bottom fibers at a section 5 ft. from the free end.

Horizontal dimensions are given first. A 4-in. by 6-in. beam is 4 in. wide and 6 in. deep. Since unit stresses are required in pounds per square inch, the moment must be in inch-pounds.

$$M = \frac{S\,b\,d^2}{6},$$

$$S = \frac{6\,M}{b\,d^2} = \frac{6 \times 14{,}400}{4 \times 36} = 600 \text{ lb. per sq. in.}$$

Problems

12. A 6-in. by 10-in. beam, 12 ft. long, is supported at the ends and carries a load of 2,400 lb. at the middle. Calculate the unit stress in the top and bottom fibers at the dangerous section caused by this load.
Ans. $S_t = S_c = 864$ lb./in.^{2}

13. If the beam of Problem 12 were placed with the 10-in. faces horizontal and the load were put 4 ft. from the left end, what would be the maximum fiber stress? *Ans.* $S = 1{,}280$ lb./in.^{2}

14. An 8-in. by 12-in. beam is 15 ft. long and is supported at the ends. It carries 1,000 lb. 6 ft. from the left support and a distributed load of 240 lb. per ft. Find the maximum unit stress.
Ans. $S = 630$ lb./in.^{2}

15. Solve Problem 14 if the distributed load is 400 lb. per ft. Calculate the moment by definition and check by the area of the shear diagram from the right end. *Ans.* $S = 903.1$ lb./in.^{2}

16. A 4-in. by 6-in. beam, 8 ft. 4 in. long, is supported at the ends and carries a load at the middle which makes the maximum stress 900 lb. per sq. in. Find the load. *Ans.* $P = 864$ lb.

17. The beam of Problem 16 has the 6-in. faces horizontal. What is the stress for a load of 864 lb. at the middle?

75. Fiber Stress in a Beam of Any Section.—The methods of Art. 74 are not convenient for sections which are not rectangles. There is a general method, however, which applies to any form of section. The moment of inertia and the center of gravity of plane figures are important factors in this method. As these are given in handbooks for many geometrical figures, a great saving of labor is gained by their use.

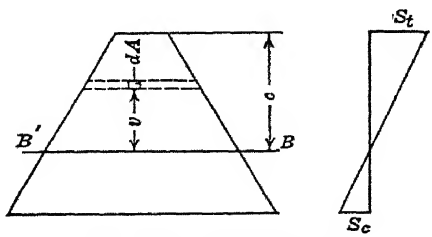


FIG. 134.—Beam section.

Figure 134 may be regarded as representing a section of any form. $B\,B'$ is the neutral axis. An element of area dA is at a distance v from the neutral axis. (The letter v will be used to represent distance from the neutral axis in a section, and y will be reserved to represent deflection of the axis from its original position.) The area dA may be infinitesimal in two dimensions or it may extend entirely across the section parallel to the neutral axis as shown by the broken lines.

Since the unit stress varies as v, it may be represented by $k\,v$, in which k is the unit stress at unit distance from the neutral

axis. The total stress on the element dA is the unit stress times the area.

$$\text{Total stress} = k\,v\,dA. \tag{1}$$

The moment of this stress on dA about the neutral axis, is

$$dM = k\,v^2 dA. \tag{2}$$

Since v^2 is positive when v is positive or negative, the sign of increment of moment is the same whether the element is above or below the neutral axis.

$$M = k\int v^2 dA = k\,I, \tag{3}$$

in which I is the moment of inertia of the section with respect to the neutral axis. Since

$$s = k\,v, \quad k = \frac{s}{v}, \tag{4}$$

which substituted in Equation (3) gives

$$M = \frac{s\,I}{v}. \tag{5}$$

Equation (5) gives the unit stress at any distance from the neutral axis. The most important stress is the stress in the extreme outer fibers where v is a maximum and the unit stress is the greatest. If the maximum unit stress be represented by S and the distance to the outer fiber from the neutral axis be represented by c, the equation becomes

$$M = \frac{S\,I}{c}. \qquad \text{Formula XV}$$

This formula is so important that it is desirable to memorize it also in the form

$$S = \frac{M\,c}{I}. \qquad \text{Formula XVI}$$

76. Location of the Neutral Axis.—The values of I and c in Formula XV depend upon the location of the neutral axis. This is found from the condition that the total tensile stress across the part of the section on one side of the neutral axis is equal to the total compressive stress across the part of the section on the other side of the axis. On an element dA,

$$\text{Total stress} = k\,v\,dA. \tag{1}$$

$$\text{Total stress on entire section} = k\int v\,dA = 0. \tag{2}$$

The constant k is not zero when the beam is bent; consequently $\int v\,dA$ must be zero.

The center of gravity of a plane area is given by

$$\bar{v} = \frac{\int v\,dA}{A}; \tag{3}$$

$$\bar{v}\,A = \int v\,dA = 0. \tag{4}$$

Since A is not zero,

$$\bar{v} = 0. \tag{5}$$

The neutral axis of a beam of any section passes through the center of gravity of the section.

77. Section Modulus.—The expression $\frac{I}{c}$, in which c is the distance from the neutral axis to the extreme outer fiber, is called the *section modulus* or *modulus* of the *section*. If the section modulus is represented by Z, Formula XVI becomes

$$S = \text{unit stress in outer fibers} = \frac{M}{Z}. \tag{1}$$

The section moduli for rolled shapes and for the principal geometric figures are given in the handbooks of the steel manufacturers. The *Carnegie Pocket Companion* prints these under the title Elements of Sections. The *A.I.S.C. Handbook* classes them under Properties of Various Sections. Both of these handbooks use S for section modulus, and f for unit stress.

Most modern American textbooks represent stress by S and s. Several use the combination $\frac{I}{c}$ for the section modulus.

$$\text{For a rectangular section, } I = \frac{b\,d^3}{12} \text{ and } c = \frac{d}{2},$$

$$Z = \frac{I}{c} = \frac{b\,d^2}{6}. \tag{2}$$

When this value of Z is substituted in Formula XV,

$$M = \frac{S\,b\,d^2}{6},$$

which is Formula XIV.

Always use the section modulus as given in the handbook or $\frac{bd^2}{6}$ for a rectangular section, instead of $\frac{I}{c}$. Beams of unsymmetrical section, such as T-beams with stems vertical or channels with loads perpendicular to the web, have two different values of c and two corresponding values of section modulus. The handbook gives the modulus for the larger c and the largest S in Formula XVI. It is necessary, sometimes, to compute Z for the smaller c.

Problems

1. Look up in the handbook the moment of inertia of a 12-in. 50-lb. standard I-beam section with respect to the axis perpendicular to the web. Calculate the section modulus. Solve also for the section modulus with respect to the axis parallel to the web.
2. Look up the moment of inertia of a 15-in. 50-lb. standard channel section with respect to the axis perpendicular to the web. Calculate the section modulus. Solve also for the section modulus with respect to the axis through the center of gravity parallel to the web.
3. Solve Problem 2 for a 12-in. 41.1-lb. ship channel.
4. Look up the moment of inertia of a 6-in. by 4-in. by 1-in. angle section with respect to the axis through the center of gravity parallel to the 4-in. leg. Solve for the section modulus.
5. Look up the moment of inertia of a 4-in. by 5-in. by ½-in. T-beam section with respect to the axis through the center of gravity parallel to the flange. Solve for the section modulus. Solve also for the section modulus with respect to the other *principal axis*.
6. Calculate the section modulus of a 6-in. by 8-in. rectangular beam with respect to the axis parallel to the 6-in. faces. Which dimension is b and which is d of the equation? *Ans.* $Z = 64$ in.³
7. Solve Problem 6 for the section modulus with respect to the axis parallel to the 8-in. faces. *Ans.* $Z = 48$ in.³
8. A 6-in. by 8-in. beam, 12 ft. long, is supported at the ends and carries a load of 800 lb. on the middle. Find the maximum unit stress caused by this load.

$$\textit{Ans. } S = \frac{400 \times 72}{64} = 450 \text{ lb./in.}^2$$

9. Solve Problem 8 if the beam has the 8-in. faces horizontal.
10. A 6-in. by 10-in. beam, 15 ft. long, is supported at the ends and carries a load of 240 lb. per ft. Find the maximum unit stress caused by this load.
11. A 15-in. 60.8-lb. standard I-beam, 20 ft. long between centers of supports, carries a uniformly distributed load of 1,200 lb. per ft. and a load of 6,000 lb. 6 ft. from the left support. Find the maximum fiber stress caused by these loads. Check moment.

$$\textit{Ans. } S = \frac{952{,}200}{81.2} = 11{,}726 \text{ lb./in.}^2$$

12. Find the unit stress in the outer fibers of the beam of Problem 11 under the load and at the middle. What is the stress at the dangerous section 1 in. from the top?

Ans. 11,172 lb./in.^{2}, 11,527 lb./in.^{2}, 10,163 lb./in.^{2}

13. If the allowable unit stress for structural steel is 18,000 lb. per sq. in. and the compression flanges are held laterally, what is the minimum section modulus for the loads of Problem 11 in addition to the weight of the beam?

$$\textit{Ans. } Z = \frac{952{,}200}{18{,}000} = 52.9 \text{ in.}^3$$

A 15-in. 42.9-lb. standard I-beam has $Z = 58.9$, which is so much greater than 52.9 that it is not necessary to recalculate for the effect of the weight of the beam.

A 12-in. 55-lb. standard I-beam has $Z = 53.2$. If the additional 3 in. of head room is worth the extra cost of this heavier beam, it is necessary to calculate for the moment of the beam's weight. The dangerous section with the weight included is not exactly 162 in. from the end but it may be assumed not to have been changed for an approximation. The additional moment is 27,852 in.-lb.

$$\frac{952{,}200 + 27{,}852}{18{,}000} = 54.45 = Z.$$

The 12-in. 55-lb. I-beam is too small.

14. A 15-in. 70-lb. standard I-beam, 25 ft. long, has a 14-in. by 1-in. plate welded to the bottom. Find the distance of the center of gravity of the combined beam from the top. Calculate the moment of inertia of the combined beam with respect to the axis perpendicular to the web through the center of gravity. Find the section modulus with respect to the top fiber of the beam and the bottom fiber of the plate.

$$\textit{Ans. } Z = \frac{1{,}189.3}{10.758} = 110.4;\ Z = \frac{1{,}189.3}{5.242} = 227.6 \text{ in.}^3$$

78. Relation of Stress to Deformation.—Figure 135 represents a bent beam which is concave upward. The amount of bending is greatly exaggerated. The deflection in a beam of these relative dimensions at stresses below the elastic limit would be so small as to be scarcely noticeable. $E\,F\,G$ is a plane section with neutral axis $B\,B'$. The broken lines $K'\,M'$, $M'\,N'$ indicate the position, before the beam was bent, of a plane section parallel to $E\,F\,G$ at a distance Δl from that section. The section $E\,F\,G$ may be regarded as fixed in position and direction and the parts of the beam on each side of this fixed section may be considered as bent upward. The plane $K'\,M'\,N'$ is rotated about its neutral axis $C\,C'$ through an angle $\Delta\theta$ to the position $K\,M\,N$. (There is a slight shift upward, but this does not affect the problem.) Since there is no elongation in the neutral surface, the distance

between the neutral axes $B\,B'$ and $C\,C'$, measured along the curved surface, remains unchanged.

It is assumed that a plane section in a beam remains plane when the beam is bent; therefore the plane section $K'\,M'N'$ remains a plane section after it has been rotated into the position $K\,M\,N$. A filament of section dA, at a distance v from the neutral surface, extends from the plane $E\,F\,G$ to the plane $K\,M\,N$. When the beam is bent, and the plane $K\,M\,N$ is rotated about the neutral axis $C\,C'$ through an angle θ, this filament is shortened by an amount $v\,\Delta\theta$. A similar filament at a

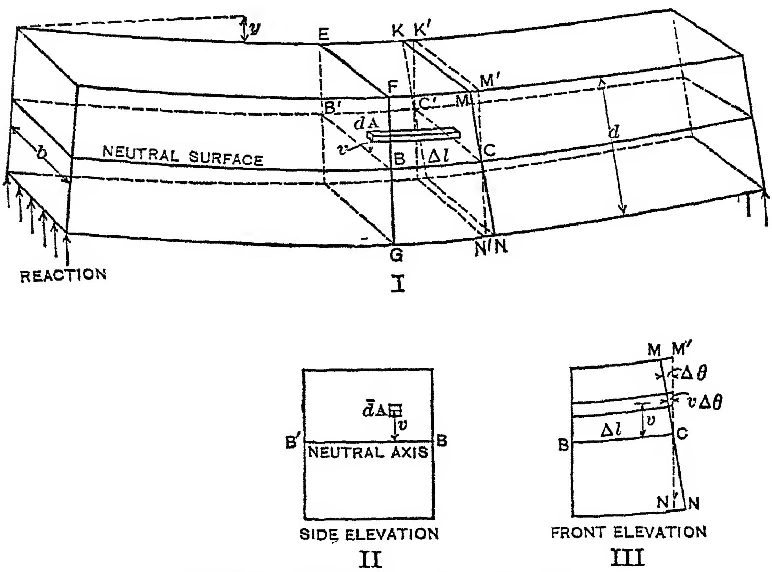


Fig. 135.—Deformation of a bent beam.

distance v below the neutral surface would be elongated $v\,\Delta\theta$. The unit deformation of the filament is given by

$$\delta = \frac{v\,\Delta\theta}{\Delta l}. \qquad (1)$$

Under the condition that the deformations are such that no stress exceeds the proportional elastic limit, the unit stress varies as the unit deformation; and since the unit deformation varies as v, the unit stress varies as the distance from the neutral axis, as was assumed in Art. 73.

Since unit stress is equal to $E\delta$, the unit stress above the neutral surface is given by

$$s_c = E_c\, v \frac{\Delta\theta}{\Delta l}. \tag{2}$$

Below the neutral surface

$$s_t = E_t\, v \frac{\Delta\theta}{\Delta l}. \tag{3}$$

In most cases **it is assumed** (and is practically true) **that the modulus of elasticity is the same in both compression and tension.** With this assumption,

$$s = E\, v \frac{\Delta\theta}{\Delta l} \tag{4}$$

is the expression for the unit stress in the beam, at any element of area.

On an element of area,

$$\text{Total stress on } dA = E\, v \frac{\Delta\theta}{\Delta l}\, dA. \tag{5}$$

The moment of this stress with respect to the neutral axis $B\,B'$ is the product of the total stress on dA by the moment arm v;

$$dM = E\, v^2 \frac{\Delta\theta}{\Delta l} dA = E \frac{\Delta\theta}{\Delta l} v^2 dA. \tag{6}$$

The total moment of all the filaments which make up the beam is the integral of dM over the section $E\,F\,G$. Integrating over this area, $\frac{\Delta\theta}{\Delta l}$ remains constant and

$$M = E \frac{\Delta\theta}{\Delta l} \int_{c_1}^{c_2} v^2 dA = E \frac{\Delta\theta}{\Delta l} I, \tag{7}$$

in which c_1, c_2 are the distances of the upper and lower surfaces of the beam from the neutral surface, I is the moment of inertia of the cross section $E\,F\,G$ or $K\,M\,N$ with respect to its neutral axis, and $\Delta\theta$ is the change in slope, in the length Δl, of the normal to the beam or the equal change in slope of the tangent to the beam.

Example

A 6-in. by 6-in. wooden beam rests on two supports, which are 50 in. apart, and overhangs each support 30 in. Equal loads are placed on each

overhanging end at 20 in. outside the supports. It is found that two sections between the supports, which are 40 in. apart and were parallel to each other before the beam was loaded, make an angle of 1° with each other after the loads are applied. Find the bending moment between the supports and find the loads, if E is 1,500,000 lb. per sq. in.

By means of Eq. (7), find the bending moment between the supports and find the loads, if $E = 1{,}500{,}000$ lb. per sq. in.

$$I = 108 \text{ in.}^4; \quad \Delta l = 40 \text{ in.}; \quad \Delta\theta = \frac{\pi}{180} \text{ radian.}$$

$$M = \frac{1{,}500{,}000 \times 108 \times \pi}{40 \times 180} = 22{,}500\,\pi = 70{,}686 \text{ in.-lb.}$$

Since the section of this beam is a rectangle, it may be solved without using Eq. (7). The deformation at the top fibers is $\frac{\pi}{60}$ and the unit deformation is $\frac{\pi}{2{,}400}$. The unit stress at the outer fibers is $\frac{1{,}500{,}000\,\pi}{2{,}400} = 1{,}963.5$ lb. per sq. in. The average compression in the upper half of the section is one-half of this and the total compression is $\frac{1{,}963.5 \times 18}{2} = 17{,}671$ lb. The moment arm of the total compression is 2 in. The total moment of the tension and compression is $17{,}671 \times 4 = 70{,}684$ in.-lb.

Problems

1. A 5-in. 10-lb. I-beam rests on supports 6 ft. apart and carries a load midway between the supports. Measurements are taken at the top and bottom of the beam on two 8-in. gage lengths. Gage length B begins 4 in. to the right of the middle, and gage length A starts from the right end of B. What is the stress in the outer fibers at the middle of the beam when a load of 3,000 lb. is applied? What is the average stress in the outer fibers for gage length B? For gage length A?
   Ans. 11,250 lb./in.^{2}, 8,750 lb./in.^{2}, and 6,250 lb./in.^{2}
2. What is the maximum stress in the beam of Problem 1 when the total load is 6,200 lb.? *Ans.* 23,250 lb./in.^{2}
3. What is the change of average compressive stress in the upper fibers of gage length B when the load changes from 200 lb. to 6,200 lb.? In the lower fibers of B? In the upper fibers of A? In the lower fibers of A?
   Ans. 17,500 lb./in.^{2} and 12,500 lb./in.^{2}
4. In the beam of Problem 1, when the load changes from 200 lb. to 6,200 lb., the gage length B lengthens 0.00476 in. at the bottom and shortens the same amount at the top. Calculate E. *Ans.* $E = 29{,}410{,}000$ lb./in.^{2}
5. In the beam of Problem 1, the gage length A shortens 0.00342 in. at the top and lengthens 0.00340 in. at the bottom when the load changes from 200 lb. to 6,200 lb. Taking the average of these readings, calculate E.
   Ans. $E = 29{,}300{,}000$ lb./in.^{2}
6. Calculate θ from the readings of Problem 4 and solve for E by Eq. (7).
7. Calculate θ from the average of the readings of Problem 5 and solve for E.

8. Through what angle may a steel plate be bent in a length of 20 in. if the plate is 0.1 in. thick, E is 30,000,000, and the ultimate strength is 90,000 lb. per sq. in.?

$$90{,}000 = 30{,}000{,}000 \times 0.05 \frac{\Delta\theta}{\Delta l};$$

$$\frac{\Delta\theta}{\Delta l} = 0.6; \quad \Delta\theta = 1.2 \text{ radians} = 68^\circ\ 45'.$$

9. A steel hacksaw blade 0.023 in. thick was bent 45° in a length of 4 in. by a constant moment. Find the unit stress in the outer fibers.

Elongation measurements are made on a beam to locate the neutral surface. If l_e is the elongation of the lower gage length, and l_c is the compression of the upper gage length (Fig. 136), the neutral axis divides the vertical distance h between the gage lines in the ratio $l_e:l_c$.

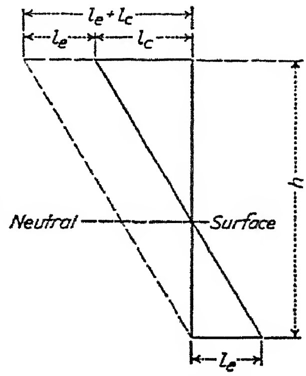


FIG. 136.—Location of neutral surface.

Problems

10. The elongation of an 8-in. gage length at the bottom of a 3-in. T-section is 0.00150 in. and the compression of an equal gage length directly above at the top of the stem is 0.00453 in. How far is the neutral surface from the bottom of the flange? *Ans.* $y = \frac{3 \times 150}{603} = 0.746$ in.

11. A reinforced-concrete beam, 12 in. high, has one 20-in. gage length 1 in. from the top and another similar gage length 1 in. from the bottom. The upper gage shortens 0.0096 in., while the lower gage lengthens 0.0134 in. How far is the neutral surface from the extreme compression fibers? *Ans.* 5.18 in.

12. A 4-in. by 5-in. by ½-in. T-beam (Carnegie T57) rested on supports 64 in. apart with flange down and carried a load at the middle. There were two gage lengths B and A on the east side of the middle and two equal gage lengths on the west side of the middle, arranged as in Problem 1. The deformation of each gage length was measured by two strain gages which were clamped lightly to the beam during each set of readings.

When the load changed from 200 lb. to 4,200 lb., the lower strain gage on B east lengthened 91 divisions and the upper strain gage shortened 201 divisions. How far was the neutral axis from the bottom of the flange? Compare with x of the handbook.

Ans. $\frac{91 \times 5}{292} = 1.558$ in.

13. In Problem 12, each unit of deformation was 0.00002 in. What was the unit deformation at the top? What was the average unit stress in the gage length at the top. Calculate E.

Ans. 0.0005025; $S_c = 15{,}484$ lb./in.^{2}; $E = 30{,}810{,}000$, using the two-figure Z from the handbook.

14. In Problem 12, find the unit stress and the unit deformation at the bottom. Calculate E. Compute Z from the I and x of the handbook.
Ans. 0.0002275; S_t = 6,936; E = 30,380,000.

15. In Problem 12, calculate $\Delta\theta$ and $\frac{\Delta\theta}{\Delta l}$, and find E by Eq. (7) of Art. 78.

$$Ans.\ \Delta\theta = \frac{292 \times 0.00002}{5} = 0.001168 \text{ radian}; \frac{\Delta\theta}{\Delta l} = 0.000146;$$

$$E = \frac{48{,}000}{10.8 \times 0.000146} = 30{,}440{,}000.$$

Other measurements of the beam of Problem 12 were:

	A east	A west	B west
Upper....................	63	60	94
Lower....................	140	133	204

16. Find the distance from the neutral axis to the flange for each gage length. *Ans.* A east, 1.552; A west, 1.554; B west 1.577.

17. Find the unit deformation for each gage length and solve for E as in Problems 13 and 14. Calculate the section modulus for the stem instead of using 3.1 of the handbook.

18. Calculate E for each gage length by means of Eq. (7) of Art. 78.
Ans. A east, 29,360,000; A west, 30,830,000; B west, 29,520,000.

79. Graphic Representation of Stress Distribution.—The unit stress in a beam, provided it does not exceed the elastic limit,

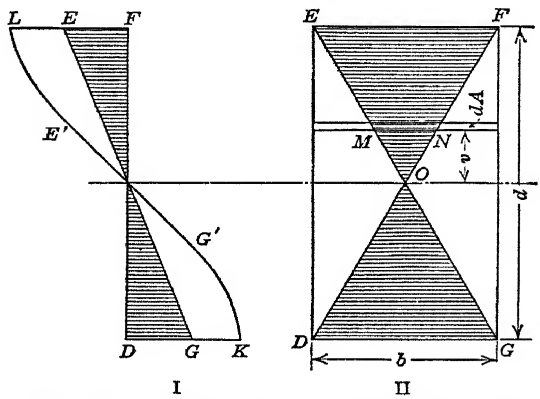


FIG. 137.—Stress distribution in a rectangular section.

varies as the distance from the neutral axis. Stress may be represented by the straight line $G E$ (Fig. 137, I), in which the magnitude is proportional to the horizontal distance from the

vertical straight line $D F$. This straight line is really a part of the stress-strain diagram for the material in both tension and compression, with the vertical line $D F$ as the X axis. If the unit stress is carried beyond the elastic limit, it may be represented by the line $K G' E' L$, which also is a stress-strain diagram with the vertical scale changed.

In a beam of rectangular section, the total stress on an area dA, extending across the section, is proportional to the unit stress.

The shaded area of Fig. 137, I, may represent the *total force* across a *rectangular section*, as well as the *unit stress* on a *section of any form*. It is often convenient to represent total force across a rectangular section by a figure similar to the shaded area

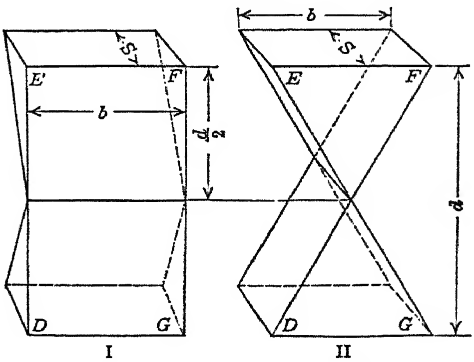


FIG. 138.—Stress-distribution solids for a rectangular section.

of Fig. 137, II. This is equivalent to Fig. 137, I, with the X axis oblique. The length of the line $E F$ represents the breadth of the section and also the total force in the extreme upper fibers. It is evident from the similar triangles that the total force on the area dA, extending across the section at a distance v above the neutral axis, will be to the total force at the top as the length of $M N$ is to the length of $E F$.

If the unit stress in the top fibers is unity, the area of the shaded triangle $O E F$ may represent the total force across the rectangular section above the neutral axis. In like manner the area of the shaded triangle $O D G$, may represent the total force across the rectangular section below the neutral axis. Since the stresses above and below the neutral axis have opposite signs, and the areas of the shaded triangles are equal, the total force across the section is zero.

The shaded triangles of Fig. 137, II, may be regarded as solids of uniform thickness. Figure 138, I, represents the distribution

of stress in the same rectangular section by the stress-distribution wedges of Art. 74. This figure differs from Fig. 132 in one respect. Both tension and compression are on the same side of the vertical plane of the rectangle $E F G D$ which represents the section of the beam.

Figures 138, I, and 138, II, show two forms of triangular wedges which may represent the unit stress and the total force in a beam of rectangular section. In Fig. 138, II, the *width* of the wedge varies as the distance from the neutral axis, while the *thickness* S is constant. In Fig. 138, I, the *width* is constant and equal to the width of the section while the thickness varies as the distance from the neutral axis. In either figure the volume of one wedge is $\frac{S\,b\,d}{4}$, and its center of gravity, which is the location of the resultant force across that half of the section, is $\frac{d}{3}$ from the neutral axis. The moment of each wedge about the neutral axis is $\frac{S\,b\,d^2}{12}$, and the moment of the two wedges representing tension and compression in the section is $\frac{S\,b\,d^2}{6}$. Figure 138, I, shows the actual distribution of stress in the section. It may be called the *stress-distribution solid.* The shaded area of Fig. 137, II, represents a portion of the area of the section on which, if a uniform stress should be applied equal to the stress in the outer fibers of the section, the total force and moment with respect to the neutral axis would be the same as that of the actual distribution. This is called the *stress-distribution diagram* or *modulus figure.*

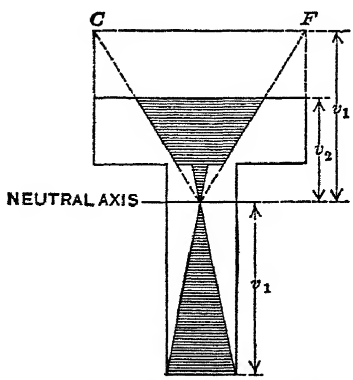


FIG. 139.—Distribution in a T-section.

Figure 139 is the stress-distribution diagram for a T-beam section, in terms of the unit stress at the bottom of the stem. The lower triangle is drawn as in Fig. 137, II. The unit stress at the top of the flange is to the unit stress at the bottom of the stem as v_2 is to v_1. For the graphical solution, a line is drawn above the flange at a distance v_1 from the neutral axis. The projection of the flange upon this line is $C F$. From the middle

of the stem, on the neutral axis, lines are drawn to C and F. The shaded trapezoid between these lines is the modulus figure for the flange. The small portion of the modulus figure for the stem above the neutral axis may be drawn by extending the lines which come up from below, or the width of the stem might have been projected upon the line $C\,F$ and ends of the projection connected with the middle of the stem at the neutral axis.

Figure 140 is the stress-distribution diagram for an I-beam section. The thickness of the web projected on the top of the flange is $J\,K$. From O, on the neutral axis at the middle of the stem, lines are drawn to J and K. The shaded area between these lines forms the modulus figure for the web. To find the portion of the modulus diagram for an element $S\,T$ in the lower flange, the line $S\,T$ is projected on to the bottom of the flange, which is the same distance from the neutral axis as the top of the upper flange. From the ends of this projection, lines are drawn to O. The part of $S\,T$ between these lines is the corresponding element of the modulus figure.

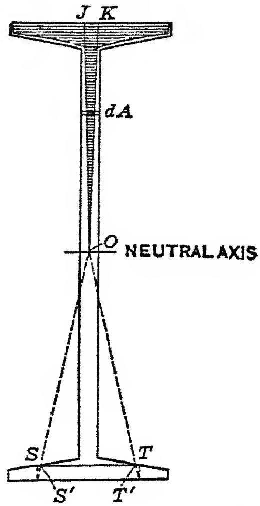


FIG. 140.—Stress distribution in an I-beam.

A number of such elements must be constructed in order to complete the stress-distribution diagram.

Problems

1. Construct the stress-distribution diagram for Problem 3 of Art. 74, similar to Fig. 139. Solve in terms of the unit stress at the bottom of the stem. Calculate the area of the lower triangle. Calculate the area of the triangle and trapezoid above the neutral axis. Compare.
2. Find the moment of the two triangles and the trapezoid of Problem 1 with respect to the neutral axis. (Divide the trapezoid into two triangles.) Add these moments; multiply by the unit stress at the bottom of the stem. Compare with Problem 8 of Art. 74.
3. Construct the modulus figure for Problem 1 in terms of the unit stress at the top of the flange.
4. Construct the stress-distribution diagram for an isosceles triangle of base 4 in. and height 6 in. in terms of the unit stress at the base. Construct elements $\frac{1}{2}$ in. apart.
5. Solve Problem 4 in terms of the stress at the vertex.

80. Stress beyond the Elastic Limit.—In the discussion of beams, it has been assumed that the unit stress is proportional to the unit deformation. This assumption is correct for allowable stresses, since these stresses are always considerably below the elastic limit. However, in order to limit more definitely the factor of safety, and to interpret correctly the results of tests in which beams are loaded to destruction, it is necessary to study the stresses and resisting moments in a beam when the stress in part of the beam exceeds the elastic limit.

In Fig. 141, the horizontal lengths represent the unit stresses in a beam, while the vertical lengths represent unit deformations and distances from the neutral axis. The curved line from G to H is really a stress-strain diagram with the X axis vertical and the Y axis horizontal. Since the total stress in a rectangle varies as the unit stress, the shaded area of Fig. 141 may also represent the stress distribution, or *modulus figure*, for a rectangular section. The triangles of Fig. 141 may represent the stress distribution in a beam of rectangular section, in which the stress in the outer fibers is below the elastic limit. The shaded areas may represent the distribution in another beam, which has the same cross section, the same deflection and unit deformation, the same modulus of elasticity for small stresses, but has a lower elastic limit. For about one-half the distance from the neutral axis to the outer fibers, the two diagrams coincide and the unit stress is the same for both beams. In the extreme outer fibers, the unit stress in the first beam is CF and in the second beam is CH. With the same deflection, the total stress and the total moment are greater in the first beam than in the second.

Figures 141 and 142 give comparisons of two beams which have equivalent sections and equal moduli of elasticity at small loads. For one beam, the stress in which is represented by a triangle, the elastic limit is above the unit stress in the outer fibers. In the other beam, represented by the shaded area, the stress in a considerable portion of the section is above the elastic limit. In Fig. 141, the *unit deformations* are the same for both beams, while the total force and the moment represented by the triangle are greater than those of the shaded area. In Fig. 142, the unit deformations are different, while the resisting moment of the curved area $O\ M\ K\ H\ C$ is equal to that of the triangular area $O\ F\ C$. From the center of the section to the point K, the curve lies outside the straight line. The unit stress in the fibers near

the neutral surface is greater than it would be if the modulus of elasticity were constant and the resisting moment were the same. The moment of the dotted area $O\,M\,K$ (or the shaded area $O\,N\,L$) is equal to the moment of $K\,F\,H$ or $L\,G\,E$.

Figure 142 might apply to two steel beams which have the same section, the same modulus of elasticity of 30,000,000 pounds per square inch, but different elastic limits and unit deformations. The unit deformation in one of these beams may be 0.003 in the outer fibers and the elastic limit be above 90,000 pounds per square inch. The length would then represent the unit stress

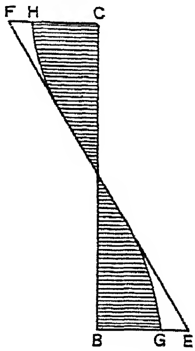


FIG. 141.—Stress-distribution diagram beyond the elastic limit.

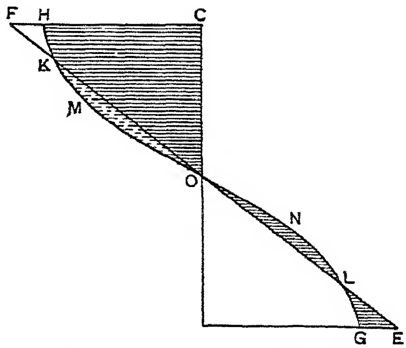


FIG. 142.—Actual and calculated unit stress.

in the outer fibers and the triangle $O\,F\,C$ would be the distribution diagram for a rectangular section. The curve $O\,M\,K\,H$ would represent the unit stress in the other beam for which the elastic limit is about 30,000 pounds per square inch. The unit deformation in the outer fibers of this beam would be over 0.004. At about one-fourth the distance from the neutral axis to the outer fibers, the stress would be proportional to the unit deformation and would be one-third greater than that of the first beam. At the extreme outer fibers, the stress in this beam would be to the stress in the first beam as the length CH is to the length CF.

81. Modulus of Rupture.—When a beam is broken by bending, the stress-distribution diagram $O\,M\,K\,H$ of Fig. 142 is similar to the complete tension or compression stress-strain diagram of the material. The actual unit stress in the outer fibers is less than that obtained by the equation

$$S = \frac{M}{Z}. \qquad \text{Formula XVI}$$

in the ratio of CH to CF (Fig. 142). The *calculated* value of the stress in the outer fibers computed from Formula XVI is called the *modulus* of *rupture*, or the transverse ultimate strength of the material. It is called also the *extreme fiber stress* in bending.

Another factor which makes the calculated modulus of rupture different from the actual unit stress is the shifting of the neutral axis. In sections which are not symmetrical with respect to the axis, the remote fibers on one side reach the elastic limit before those on the other. Figure 143 represents a T-section. Figure 143, II, shows part of the stress-strain diagram for both tension

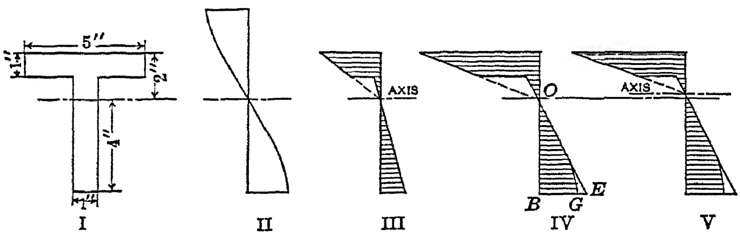


FIG. 143.—Displacement of the neutral axis.

and compression. Figure 143, III, shows the distribution of stress for small deformation which produces no stress beyond the proportional elastic limit. The neutral axis passes through the center of gravity of the section. Figure 143, IV, shows the distribution when the deformation is doubled, on the assumption that the neutral axis is not shifted. The lower half of the stem has passed the elastic limit and the unit stress in it is *not* proportional to the distance from the neutral axis. The shaded area below the axis is smaller than that above (which is equal to the triangle $O\ B\ E$) and consequently the neutral axis cannot pass through the point O but must be moved upward away from the center of gravity of the section. Figure 143, V, is the actual diagram with the axis shifted. The area above the axis is diminished and that below increased. With a still greater deformation the upper fibers will also pass the elastic limit, and it may happen that, with some forms of stress-strain diagrams, the neutral axis may move backward toward the center of gravity of the section.

The neutral axis may be shifted in a *symmetrical section*, if the tension and compression curves are not alike. In cast iron, for instance, the stress-strain diagram for tension differs greatly

from the diagram for compression. The compressive strength of cast iron is three or four times as great as the tensile strength. Beams of this material should be made of T-section, or equivalent, and so loaded as to bring the stem in compression and the flange in tension. The remote fibers on the compression side should be *two* or *three* times as far from the center of gravity of the section as those on the tension side.

While the modulus of rupture does not give the actual unit stress in the outer fibers, it makes it possible to compare stresses in similar *sections*. If the modulus of rupture of a given material is obtained from tests of beams of *rectangular* section, this modulus may be used in computing the ultimate strength of beams of this material of *any rectangular* section. The results may also be used with little error for beams of other shapes, provided they are symmetrical with respect to the neutral axis. With unsymmetrical sections, such as angles, it is better to make tests and obtain the modulus of rupture for each shape.

The student will remember, however, that these statements apply to the stress beyond the elastic limit. Since allowable stresses are below the elastic limit, Formula XVI is strictly correct for allowable loads. The change in the stress-distribution diagram when the stress passes the elastic limit *affects* the *factor* of *safety only*.

Strictly speaking, ductile materials, such as soft steel, have no modulus of rupture, since beams of such material may be bent double without breaking.

Problems

1. A rectangular beam of western spruce, tested at the Bureau of Standards, was 1.75 in. wide and 1.78 in. high. The beam was on supports 24 in. apart and loaded at the middle. When the total load changed from 31 lb. to 597 lb., the deflection at the middle increased 0.1326 in. The beam failed at a load of 973 lb. Find the unit stress at 597 lb. load, and find the modulus of rupture. *Ans.* 3,879 lb./in.^{2} and 6,304 lb./in.^{2}
2. A rectangular bar of cast iron, 1.04 in. wide and 0.80 in. thick, was placed on supports 12 in. apart and was broken by a load of 1,635 lb. midway between the supports. Find the modulus of rupture.
3. A cast-iron beam tested in bending had the same section as the T-beam, T87, formerly rolled by the Carnegie Company. Width of flange, 2 in.; stem depth (total), $1\frac{1}{2}$ in.; thickness of flange and stem, $\frac{1}{4}$ in. to $\frac{5}{16}$ in.; moment of inertia, axis 1–1, 0.16 in.^{4}; distance axis 1–1 to back of flange, 0.42 in.; area, 0.91 sq. in. When this beam was placed on supports 20 in. apart, with flange down, and loaded with 2,400 lb. at

the middle, it did not fail. What was the calculated stress at the bottom of the flange and at the top of the stem at this load?

Ans. $S_t = 31{,}500$; $S_c = 81{,}000$ lb./in.^{2}

4. When the T-beam of Problem 3 was turned over to bring the stem to the bottom, it failed under a load of 1,510 lb. Find the modulus of rupture. Find the unit compressive stress in the flange.
Ans. Tension modulus of rupture = 50,960 lb./in.^{2}; S_c = 19,820 lb./in.^{2}
The tension crack in the stem of this beam was perpendicular to the length. In similar tests the loading was stopped before the crack reached the flange. The beam was then turned flange down and loaded. It was found still to carry a load of 2,400 lb.
5. One-half of the beam of Problem 4 was placed flange down across a 10-in. span. The load at failure was 7,610 lb. Find the modulus of rupture. Find the unit tensile stress in the flange.
Ans. Compression modulus of rupture = 128,400 lb./in.^{2}; S_t = 49,940 lb./in.^{2}
Caution: In these tests of brittle cast iron the test piece must be shielded to avoid injury from flying pieces.
6. The other half of the test piece of Problem 4 was tested flange up. The span was 10 in. and the load at failure was 3,170 lb. Calculate stresses and compare with Problem 4.

The next problems have been taken from *Bulletin* No. 41 of the University of Illinois Engineering Experiment Station. This bulletin, by Prof. A. N. Talbot, describes tests of timber stringers, such as would meet the ordinary requirements for railroad use. These large beams were supported near the ends and symmetrically loaded at the "third points." Neglecting the weight of the beam, the moment of the concentrated loading is constant over the middle third of the length. A small defect anywhere in this middle third will cause failure, while the same defect might not cause failure if the load were uniformly distributed or concentrated at the middle, because the weak section might be so far from the middle as to be under much less than the maximum stress.

All of these beams were 14 ft. long. They were supported 13 ft. 6 in. apart, and the two loads were 4 ft. 6 in. from the supports.

7. Loblolly pine beam C-14, 6.87 in. by 16 in., had four knots under south load. It failed by tension when the total load was 48,860 lb. With the weight of the beam neglected, the modulus of rupture is given in the bulletin as 4,500 lb. per sq. in. Verify this calculation.
8. Shortleaf yellow-pine beam C-10, 7.12 in. by 16.25 in., was cross grained and had been creosoted. It failed in tension under a total load of 55,000 lb. Calculate the modulus of rupture. *Ans.* 4,740 lb./in.^{2}
9. If the beam of Problem 8 had been turned over to make the 16.25-in. faces horizontal, about what uniformly distributed load would it have carried?
10. Creosoted shortleaf yellow-pine beam C-11, 7 in. by 15.87 in., had one knot under south load point. It failed by tension under a total load of 37,000 lb. Calculate the modulus of rupture.

11. The unit stress at the elastic limit for beam C-11 is given as 2,530 lb. per sq. in. What was each of the symmetrical loads at the elastic limit? *Ans.* 13,770 lb.
12. Longleaf yellow-pine beam A-16, 6.60 in. by 11.75 in., was a clear stick. It failed under a total load of 39,400 lb. Calculate the modulus of rupture. *Ans.* 7,005 lb./in.^{2}
13. If the beam of Problem 12 were supported at points 12 ft. apart and had 8,000 lb. at 4 ft. from the left support and a uniformly distributed load, what would be the maximum value of this distributed load with a factor of safety of 4?
14. Longleaf yellow-pine beam A-10, 7 in. by 14 in., was cross grained and knotty. It failed under a total load of 25,000 lb. Find the modulus of rupture. *Ans.* 2,952 lb./in.^{2}
15. Douglas-fir beam F-8, very knotty, cross grained, 7.7 in. by 15.60 in., failed under a total load of 35,000 lb. Calculate the modulus of rupture. *Ans.* 3,026 lb./in.^{2}
16. Douglas-fir beam F-16, 8.13 in. by 15.88 in., had numerous small knots. It failed under a total load of 61,350 lb. Find the modulus of rupture.
17. A stoneware rod, average diameter 0.949 in., was supported at points 4.2 in. apart and carried two equal loads, each 4.2 in. outside the adjacent support. It failed under a total load of 102 lb. Find the modulus of rupture. *Ans.* 2,553 lb./in.^{2}

82. Allowable Bending Stress.—Table XX gives some allowable bending stresses which are *to be memorized.* Some of these are from official specifications. Since the allowable stresses for timber vary greatly with the quality, one stress, which applies to good (but not select) material is given as an approximate value for the structural timbers in most common use. For other timber the student will use the handbook. The bending stress in compression in the extreme fibers of reinforced concrete applies to material which has an ultimate strength of 2,000 pounds per square inch at 28 days.

TABLE XX.—BENDING STRESSES IN EXTREME FIBERS

Material	Pounds per Square Inch
Rolled structural steel (A.R.E.A.)	16,000
Rolled structural steel (A.I.S.C.)	18,000
Structural-steel pins (A.R.E.A.)	24,000
Cast steel	16,000
Wrought iron	12,000
Cast iron, tension in extreme fiber	3,000
Cast iron, compression in extreme fiber	10,000
Douglas fir, red and white oak, redwood, southern yellow pine	1,200
Concrete, compressive stress in outer fibers caused by bending	700

Problems

(For material not given in Table XX, and for material of designated quality, use the handbook.)

1. Find the allowable load, uniformly distributed, on a 12-in. 50-lb. standard I-beam, which is 15 ft. long, center to center of supports (A.R.E.A. stress).
2. Southern yellow-pine joists, 2 in. thick and h in. high, are used to carry a load of 120 lb. per ft. Find h for a span of 12 ft.
3. Solve Problem 2 if the pine is dense, select.
4. Solve Problem 2 for common Norway pine.
5. A trapezoidal cast-iron beam is 3 in. wide at the bottom, 1 in. wide at the top, and 4 in. high. It spans 5 ft. and carries a load 2 ft. from one end. Find the total safe load.
6. A Carnegie T-section, 5 in. by 3 in. by $\frac{3}{8}$ in. is supported at points 5 ft. apart and carries a load at the middle. Find the maximum safe load.
7. Solve Problem 6 if the beam were made of cast iron and used stem down; also when used flange down.
8. A box beam is made of 2-in. select Douglas fir. It is 12 in. high and 10 in. wide outside. What is the maximum safe span for a load of 360 lb. per ft.?

83. Neutral Axis for an Unsymmetrical Section.—Figure 144 shows an angle section which is unsymmetrical with respect to all vertical axes. Figure 144, I, shows the usual construction of the stress-distribution diagram on the assumption that the beam bends about the horizontal axis through the center of gravity. Figure 144, II, differs from Fig. 144, I, in that two centers, O and O', on the neutral axis are used in the construction. One center O lies on the vertical line through the center of the vertical leg and the other center O' lies in the vertical line through the center of gravity of the portion of the horizontal leg which lies to the right of the vertical leg. Each method gives the total stress in terms of the unit stress in the fibers at the bottom of the beam, and each shows the true distance of the resultant forces from the horizontal axis. The true position C of the resultant force in the upper portion is at the center of gravity of the triangular and the trapezoidal areas of Fig. 144, II. It is at the center of gravity of the upper wedge of the stress-distribution solid of Fig. 144, III.

The forces H and C perpendicular to the plane of the paper form a couple, which is the resisting moment. The plane of this couple is not vertical. To bend the beam in a vertical plane the external moment must lie in the same plane as the resisting moment. A vertical load will not bend a beam of this form in a vertical direction. If there were two equal angles

fastened together with their vertical legs back to back, then the combination would be symmetrical with respect to a vertical axis, and the neutral axis for vertical loads would be horizontal.

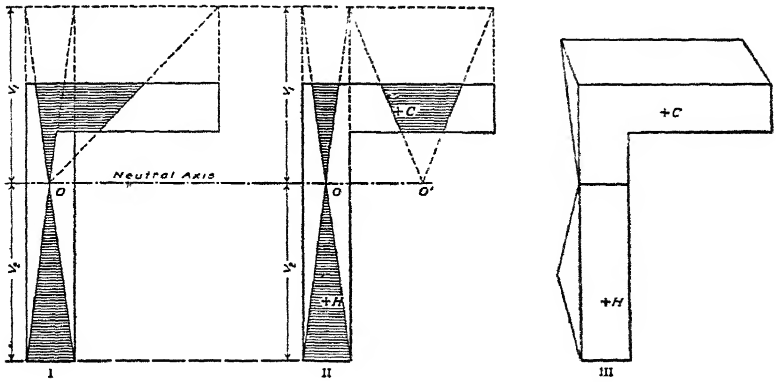


FIG. 144.

In Fig. 145, $A\ B\ C\ D$ is a rectangular section with diagonal horizontal. It may be regarded as the end of a cantilever perpendicular to the plane of the paper. If a vertical load P is placed on the end of this cantilever, the deflection will not be vertically downward, but the section will be displaced into a position such as shown in the figure.

84. Bending Moment about a Secondary Axis.—Figures 144 and 145 are special cases of the general problem in which the bending moment does not lie in the plane of one of the principal moments of inertia. In other words, the bending moment is not about an axis for which the moment of inertia is a maximum or a minimum. If an axis perpendicular to the plane of the bending moment is a principal axis of the cross section, then the beam will bend about this axis, and the deflection will be in the direction of the applied forces. For instance, if an I-beam is placed with the web vertical, or a rectangular beam is placed with the long sides of the rectangle vertical, the axis of maximum moment of inertia is horizontal, and a vertical load will deflect the beam vertically downward. If the I-beam or rectangular beam were turned 90 degrees, to make the axis of minimum moment hori-

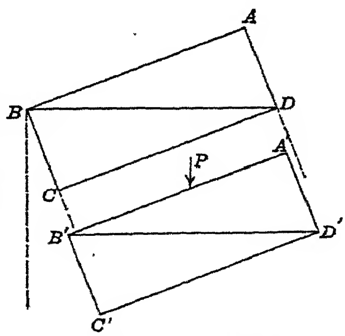


FIG. 145.—Rectangular beam with load perpendicular to diagonal.

zontal, the beam again would deflect vertically downward under a vertical load, and the neutral axis would coincide with the principal axis of inertia. When the section of the beam is an equilateral triangle, a square, or any other regular polygon, the moment of inertia is the same for every axis through the center of gravity. Such beams deflect in the plane of the bending moment, no matter what the position of the section may be.

The method of finding the unit stress when the bending moment is not in the plane of a principal axis of inertia is very simple. *Resolve the bending moments or the applied forces into two components perpendicular to the two principal axes of inertia, and compute the stress separately for each component. The actual stress at any point is the algebraic sum of the stresses caused by the two components.*

The proof of this proposition is given in Art. 232.

Example

A cantilever beam of rectangular section is 4 in. by 3 in. and is 5 ft. long. The beam is placed with the 4-in. faces at 30° with the horizontal and a load of 120 lb. is put on the free end. Find the unit stress at each corner, and find the direction of the neutral axis.

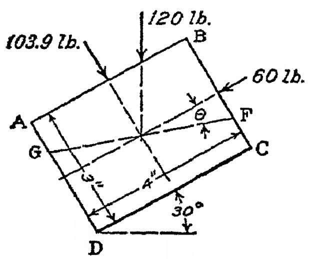


FIG. 146.

The load of 120 lb. is resolved into 103.9 lb. perpendicular to the 4-in. faces and 60 lb. perpendicular to the 3-in. faces. From the first component, the stress in $A\,B$ and in $C\,D$ of Fig. 146 is

$$S = \frac{103.9 \times 60}{6} = 1{,}039 \text{ lb. per sq. in.}$$

From the second component,

$$S = \frac{60 \times 60}{8} = 450 \text{ lb. per sq. in.}$$

At B both stresses are tensile and at D both are compressive. The unit stress at these corners is $1{,}039 + 450 = 1{,}489$ lb. per sq. in. At A the stress caused by the 60-lb. component is compressive, while that caused by the other component is tensile. The tensile stress at A and the compressive stress at C are $1{,}039 - 450 = 589$ lb. per sq. in.

The location on the line $C\,B$ of the point F at which the stress is zero is found by dividing the distance from C to B in the ratio of 589 to 1,489.

$$CF = \frac{589 \times 3}{589 + 1{,}489} = \frac{1{,}767}{2{,}078} = 0.850 \text{ in.}$$

At G on the line AD at a distance of 0.850 in. from A, the unit stress is zero. The line $G\,F$ through the center of the section is the neutral axis.

The angle θ between the neutral axis $G F$ and a line parallel to the 4-in. faces is given by

$$\tan \theta = \frac{0.65}{2} = 0.325; \; \theta = 18°, \text{ nearly.}$$

The neutral axis makes an angle of 12° with the horizontal.

Problems

1. A rectangular cantilever, 6 in. by 10 in., is 6 ft. long. The 10-in. faces make an angle of 20° with the vertical. The beam carries a load of 1,200 lb. on the free end. Find the unit stress at each corner and the angle which the true neutral axis makes with the horizontal.

   Ans. 1,304 lb./in.^{2}; 320 lb./in.^{2}, 25° 16′.
2. A rectangle of sides b and d is placed with one diagonal horizontal and subjected to a vertical load. Find the fiber stress at the corners A and C (Fig. 147).

   $$Ans.\ S = \frac{6M\sqrt{d^2 + b^2}}{b^2d^2}.$$

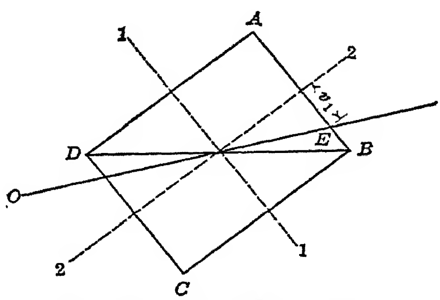


FIG. 147.—Rectangular beam.

3. Show that the result of Problem 2 is the same as would be found if the neutral axis coincided with the horizontal diagonal $D B$.
4. A 10-in., 40-lb. standard I-beam, 10 ft. long between supports, carries a uniformly distributed load of 900 lb. per ft. The web makes an angle of 7° to the right of the vertical upward. Find the unit stress at each corner at the middle of the span.

   Ans. Upper left, 8,687 lb./in.^{2} compression; upper right 207 lb./in.^{2} tension.
5. At what angle with the vertical will a 12-in. 31.8-lb. I-beam have zero stress at two corners? *Ans.* Arc tan 0.10556 = 6° 02′.
6. An 18-in. × 7½-in. 47-lb., wide-flange section is 20 ft. long between supports and carries 12,000 lb. 8 ft. from one support. The web makes an angle of 6° with the vertical. Find the unit stress at the upper corners caused by this load.

   Ans. 16,380 lb./in.^{2} compression; 326 lb./in.^{2} compression.
7. A 10-in. 15.3-lb. standard channel, 12 ft. long between supports, carries a load of 400 lb. per ft. The web makes an angle of 5° to the right of

the vertical upward. The flanges are on the right. Find the unit stress at each corner at the dangerous section.

Ans. Upper left, 8,518 compression; upper right, 147 compression; lower left, 4,328 tension; lower right, 12,699 tension.

8. A 6-in. by 6-in. timber cantilever, 5 ft. long, carries a load of 180 lb. per ft. Find the maximum unit stress when the faces are vertical. Find the maximum unit stress when one diagonal is vertical.

Ans. 750 lb./in.^{2}; 1,061 lb./in.^{2}

9. Solve Problem 8 if two faces make an angle of 30° with the horizontal. Solve by resolving the load into components. Solve also by calculating the distance c for this position. *Ans.* 1,024 lb./in.^{2}

10. In Problem 9 find the unit stress at the other corners.

11. A cantilever beam 4 ft. long carries a load of 200 lb. on the free end. Each section is an equilateral triangle 4 in. on each side. Find the stress at each vertex if the lower base is horizontal. *Ans.* 4,800 lb./in.^{2}; 2,400 lb./in.^{2}

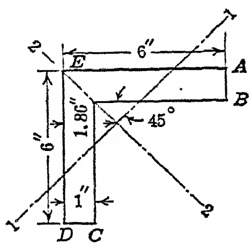


FIG. 148.—Angle section.

12. Solve Problem 11 if the lower base makes an angle of 20° with the horizontal.

13. A 6-in. by 6-in. by 1-in. standard angle, 10 ft. long, is used as a beam supported at the ends. The angle is placed with legs horizontal and vertical and a load of 1,000 lb. is applied at the middle, over the center of gravity of the section. Find the unit stress at the corners. Here the principal axes are 1–1 for which the moment inertia is 14.78, and 2–2 for which the moment of inertia is 56.14. The bending moment for each axis is $15{,}000\sqrt{2}$.

$$\text{Unit stress at } E = \frac{15{,}000 \times \sqrt{2} \times 1.86 \times \sqrt{2}}{14.78} = 3{,}775 \text{ lb. per sq. in.}$$

Ans. Unit tensile stress at $C = 3{,}329 + 1{,}336 = 4{,}665$ lb./in.^{2}

85. Bending Moment in Different Planes.—It frequently happens that a beam is subjected to forces not all of which are parallel. If the sections are circles or regular polygons so that the moment of inertia is the same in all directions, the resultant moment may be calculated at any section, and this moment may be used to find the fiber stress. If the two principal moments of inertia are not equal, the forces or moments should be resolved in the directions of the principal axes, and the stress at any point calculated as in Art. 84.

Example

A horizontal cantilever 5 ft. long carries a load of 120 lb. per ft. and is subjected to a horizontal pull, perpendicular to its length, of 400 lb. at the free end. Find the expressions, in inch-pounds, for the moment at any section. *Ans.* $M_y = 5x^2$; $M_z = 400x$; resultant $M = 5x\sqrt{x^2 + 6{,}400}$,

in which M_y is the moment in the vertical plane and M_z is the moment in the horizontal plane.

Problems

1. In the foregoing example, find the direction and magnitude of the resultant moment at the fixed end.
   Ans. 30,000 in.-lb. in a plane at an angle of 36° 52′ with the horizontal.
2. If the cantilever of Problem 1 is of circular section, 4 in. in diameter, find the maximum fiber stress. *Ans.* 4,775 lb./in.^{2}
3. In Problem 1 find the magnitude and direction of the resultant moment at 30 in. from the free end.
   Ans. 12,816 in.-lb. at angle of 20° 34′ with the horizontal.
4. A horizontal shaft 10 ft. long, weighing 24 lb. per ft., is supported at the ends and carries a vertical load of 60 lb. 2 ft. from the left end and a vertical load of 40 lb. 3 ft. from the right end. A horizontal force of 160 lb., perpendicular to the shaft, is applied 3 ft. from the left end. Find the resultant moment at 3 ft. from the left end and at the middle.
   Ans. 501 ft.-lb. at 3 ft.; 484 ft.-lb. at 5 ft.
5. Write an expression for the moment in each plane and for the resultant moment between 3 ft. and 7 ft. from the left support. Differentiate the expression for the resultant moment to derive an equation for finding the position of maximum moment.

Figure 149 shows the diagram for the moment in the vertical and horizontal planes for Problem 4. The maximum moment in the vertical plane is at 5 feet, and the maximum in the horizontal

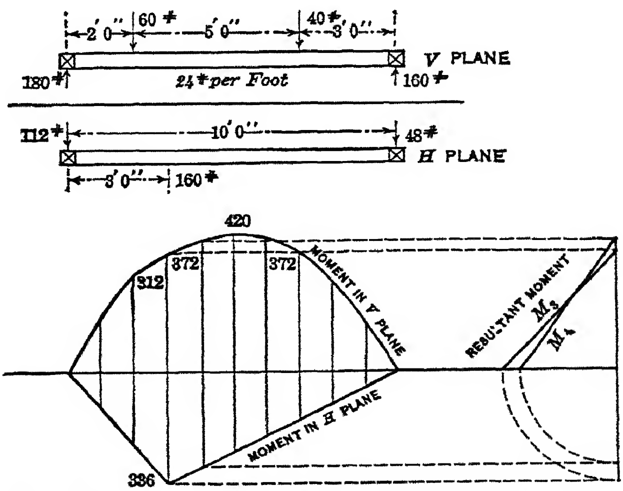


FIG. 149.—Beam with horizontal and vertical loading.

plane is at 3 feet. The maximum resultant moment is between 3 feet and 5 feet. The resultant moment at any section may

be determined graphically from the diagonal of the right-angled triangle, the legs of which are the horizontal and vertical moments.

Problems

6. A 6-in. by 10-in. beam, 15 ft. long between vertical and horizontal supports, carries a load of 240 lb. per ft. and resists a transverse force of 720 lb. 5 ft. from one end. Find the maximum stress for each 10-in. interval between the dangerous section for the vertical load and the dangerous section for the horizontal force.

Ans. 1,200 lb./in.^{2}, 1,210 lb./in.^{2}, 1,200 lb./in.^{2}, 1,170 lb./in.^{2}

7. A 20-in. 75-lb. standard I-beam, 24 ft. long, is supported vertically 3 ft. from each end and carries 2,400 lb. per ft. between the supports. It is held horizontally 1 ft. from each end. A force of 3,000 lb. at an angle of 60° below the horizontal, in a transverse plane perpendicular to the length of the beam, is applied 9 ft. from one end. Find the maximum stress at 10-in. intervals in the region of maximum moment.

Ans. Maximum stress = 18,950 lb./in.^{2}

CHAPTER VIII

DEFLECTION OF BEAMS

86. Relation of Moment to Curvature.—In Equation (7) of Art. 78, it was shown that

$$M = E\,I\frac{\Delta\theta}{\Delta l} = E\,I\frac{d\theta}{dl}, \tag{1}$$

for infinitesimal lengths measured along the neutral surface of the bent beam. The angle $d\theta$ is the change in slope of the tangent to the neutral surface in the length dl. In Figs. 150

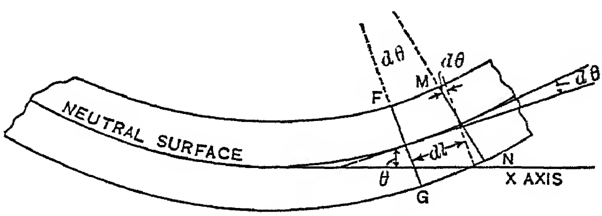


Fig. 150.—Curvature of beam.

and 135, two lines $F\,G$ and $M\,N$ are drawn perpendicular to the neutral surface at a distance dl apart. The broken line which intersects $M\,N$ at the neutral surface is parallel to $F\,G$. The lines $F\,G$ and $M\,N$ make an angle $d\theta$ with each other, since they are normal to the neutral surface and intersect at some point beyond the drawing, at a distance ρ from the neutral surface. This distance ρ is the radius of curvature of the neutral surface.

By geometry:

$$\rho\, d\theta = dl, \tag{2}$$

$$\frac{d\theta}{dl} = \frac{1}{\rho}, \tag{3}$$

Substituting in Equation (1):

$$\frac{M}{E\,I} = \frac{1}{\rho}, \quad M = \frac{E\,I}{\rho}. \tag{4}$$

If M is constant, or if I varies as M, ρ is constant, and the curve of the beam is an arc of a circle which may be computed by trigonometry.

Problems

1. A 3-in. by 1-in. steel beam, 10 ft. long, rests on two supports, each 30 in. from an end, and carries a load of 200 lb. on each end (Fig. 151). If the weight of the beam is neglected, what is the bending moment for the portion between the supports? If the modulus of elasticity is 30,000,000 lb. per sq. in., what is the radius of curvature? How much is the middle of the beam deflected upward above the supports? Solve for the deflection by geometry, assuming that the chord is equal to the

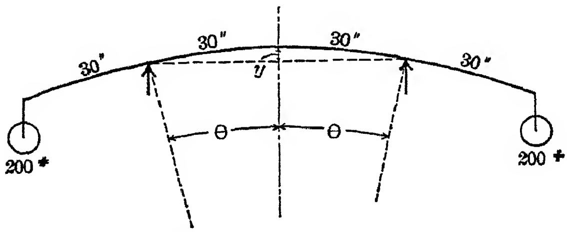


FIG. 151.—Curvature constant.

arc. Solve also by trigonometry. If tables do not give the cosine with sufficient accuracy, calculate it by the series.
Ans. M = 6,000 in.-lb.; ρ = 1,250 in.; y = 0.36 in.

2. A steel plate, 1 in. wide and 0.1 in. thick, is bent through an angle of 45° in a length of 30 in. by a constant moment. If E is 30,000,000, what are the moment, the radius of curvature, and the deflection at the middle of the 30 in.? *Ans.* M = 65.45 in.-lb.; ρ = 38.2 in.; y = 2.90 in.

3. A 4-in. by 6-in. timber beam, 18 ft. long, rests on supports 10 ft. apart and overhangs each support 4 ft. Calculate the radius of curvature for the portion between the supports when a load of 500 lb. is placed on each end. The modulus of elasticity is 1,600,000 lb. per sq. in.
Ans. 400 ft.

4. In Problem 3, what is the difference in the slope of the beam at the supports? What is the deflection of the middle upward?
Ans. θ = 0.025 radian = 1° 26′ in a length of 10 ft. y = 0.384 in., using five place cosines = 0.375 in., using two terms of cosine series.

87. Change of Slope in Rectangular Coördinates.—When Equation (1) of Art. 86 is integrated with θ and l as the variables, the integral gives the change of slope in radians. For most purposes, it is desirable to express the slope in rectangular coördinates. The curved line of Fig. 152 represents the neutral surface of a bent beam with the deflection greatly exaggerated. In a floor beam for instance, the deflection should not be more than 1 inch in 30 feet. A deflection as great as would be per-

missible in an engineering structure would not be noticeable in a drawing. For this reason, in the discussion of deflection which follows, a beam has been drawn in each figure as it would appear to the eye, and then a heavy line has been drawn below to represent the position and slope of the neutral surface with all vertical distances magnified.

In Fig. 152, dl is a small length measured along the neutral surface, and dx is the projection of this length on the horizontal.

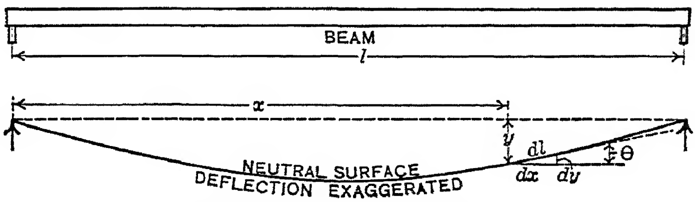


FIG. 152.—Slope and deflection.

$dx = dl \cos \theta$, in which θ is the slope of the tangent to the beam. With the small deflections allowable in an engineering structure, $\cos \theta$ is nearly unity and dx is practically equal to dl. For instance, if the slope were 1 part in 100 (which is relatively large) a triangle could be drawn with a base of 1 unit and an altitude of 0.01 unit. The hypotenuse of this triangle is $\sqrt{1.0001}$, which is 1.00005, and the error in the assumption that dx equals dl is 1 part in 20,000. When dx is substituted for dl, Equation (1) of Art. 86 becomes

$$M = EI\frac{d\theta}{dx}. \tag{1}$$

Since θ is a small angle, its value in radians is practically equal to its tangent:

$$\theta = \tan \theta = \frac{dy}{dx}; \tag{2}$$

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2}; \tag{3}$$

$$M = EI\frac{d^2y}{dx^2}. \qquad \text{Formula XVII}$$

This formula may be derived in a slightly different manner, which will show the magnitude of the approximations. From

the calculus, the reciprocal of the radius of curvature is

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}. \tag{4}$$

When this expression is substituted in Equation (4) of Art. 86, the result is

$$M = \frac{E\,I\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}. \tag{5}$$

When $\frac{dy}{dx}$ is small, $\left(\frac{dy}{dx}\right)^2$ is much smaller and may be neglected. The denominator of the second member of Equation (5) then becomes unity, and Equation (5) is equivalent to Formula XVII. Mathematically, Equation (5) is reduced to Formula XVII by multiplying by $\sec^3\theta$ or dividing by the $\cos^3\theta$, when $\tan\theta = \frac{dy}{dx}$.

In some problems in which the deflections are large, the exact formula of Equation (5) is required. The application of this equation was considered in Appendix D of the Third Edition. For all ordinary problems of beam and column deflection, Formula XVII is ample. In the application of the formula the X axis is taken parallel to the unbent beam, and the deflection is so small that $\left(\frac{dy}{dx}\right)^2$ is negligible compared with unity.

88. Solution of the Differential Equation of Deflection.—Before solving Formula XVII for the deflection of a beam or column, all the factors must be expressed in terms of x, y, and *constants*. The modulus of elasticity is constant, provided the unit stress does not exceed the proportional elastic limit. The formulas for deflection are valid only below this limit. For beams of uniform section, I is constant; for beams of variable section, it is expressed as a function of x. The moment is expressed as a function of x and y. In beams it is usually a function of x only, as in equations of Art. 69.

When the expressions for M and I do not depend upon the deflection y, Formula XVII becomes

$$\frac{d^2y}{dx^2} = \text{function of } x. \tag{1}$$

The first integration of Equation (1) gives $\frac{dy}{dx}$ as a function of x with the addition of an integration constant. If $\frac{dy}{dx}$ is known for any one value of x, these values may be substituted in the integral and the integration constant determined.

The second integration gives the deflection y as a function of x with the addition of a second integration constant. If y is known for some one value of x, these values may be substituted in the second integral and the second integration constant determined. The final integral with the integration constants given in terms of the loads and dimensions of the beam is called the *equation of the elastic line.*

If $\frac{dy}{dx}$ is not known for any one value of x, the first integration constant must be carried through the second integration, and the value of y must be known for two values of x to complete the solution.

89. Cantilever Loaded at Free End.—Figure 153 represents a cantilever beam fixed at the right end and loaded at the left end.

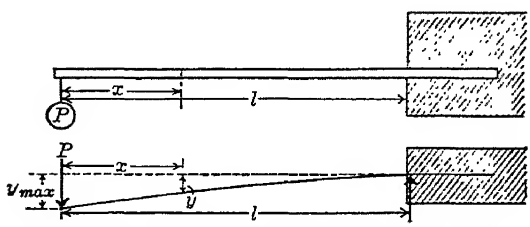


FIG. 153.—Cantilever with load on free end.

The beam was horizontal before the load was applied and remains horizontal at the wall when loaded. The origin of coördinates is taken at the position of the left end before the load was applied. The moment at *any* distance x from the origin is $-P\,x$. The differential equation is

$$E\,I\frac{d^2y}{dx^2} = -P\,x. \tag{1}$$

$$E\,I\frac{dy}{dx} = -\frac{P\,x^2}{2} + C_1. \tag{2}$$

At the wall, where $x = l$, the beam is horizontal and $\dfrac{dy}{dx} = 0$;

$$C_1 = \frac{P\,l^2}{2}. \tag{3}$$

$$E\,I\frac{dy}{dx} = -\frac{P\,x^2}{2} + \frac{P\,l^2}{2}. \tag{4}$$

$\dfrac{P\,l^2}{2\,E\,I}$ is the slope of the beam at the left end, where $x = 0$.

$$E\,Iy = -\frac{P\,x^3}{6} + \frac{P\,l^2x}{2} + C_2. \tag{5}$$

At the wall $x = l$, $y = 0$;

$$0 = -\frac{P\,l^3}{6} + \frac{P\,l^3}{2} + C_2;$$

$$C_2 = -\frac{P\,l^3}{3}. \tag{6}$$

$\dfrac{C_2}{E\,I}$ is the deflection at the left end, where $x = 0$.

$$E\,I\,y = -\frac{P\,x^3}{6} + \frac{P\,l^2x}{2} - \frac{P\,l^3}{3}; \tag{7}$$

$$y = -\frac{P}{6\,E\,I}(2\,l^3 - 3\,l^2x + x^3). \tag{8}$$

The maximum deflection is at the free end, where $x = 0$;

$$y_{\max} = -\frac{P\,l^3}{3\,E\,I}. \qquad \text{Formula XVIII}$$

If $x = k\,l$, in which k is a fraction less than unity,

$$y = -\frac{P\,l^3}{6\,E\,I}(2 - 3\,k + k^3). \tag{9}$$

Problems

(*Memorize no equations except Formula* XVIII.)

1. A 4-in. by 6-in. cantilever, 10 ft. long, carries a load of 240 lb. on the free end. Find the deflection at the free end if $E = 1{,}500{,}000$ lb. per sq. in. Find the maximum unit stress.

Ans. $y_{\max} = 1.28$ in.; $S = 1{,}200$ lb./in.^{2}

2. What would be the deflection at the end and the maximum stress for the beam of Problem 1 if it were turned 90° to bring the 4-in. faces vertical?

3. A 10-in. 30-lb. standard I-beam, as a cantilever 8 ft. 6 in. long, is deflected 0.357 in. at the free end by a load of 4,000 lb. Find E and the maximum stress. *Ans.* $E = 29{,}170{,}000$; $S = 15{,}280$ lb./in.^{2}

4. A 2-in. by 3-in. timber cantilever, 6 ft. 8 in. long, is deflected 1.24 in. at the end by a load of 40 lb. on the end. Find E and the maximum stress.
Ans. $E = 1{,}220{,}000$; $S = 1{,}067$ lb./in.^{2}

5. Find the deflection of the cantilever of Problem 1 at 20 in. from the free end and at 40 in. from the free end. Solve by Eq. (8). Solve again by Eq. (9) and the answer of Problem 1.

6. Find the deflection of the beam of Problem 3 at 30 in. from the free end.
Ans. $y = 0.204$ in.

7. Derive the expression for the deflection of a cantilever of length l at one-third the length and one-half the length from the free end.
Ans. $\frac{14\,P\,l^3}{81\,E\,I}$; $\frac{5\,P\,l^3}{48\,E\,I}$

8. For Problem 1, find the slope at the free end, at one-fourth the length from the free end, and at one-half the length from the free end.
Ans. $\frac{dy}{dx} = 0.016, 0.015,$ and $0.012.$

9. At what distance from the end of a cantilever loaded at the end is the slope one-half the slope at the end? *Ans.* $x = 0.7071\,l.$

10. A cantilever of length l and depth d, with sections symmetrical with respect to the neutral axis, is deflected by a load at the end. Derive an expression for deflection at the end in terms of the maximum stress S, the modulus of elasticity, and the dimensions l and d.
Ans. $y_{\max} = \frac{2\,S\,l^2}{3\,E\,d}.$

11. If $E = 29{,}000{,}000$ lb. per sq. in., and the allowable stress is 18,000 lb. per sq. in., find the deflection at the end of any 10-in. I-beam or channel as a cantilever 6 ft. 8 in. long. *Ans.* $y_{\max} = 0.2648$ in.

12. A beam in the form of an isosceles triangle 6 in. high has a modulus of elasticity of 1,400,000 lb. per sq. in. and an allowable stress of 1,100 lb. per sq. in. The length is 6 ft. Derive a formula and find the maximum deflection at the end. *Ans.* $y_{\max} = 3.3394$ in.

13. A cantilever of length l is fixed at the left end and carries a load P at the right end. With the origin of coördinates at the fixed end and x positive toward the right, derive the equation of the elastic line.

$$\textit{Ans. } M = -P(l - x) = -P\,l + P\,x; \tag{10}$$

$$E\,I\frac{dy}{dx} = -P\,l\,x + \frac{P\,x^2}{2} + [C_1 = 0]; \tag{11}$$

$$E\,I\,y = -\frac{P\,l\,x^2}{2} + \frac{P\,x^3}{6} + [C_2 = 0]. \tag{12}$$

14. Why are C_1 and C_2 equal to zero in Problem 13? Find the slope at the right end and at the middle. Compare with Problem 8. Find the deflection at the middle and at one-third the length from the free end.

15. Solve Problem 13 using the first expression for M in Eq. (10).

$$\textit{Ans. } E\,I\frac{dy}{dx} = \frac{P(l - x)^2}{2} - \frac{P\,l^2}{2}; \tag{13}$$

$$E\,I\,y = -\frac{P(l - x)^3}{6} - \frac{P\,l^2\,x}{2} + \frac{P\,l^3}{6}. \tag{14}$$

16. Expand Eq. (13) and compare with Eq. (11). Expand Eq. (14) and compare with Eq. (12). Find the slope and deflection at the free end. How do the signs of the slope compare with those from Eq. (4)? Explain.

90. Cantilever with Load at Any Point.—Figure 154 shows a cantilever, which is fixed at the right end and carries a concentrated load P at a distance a from the left end. The portion to

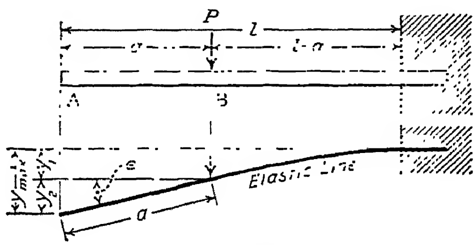


FIG. 154.

the left of the load remains straight (if the weight of the beam is neglected). The portion to the right of the load is a cantilever of length $l - a$ which is loaded at the end. The deflection of this portion is

$$y_1 = -\frac{P(l - a)^3}{3\,E\,I} \tag{1}$$

The additional deflection of the portion to the left of the load is $y_2 = -a \sin \theta$, in which θ is the slope at the load. For a small angle, $\sin\ \theta = \tan\ \theta = \frac{dy}{dx}$. From Equation (4) of Art. 89, the slope under the load is

$$\frac{dy}{dx} = \frac{P(l - a)^2}{2\,E\,I}. \tag{2}$$

$$y_2 = -\frac{P\,a(l - a)^2}{2\,E\,I};$$

$$y_{max} = -\frac{P(l-a)^3}{3\,E\,I} - \frac{P\,a(l-a)^2}{2\,E\,I} = -\frac{P}{6\,E\,I}(2\,l^3 - 3\,l^2a + a^3). \tag{3}$$

Problems

1. A cantilever of length l has a load P at one-third the length from the free end. Find the deflection at the free end and under the load.

Ans. $y_{max} = -\frac{14\,P\,l^3}{81\,E\,I}$; $y = -\frac{8\,P\,l^3}{81\,E\,I}$.

2. Solve Problem 1 if the load is at the middle.

$$Ans.\ y_{\max} = -\frac{5\,P\,l^3}{48\,E\,I};\ y = -\frac{P\,l^3}{24\,E\,I}.$$

3. A 6-in. by 4-in. longleaf yellow-pine cantilever, 6 ft. long, carries 200 lb. 1 ft. from the free end and 120 lb. 2 ft. from the free end. Find the deflection at the free end if $E = 1{,}200{,}000$.
4. In Problem 3 find the deflection under each load. Find the maximum stress.
5. Solve Problems 3 and 4 if the beam were turned 90° to make the 6-in. faces vertical. Use answers of preceding problems.

91. Maxwell's Theorem.—As long as the stress remains below the proportional elastic limit, the deflection caused by several loads is equal to the sum of the deflections which would be caused by the loads acting separately. This statement, which is called the principle of *superposition,* may be regarded as an axiom amply verified by experiment and by theoretical investigation of special cases. In Art. 203, starting from the principle of superposition, Maxwell's theorem is derived. This law of the equivalence of reciprocal deflections is: *If A and B are two points on a beam (or any elastic structure), the deflection at A caused by a given load at B is equal to the deflection at B caused by the same load as A.*

If x is substituted for a in Equation (3) of Art. 90, the result is Equation (8) of Art. 89. In order to find the deflection at the end of a beam which is caused by a load at a distance x from the end, simply assume that the load is applied at the end and find the deflection at a distance x from the end by Equation (8) or Equation (9) of Art. 89.

Example

A 5-in. 14.75-lb. I-beam as a cantilever 7 ft. 6 in. long carries 1,000 lb. 6 ft. from the fixed end and 500 lb. 5 ft. from the fixed end. Find the deflection at the free end which is caused by these loads if $E = 30{,}000{,}000$ lb. per sq. in.

$$y_{\max} = -\frac{90^3}{6\,E\,I}\left\{1{,}000\left(2 - \frac{3}{5} + \frac{1}{125}\right) + 500\left(2 - 1 + \frac{1}{27}\right)\right\}.$$

$$y_{\max} = -0.00027\ (1{,}000 \times 1.408 + 500 \times {}^{28}\!/_{27}) = 0.52016 \text{ in.}$$

Problems

1. Find the deflection under the 1,000-lb. load of the example above.
2. Find the deflection under the 500-lb. load of the example above. Find the maximum unit stress including the weight of the beam.
3. Calculate the deflection at the end of the cantilever of Problem 1 of Art. 89 if an additional load of 300 lb. is placed 3 ft. from the free end.

92. Cantilever with Load Uniformly Distributed.—Figure 155 shows a cantilever which is fixed at the right end and carries a

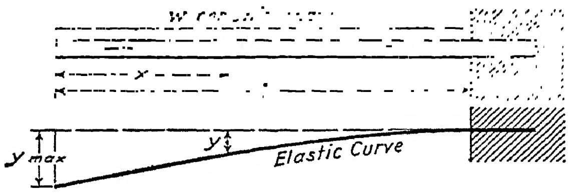


FIG. 155.—Cantilever with distributed load, fixed at right end.

load of w per unit length. The origin of coördinates is taken at the position of the free end before the load was applied.

The moment at any section at a distance x from the free end is $-\frac{w\,x^2}{2}$

$$E\,I\frac{d^2y}{dx^2} = -\frac{w\,x^2}{2}, \tag{1}$$

$$E\,I\frac{dy}{dx} = -\frac{w\,x^3}{6} + C_1. \tag{2}$$

$$E\,I\frac{dy}{dx} = -\frac{w\,x^3}{6} + \frac{w\,l^3}{6}. \tag{3}$$

$$E\,I\,y = -\frac{w\,x^4}{24} + \frac{w\,l^3x}{6} + C_2, \tag{4}$$

$$E\,I\,y = -\frac{w\,x^4}{24} + \frac{w\,l^3x}{6} - \frac{w\,l^4}{8}. \tag{5}$$

$$y = -\frac{w}{24\,E\,I}(3\,l^4 - 4\,l^3x + x^4), \tag{6}$$

which is the equation of the elastic line.

If $x = k\,l$, in which k is a fraction smaller than unity, Equation (6) may be written

$$y = -\frac{w\,l^4}{24\,E\,I}(3 - 4\,k + k^4) = -\frac{W\,l^3}{24\,E\,I}(3 - 4\,k + k^4). \tag{7}$$

At the free end

$$y_{\max} = -\frac{w\,l^4}{8\,E\,I} = -\frac{Wl^3}{8\,E\,I}. \qquad \text{Formula XIX}$$

In these equations $W = w\,l$, which is the total distributed load.

Since modulus of elasticity is expressed in pounds per square inch, and moment of inertia in inches^4, w must be in pounds per

inch. To find the total load W, pounds per foot may be used with length in feet.

Problems

1. A 6-in. by 10-in. timber cantilever, 10 ft. long, is subjected to a load of 180 lb. per ft. What is the deflection at the free end if $E = 1,600,000$ lb. per sq. in.? What is the maximum fiber stress at the fixed end? Use the total load W. *Ans.* $y_{max} = -0.486$ in.; $S = 1,080$ lb./in.^{2}
2. In Problem 1, what is the deflection 3 ft. from the free end? Solve by Eq. (6) and also by Eq. (7). *Ans.* $y_{max} = -0.2929$ in.
3. A 2-in. by 3-in. timber beam, weighing 24 lb., has 2 ft. of its length clamped between horizontal steel plates and projects 10 ft. as a cantilever. The deflection caused by its weight brings the lower edge at the free end 0.48 in. below the plane of the upper surface of the lower steel plate. Assuming that the beam was originally straight, find E. *Ans.* $E = 2,000,000$ lb./in.^{2}
4. Assuming that the beam of Problem 3 is turned 180° and the deflection of the free end is apparently 0.52 in., calculate the corrected E.
5. If a load of 25 lb. on the beam of Problem 3 at 1 ft. 8 in. from the free end produces an additional deflection of 0.81 in., find E.
6. If a given cantilever is deflected 1.2 in. at the end by a uniformly distributed load, how much would it be deflected at the end if an equal total load were placed on the end? Solve without writing.
7. A cantilever, uniformly loaded, is deflected 1.2 in. at the end when the maximum unit stress is 800 lb. per sq. in. How much would this cantilever be deflected by a load on the end which would cause the same maximum stress? Solve without writing.
8. Find the slope at the free end, at one-third the length from the free end, and at the middle of a uniformly loaded cantilever.

Ans. $\dfrac{W\,l^2}{6\,E\,I}$; $\dfrac{13\,W\,l^2}{81\,E\,I}$; $\dfrac{7\,w\,l^3}{48\,E\,I}$.

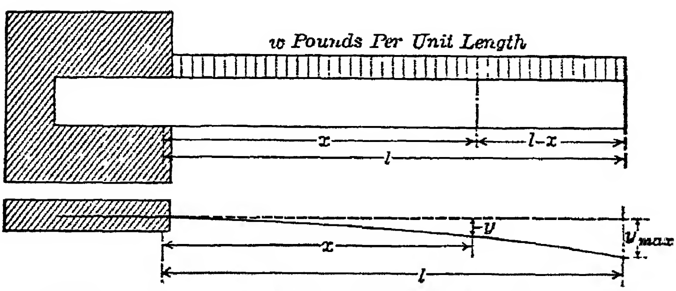


FIG. 156.—Cantilever with uniformly distributed load.

9. A cantilever with uniformly distributed load is fixed at the left end, as in Fig. 156. Show that

$$M = -\frac{w(l - x)^2}{2}; \tag{8}$$

$$E\,I\frac{dy}{dx} = \frac{w(l - x)^3}{6} - \frac{w\,l^3}{6}; \tag{9}$$

$$E\,I\,y = -\frac{w(l - x)^4}{24} - \frac{w\,l^3x}{6} + \frac{w\,l^4}{24}. \tag{10}$$

10. Expand the expression for M in Eq. (8) and integrate for the equation of the elastic line.

$$Ans.\ E\,I\,y = -\frac{w\,l^2\,x^2}{4} + \frac{w\,l\,x^3}{6} - \frac{w\,x^4}{24}. \tag{11}$$

93. Simply-supported Beam, Uniformly Loaded.—In a beam supported at the ends and uniformly loaded, each reaction is

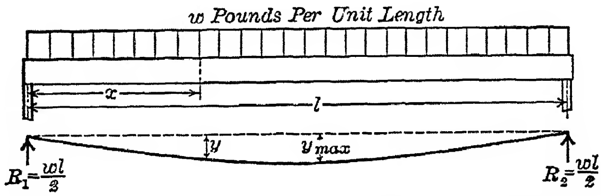


FIG. 157.—Supports at ends, load uniformly distributed.

one-half the total load $w\,l$. The moment at a distance x from the left support is

$$M = \frac{w\,l\,x}{2} - \frac{w\,x^2}{2},$$

and the differential equation becomes

$$E\,I\frac{d^2y}{dx^2} = \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}. \tag{1}$$

$$E\,I\frac{dy}{dx} = \frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6} + C_1. \tag{2}$$

From symmetry it is evident that the maximum deflection is at the middle

$$\frac{dy}{dx} = 0 \text{ when } x = \frac{l}{2};$$

$$C_1 = -\frac{w\,l^3}{24}. \tag{3}$$

$\left(\frac{C_1}{E\,I}\right.$ is the slope of the elastic line at the left support.$\left.\right)$

$$E\,I\frac{dy}{dx} = \frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6} - \frac{w\,l^3}{24}. \tag{4}$$

$$E\,I\,y = \frac{w\,l\,x^3}{12} - \frac{w\,x^4}{24} - \frac{w\,l^3x}{24} + C_2. \tag{5}$$

At $x = 0$, $y = 0$; $C_2 = 0$; (6)

$$E\,I\,y = \frac{w\,l\,x^3}{12} - \frac{w\,x^4}{24} - \frac{w\,l^3x}{24}. \tag{7}$$

When $x = \dfrac{l}{2}$ the deflection is a maximum,

$$E\,I\,y_{\max} = \frac{w\,l^4}{12}\left(\frac{1}{8} - \frac{1}{32} - \frac{1}{4}\right) = -\frac{5w\,l^4}{384};$$

$$y_{\max} = -\frac{5\,w\,l^4}{384\,E\,I} = -\frac{5\,W\,l^3}{384\,E\,I}. \qquad \text{Formula XX}$$

Substituting $x = l$ in Equation (7), the deflection at the right support is found to be zero. This condition might have been used to determine one of the constants.

This beam might be regarded as fixed at the middle where the slope is zero, and to consist of two cantilevers of length $\dfrac{l}{2}$ which are bent downward by the distributed load and bent upward by the end reactions. The deflection at one end is

Downward, $$\frac{\dfrac{W}{2}\left(\dfrac{l}{2}\right)^3}{8\,E\,I} = \frac{W\,l^3}{128\,E\,I};$$

Upward, $$\frac{\dfrac{W}{2}\left(\dfrac{l}{2}\right)^3}{3\,E\,I} = \frac{W\,l^3}{48\,E\,I};$$

Total deflection upward, $\dfrac{W\,l^3}{384\,E\,I}(8 - 3) = \dfrac{5\,W\,l^3}{384\,E\,I}.$

The deflection at any point, measured upward from the tangent at the middle, may be calculated in a similar way.

Problems

1. A 2-in. by 12-in. floor joist is 15 ft. long between supports and carries a distributed load of 180 lb. per ft. If $E = 1{,}600{,}000$ lb. per sq. in., what is the deflection at the middle and at 5 ft. from each end? What is the maximum fiber stress? *Ans.* $y_{\max} = -0.444$ in.
2. The floor of a room 30 ft. wide is carried on 24-in. I-beams spaced 12 ft. apart. The total live and dead load is 210 lb. per sq. ft. Using A.I.S.C. Specifications, select the I-beam. Calculate the maximum deflection if $E = 29{,}000{,}000$. *Ans.* $y_{\max} = -0.688$ in.

3. A 10-in. 30-lb. standard I-beam, 12 ft. long, is supported at the ends and carries a load of 1,740 lb. per ft., including its own weight. If $E = 29{,}000{,}000$ lb. per sq. in., what is the deflection at the middle and at 4 ft. from each end? What is the slope at the left end?

Ans. $y_{max} = -0.2097$ in.; $y = -0.1822$ in.; $\frac{dy}{dx} = -0.00466$.

4. Write an expression for the deflection at a distance $k\,l$ from the left end from Eq. (7).

Ans. $y = -\frac{W\,l^3}{24\,E\,I}(k - 2\,k^3 + k^4)$.

5. If the beam of Problem 3 extended 5 ft. to the left of the left support, how much would the free end be elevated when the span between the supports is loaded?

6. A beam of length l and depth d has its neutral surface midway between the top and bottom. The beam is supported at the ends and carries a uniformly distributed load which makes the maximum stress equal to S. What is the deflection at the middle?

Ans. $y_{max} = -\frac{5\,S\,l^2}{24\,E\,d}$.

7. If $E = 29{,}000{,}000$ and $S = 17{,}400$, find the maximum deflection for a 12-in. I-beam 16 ft. long. *Ans.* $y_{max} = -0.384$ in.

8. A 1-in. by ¼-in. steel bar, 10 ft. long, supported at the ends, is deflected 5 in. by its weight. Find E. Find maximum stress.

Ans. $S = 12{,}240$ lb./in.^{2}

94. Simply-supported Beam, Loaded at Middle.—If P is the load at the middle, each reaction is $\frac{P}{2}$, and the moment from the left end to the middle is $\frac{P\,x}{2}$ (see Fig. 123). For the portion of the beam between the left end and the middle,

$$E\,I\frac{d^2y}{dx^2} = \frac{P\,x}{2}, \tag{1}$$

$$E\,I\frac{dy}{dx} = \frac{P\,x^2}{4} + C_1. \tag{2}$$

At the middle, from the symmetry of the sides, $\frac{dy}{dx} = 0$;

$$C_1 = -\frac{P\,l^2}{16}; \tag{3}$$

$$E\,I\frac{dy}{dx} = \frac{P\,x^2}{4} - \frac{P\,l^2}{16}; \tag{4}$$

$$E\,I\,y = \frac{P\,x^3}{12} - \frac{P\,l^2x}{16} + C_2. \tag{5}$$

At the left support, where $x = 0$, $y = 0$:

$$C_2 = 0. \tag{6}$$

$$E\,I\,y = \frac{P\,x^3}{12} - \frac{P\,l^2x}{16}. \tag{7}$$

At the middle, where $x = \frac{l}{2}$

$$y_{\max} = \frac{P\,l^3}{96\,E\,I} - \frac{P\,l^3}{32\,E\,I} = -\frac{P\,l^3}{48\,E\,I}. \quad \text{Formula XXI}$$

Since the moment equation applies only to the left half of the beam, the formulas derived from this equation are not valid beyond the middle. From the middle to the right end, the moment is $\frac{P}{2}(l - x)$.

$$E\,I\frac{d^2y}{dx^2} = \frac{P}{2}(l - x). \tag{8}$$

$$E\,I\frac{dy}{dx} = -\frac{P(l - x)^2}{4} + \left[C_3 = \frac{P\,l^2}{16}\right]. \tag{9}$$

$$E\,I\,y = \frac{P(l - x)^3}{12} + \frac{P\,l^2x}{16} + C_4. \tag{10}$$

At the right support, where $x = l$, $y = 0$; $C_4 = -\frac{P\,l^3}{16}$.

$$E\,I\,y = \frac{P(l - x)^3}{12} + \frac{P\,l^2x}{16} - \frac{P\,l^3}{16}. \tag{11}$$

A beam supported at the ends and loaded at the middle may be regarded as two cantilevers, each of length $\frac{l}{2}$, which are bent upward by reactions $\frac{P}{2}$.

Formula XXI is largely used in the determination of the modulus of elasticity.

Problems

1. A 4-in. by 6-in. beam, 5 ft. long between supports, is deflected 0.0600 in. at the middle when the load changes from 200 lb. to 1,400 lb. Find E and the maximum stress.

   Ans. E = 1,250,000 lb./in.^{2}; S = 875 lb./in.^{2}

2. The spruce beam of Problem 1 of Art. 81 was deflected 0.1326 in. when the load changed from 31 lb. to 597 lb. Calculate E.

3. A yellow-pine beam, 1.63 in. by 1.94 in., resting on supports 20 in. apart and loaded at the middle, deflected 0.0517 in. when the load changed from 100 lb. to 600 lb. and deflected 0.0520 in. when the load changed from 300 lb. to 800 lb. Find E for each reading. The beam failed under a load of 2,400 lb. Find the modulus of rupture.
Ans. 1,625,000 lb./in.²; 1,615,000 lb./in.²; 11,740 lb./in.²

4. A cast-iron T-beam, similar to that of Problem 3 (Art. 81) and loaded and tested in the same way, was deflected 0.0332 in. at the middle when the load changed from 300 lb. to 800 lb. Find E.
Ans. $E = 15{,}800{,}000$ lb./in.²

5. A 7-in. 20-lb. I-beam, 20 ft. long, rests on supports 12 ft. apart and overhangs the left support 5 ft. When a load of 5,000 lb. is placed midway between the supports, what is the deflection under the load, and how much are the ends elevated? $E = 29{,}000{,}000$.
Ans. 0.256 in.; 0.320 in.; 0.192 in.

6. In Problem 1, find the total deflection 20 in. from the left support when the load was 1,400 lb. *Ans.* $y = -0.0596$ in.

7. Derive an expression for the deflection at a distance $k\,l$ from the left support if k is not greater than ½.
Ans. $y = -\dfrac{P\,l^3}{48\,E\,I}(3\,k - 4\,k^3).$

8. Derive an expression for the deflection at a distance $k\,l$ from the left support if k is greater than ½.
Ans. $y = -\dfrac{P\,l^3}{48\,E\,I}\left(4(1-k)^3 + 3\,k - 3\right).$

9. Derive expressions for the deflection at one-fourth the length and at one-third the length from the left end, and for the deflection at two-thirds the length and three-fourths the length from the left end. Compare.

10. If S is the allowable unit stress in a beam, d is the depth, and l is the length, and if the neutral surface is midway between the top and bottom, what is the expression for the maximum allowable deflection when the beam is loaded at the middle. By means of this formula find the maximum allowable deflection in a steel beam 20 ft. long and 18 in. deep, using the A.I.S.C. allowable stress.

95. Beam with Constant Moment.—Figure 158 shows a beam which is supported at two points at a distance l apart, overhangs each support, and carries a load on each end. The moment at the left support is $-P\,a$, and the moment at the right support is $-Q\,b$. If $P\,a = Q\,b$, the left reaction is equal to P and the right reaction is equal to Q. The loads and reactions then form two equal and opposite couples, and the moment between the supports is constant and equal to $-P\,a$(or $-Q\,b$). With the origin of coördinates at the left support,

$$E\,I\frac{d^2y}{dx^2} = -P\,a; \tag{1}$$

$$E\,I\frac{dy}{dx} = -P\,a\,x + C_1; \tag{2}$$

$$E\,I\,y = -\frac{P\,a\,x^2}{2} + C_1x + C_2. \tag{3}$$

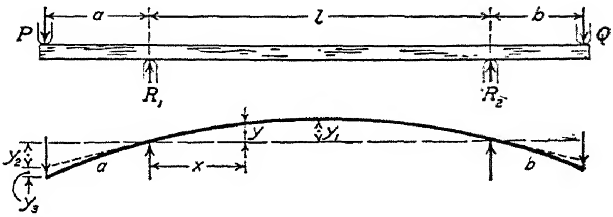


FIG. 158.—Moment constant.

At the left support, where $x = 0$, $y = 0$; hence $C_2 = 0$.
At the right support, where $x = l$, $y = 0$.

$$0 = -\frac{P\,a\,l^2}{2} + C_1l,$$

$$C_1 = \frac{P\,a\,l}{2}.$$

When this value of C_1 is substituted in Equation (2),

$$E\,I\frac{dy}{dx} = -P\,a\,x + \frac{P\,a\,l}{2}. \tag{4}$$

At the point of maximum deflection, $\frac{dy}{dx} = 0$ and $x = \frac{l}{2}$. The maximum deflection is at the middle of the span.

The equation of the elastic line is

$$y = \frac{P\,a}{2\,E\,I}(l\,x - x^2) = \frac{P\,a\,x}{2\,E\,I}(l - x). \tag{5}$$

$$y_{\max} = \frac{P\,a\,l^2}{8\,E\,I} = -\frac{M'l^2}{8\,E\,I}, \qquad \text{Formula XXII}$$

in which M' is a positive, constant moment.

If P and Q are equal and a and b are equal, it is evident that the elastic line is symmetrical with respect to the middle of the span. Under these conditions, it could be assumed that $\frac{dy}{dx}$ is zero when x is $\frac{l}{2}$, and the value of C_1 could be obtained from the first integral. If P and Q are not equal, but the product $P\,a$ equals $Q\,b$, the symmetry is not self-evident, and it is better to determine both constants from the second integral.

Problems

1. A board, 6 in. wide and 1 in. thick, rests on two supports 80 in. apart. A load of 30 lb. is placed 16 in. to the left of the left support and a load of 20 lb. is placed 24 in. to the right of the right support. If E is 1,200,000 lb. per sq. in., what is the deflection upward midway between the supports? What is the slope at each support?

Ans. $y = 0.64$ in.; $\frac{dy}{dx} = 0.032$.

2. In Problem 1, find the deflection at 20 in. to the right of the left support.

3. In Problem 1, find the radius of curvature of the portion of the beam between the supports and calculate the deflection at the middle and the slope over the supports geometrically.

The overhanging ends of Fig. 158 are cantilevers. The deflection of each load *from the tangent at the support* (y_3 of Fig. 158) is given by Formula XVIII. The deflection of the tangent at the support from the horizontal line through the supports (y_2 of Fig. 158) is the distance from the load to the support multiplied by the slope at the support.

$$y_2 + y_3 = -\frac{P\,a^2 l}{2\,E\,I} - \frac{P\,a^3}{3\,E\,I} = -\frac{P\,a^2}{6\,E\,I}(3\,l + 2\,a). \qquad (6)$$

Problems

4. In Problem 1, find the deflection of the load of 30 lb. downward from the horizontal line through the supports. *Ans.* 0.512 + 0.068 = 0.580 in.

5. A beam of total length l is supported at two points, each of which is $\frac{l}{4}$ from one end and carries a load of $\frac{P}{2}$ at each end. How much is the middle deflected upward and how much is each end deflected downward from the supports?

(*Article* 96 *may be omitted. Compare with Arts.* 101; 111.)

96. Simply-supported Beam, Loaded at Any Point.—Since the moment equation changes at the concentrated load, two differential equations of the second order must be solved to obtain the equation of the elastic line for the entire beam. For a beam supported at the ends with a load at the middle, the symmetry made it possible to obtain the integration constants from the single equation of the left half. Where the load is not at the middle, the slope is known neither at the middle nor under the load. If the moment equation for the portion of the beam between the left support of Fig. 159 and the load were integrated alone, the only known condition would be that $y = 0$ when $x = 0$. If the equation for the portion between the load and the right support were integrated alone, the known condition

would be $y = 0$ when $x = l$. Since the solution of these two equations introduces four integration constants, two other relations must be known. When the two moment equations are integrated together, these two additional relations may be found from the fact that under the load, where $x = a$, the slope $\frac{dy}{dx}$ is the same for the two first integrals, and the deflection y is the same for the two second integrals. Since the principle of this method

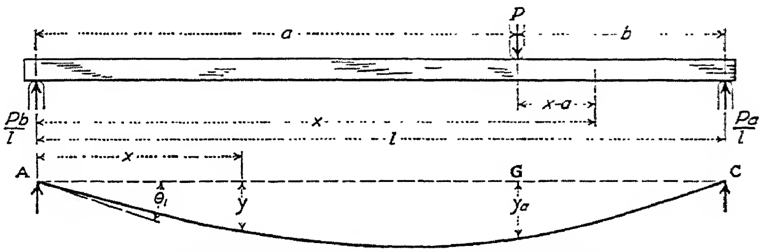


FIG. 159.—Beam with single concentrated load.

is important, it will be carried through, although briefer solutions are given in Arts. 101, 111, and 206.

Figure 159 represents a beam of length l, which is supported at the ends and carries a load P at a distance a from the left support and a distance b from the right support. The left reaction is $\frac{P\,b}{l}$. For the left portion of the beam

$$M = \frac{P\,b\,x}{l},$$

and to the right of the load

$$M = \frac{P\,b\,x}{l} - P(x - a).$$

For all points from $x = 0$ *to* $x = a$, *inclusive,*

$$E\,I\,\frac{d^2y}{dx^2} = \frac{P\,b\,x}{l}. \qquad (1)$$

$$E\,I\,\frac{dy}{dx} = \frac{P\,b\,x^2}{2\,l} + C_1. \qquad (3)$$

For all points from $x = a$ *to* $x = l$, *inclusive,*

$$E\,I\,\frac{d^2y}{dx^2} = \frac{P\,b\,x}{l} - P(x - a). \qquad (2)$$

$$E\,I\,\frac{dy}{dx} = \frac{P\,b\,x^2}{2\,l} - \frac{P(x - a)^2}{2} + C_3. \qquad (4)$$

The curve is continuous under the load with no abrupt change of slope. When $x = a$, the value of $\frac{dy}{dx}$ calculated from Equation (3) is the same as when calculated from Equation (4). This makes the first members of the two equations equal and, consequently, the second members are equal when a is substituted for x.

$$\frac{P\,b\,a^2}{2\,l} + C_1 = \frac{P\,b\,a^2}{2\,l} - \frac{P(a - a)^2}{2} + C_3;$$

$$C_1 = C_3.$$

Substituting C_1 for C_3 in Equation (4) and integrating both equations:

$$EIy = \frac{P\,b\,x^3}{6\,l} + C_1x + C_2. \quad (5)$$

When $x = 0$, $y = 0$;
hence $C_2 = 0$.

$$E\,I\,y = \frac{P\,b\,x^3}{6\,l} - \frac{P(x-a)^3}{6} + C_1x + C_4. \quad (6)$$

When $x = a$, the values of y from Equations (5) and (6) are the same and the second members of these equations are equal, from which

$$0 = C_2 = C_4.$$

When $x = l$ in Equation (6), $y = 0$;

$$C_1 = -\frac{P\,b\,l^2}{6\,l} + \frac{P(l-a)^3}{6\,l} = -\frac{P\,b}{6\,l}(l^2 - b^2). \quad (7)$$

Substituting the value of C_1 from Equation (7) in Equation (5):

$$E\,I\,y = \frac{P\,b\,x^3}{6\,l} - \frac{P\,b(l^2-b^2)x}{6\,l}. \quad (8)$$

Substituting C_1 in Equation (3) and equating to zero,

$$x^2 = \frac{l^2 - b^2}{3} = \frac{a(a+2b)}{3}, \quad (9)$$

gives the point of maximum deflection, provided b is less than a. Substituting x from Equation (9) in Equation (8),

$$y_{\max} = -\frac{P\,b(l^2-b^2)\sqrt{3(l^2-b^2)}}{27\,E\,I\,l} = -\frac{P\,b\,a(a+2b)\sqrt{3\,a(a+2b)}}{27\,E\,I\,l}. \quad (10)$$

The deflection under the load is

$$y = -\frac{P\,a^2b^2}{3\,E\,I\,l}. \quad (11)$$

Problems

1. A beam of length l is supported at the ends and carries a load P at six-tenths of the length from the left support. Find the deflection under the load, the deflection at the middle, and the maximum deflection.

Ans. $E\,I\,y = -0.0192\,P\,l^3,\ -0.01967\,P\,l^3,\ -0.01975\,P\,l^3.$

2. Find the deflection at 0.3 l and at 0.4 l from the left support of Problem 1.

Ans. $E\,I\,y = -0.0150\,P\,l^3,\ -0.01813\,P\,l^3.$

3. A 3-in. by 2-in. simply-supported rectangular beam, 10 ft. long, carries a load of 36 lb. 6 ft. from the left end. Find the maximum deflection if $E = 1{,}500{,}000$ lb. per sq. in. Find the slope at the left support. Find the slope at the right support and the deflection under the load.

Ans. $y_{\max} = -0.4096$ in.; $\dfrac{dy}{dx} = -0.009677$ at left

$\dfrac{dy}{dx} = 0.010159$ at right; $y = -0.3981$ in. under load.

CHAPTER IX

INTEGRATION BETWEEN LIMITS

(This chapter may be omitted, or this chapter may be studied and any or all of Chapter VIII, *except Art.* 91, *may be omitted.)*

97. Fundamental Operations.—Chapter VIII gives the accepted method for derivation of the equations of the elastic line for beams. For each integration there is an arbitrary constant, which must be evaluated from a known condition of slope or deflection. When the moment throughout a span is represented by a single equation, only two constants are needed. If two equations are required to express the moment of different portions of the span, as in Art. 96, four constants are necessary. With more than two equations, the method is impracticable.

The fundamental deflection formula of Art. 78 is

$$M = E\,I\,\frac{d\theta}{dl} = E\,I\,\frac{d\theta}{dx},$$

in which x is taken parallel to the unbent beam. Since the deflection of beams is small, it is customary to assume that $dl = dx$, and that $\theta = \tan\,\theta = \dfrac{dy}{dx}\cdot$ These approximations are used in the derivation of deflection equations by all methods.

After the first integration for the slope in terms of x, θ is replaced by $\dfrac{dy}{dx}$ and a second integration is performed to obtain the equation of the elastic line in rectangular coordinates.

When the moment curve is drawn with the moment as the ordinate, $M\,dx$ is an element of the area between the curve and the axis of x and $\int M\,dx$ between two limits gives the area under the curve between these limits. *The area under the moment diagram between two ordinates, divided by $E\,I$, gives the change of slope in that interval.*

98. Cantilever Loaded at End.—Figure 160, I, shows a cantilever which is fixed at the right end and loaded at the left

end. The moment diagram is a negative triangle of base l, altitude $-P\,l$, and area $-\frac{P\,l^2}{2}$.

$$M = -P\,x; \quad E\,I\,d\theta = -P\,x\,dx; \tag{1}$$

$$E\,I\,\theta = E\,I\,\theta_1 - \left[\frac{P\,x^2}{2}\right]_0^x, \tag{2}$$

in which θ is the slope at a distance x from the origin, θ_1 is the slope at the origin and $\left[\frac{P\,x^2}{2}\right]_0^x$ is the quantity which, divided by $E\,I$,

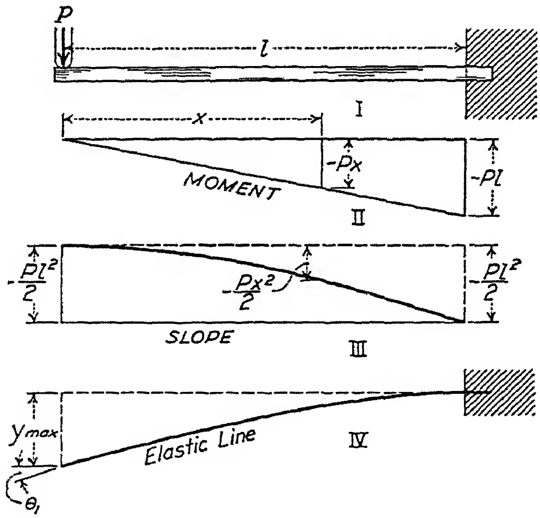


FIG. 160.—Moment and slope of cantilever.

gives the change of slope in the interval x. Since the integral of $-\frac{P\,x^2}{2}$ from 0 to x is $-\frac{P\,x^2}{2}$,

$$E\,I\,\theta = E\,I\,\theta_1 - \frac{P\,x^2}{2} \tag{3}$$

is the slope equation for the span.

Since the slope is 0 at the fixed end, the value of θ_1 may be obtained by substituting l for x in Equation (3), or by using l in place of x as the upper limit in Equation (2).

$$0 = E\,I\,\theta_1 - \frac{P\,l^2}{2}; \quad \theta_1 = \frac{P\,l^2}{2\,E\,I}. \tag{4}$$

Equation (3) now reads

$$E\,I\,\theta = \frac{P\,l^2}{2} - \frac{P\,x^2}{2}. \tag{5}$$

Equation (5) might have been written without formal integration. The area of the entire triangle of Fig. 160, II, is $-\frac{P\,l^2}{2}$, which added to $E\,I\,\theta_1$ gives the first equation of (4). The area of the triangle of base x is $-\frac{P\,x^2}{2}$, which is $E\,I$ times the difference in slope between $x = 0$ and $x = x$.

With θ replaced by $\frac{dy}{dx}$, Equation (5) now reads

$$E\,I\,\frac{dy}{dx} = \frac{P\,l^2}{2} - \frac{P\,x^2}{2}; \tag{6}$$

$$E\,I\,y = E\,I\,y_1 + \left[\frac{P\,l^2 x}{2} - \frac{P\,x^3}{6}\right]_0^x, \tag{7}$$

in which y is the deflection at any point at a distance x from the free end, and y_1 is the deflection at the origin. Since $y = 0$ when $x = l$, substitution of l as the upper limit in Equation (7) gives

$$0 = E\,I\,y_1 + \frac{P\,l^3}{3}; \quad y_1 = y_{max} = -\frac{P\,l^3}{3\,E\,I}. \quad \text{Formula XVIII}$$

Since the value of each term in the bracket of Equation (7) is zero at the lower limit, the expression retains its form when x is taken as the upper limit, and

$$E\,I\,y = -\frac{P\,l^3}{3} + \frac{P\,l^2 x}{2} - \frac{P\,x^3}{6}. \tag{8}$$

Problems

1. For a cantilever with a load on the free end, what is the ratio of the deflection at any point to the slope at that point?

Ans. $\dfrac{2\,l^2 - l\,x - x^2}{3\,(l + x)}$.

2. At what point on a cantilever loaded at the free end is the ratio of the *numerical* value of the deflection to the slope equal to one-half the length?

Ans. $x = \dfrac{l(\sqrt{33} - 5)}{4} = 0.186\,l$.

3. Find the deflection and slope when $x = 0.4\,l$ from Eqs. (6) and (8). Calculate the ratio of these quantities and check by the answer of Problem 1.

(The remainder of this article may be omitted.)

While the equation of the elastic line for a cantilever with a concentrated load on the end has been fully developed in the preceding discussion, the principles would be better understood if the investigation should include a beam fixed at the left end.

If the right portion, of length $l - x$, of the beam of Fig. 161 is taken as the free body, the moment at a distance x from the fixed end is

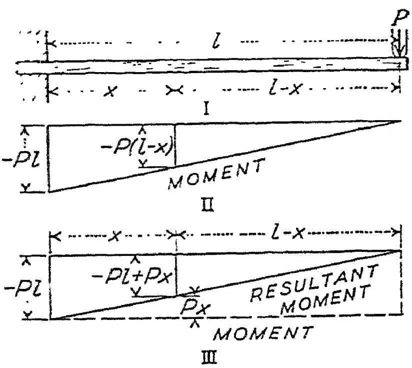


FIG. 161.—Cantilever fixed at left end.

$$M = -P(l - x); \tag{9}$$

$$E\,I\,\theta = 0 + \left[\frac{P(l - x)^2}{2}\right]_0^x; \tag{10}$$

$$E\,I\,\theta = E\,I\frac{dy}{dx} = \frac{P(l - x)^2}{2} - \frac{P\,l^2}{2}. \tag{11}$$

The integral of Equation (10) at the lower limit is $-\frac{P\,l^2}{2}$, and Equation (11) is not the same as Equation (10).

$$E\,I\,y = \left[-\frac{P(l - x)^3}{6} - \frac{P\,l^2 x}{2}\right]_0^x; \tag{12}$$

$$E\,I\,y = -\frac{P(l - x)^3}{6} + \frac{P\,l^3}{6} - \frac{P\,l^2 x}{2}. \tag{13}$$

From the general moment equation, or from the expansion of Equation (9),

$$M = -P\,l + P\,x. \tag{14}$$

This equation is represented by Fig. 161, III. The negative rectangle of altitude $-P\,l$ gives the first term. The positive triangle gives the second term. The resultant diagram, of course, is the same as Fig. 161, II.

$$E\,I\,\theta = \left[-P\,l\,x + \frac{P\,x^2}{2}\right]_0^x; \tag{15}$$

$$E\,I\,\theta = E\,I\frac{dy}{dx} = -P\,l\,x + \frac{P\,x^2}{2}. \tag{16}$$

Since both terms of Equation (15) are zero at the lower limit, Equations (15) and (16) are identical in form. After the student has acquired sufficient experience to recognize this condition, it will not be necessary for him to write both equations.

$$E\,I\,y = \left[-\frac{P\,l\,x^2}{2} + \frac{P\,x^3}{6}\right]_0^x; \tag{17}$$

$$E\,I\,y = -\frac{P\,l\,x^2}{2} + \frac{P\,x^3}{6}. \tag{18}$$

Problems

4. Derive Eq. (17) geometrically from Fig. 160. VII.
5. Find the deflection at the end by Eq. (19) and by Eq. (18). Find the deflection at the middle and at one-third the length from the free end by these equations and by Eq. (9).

99. Cantilever Uniformly Loaded.—Figure 162 shows a uniformly loaded cantilever which is fixed at the right end. The moment diagram is a negative parabola which has a maximum

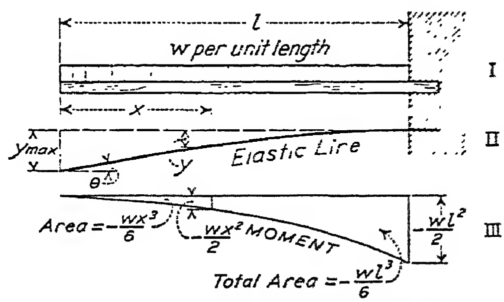


FIG. 162.—Uniformly loaded cantilever fixed at right end.

altitude of $-\frac{w\,l^2}{2}$. Since the area under a parabola which is convex toward the base is one-third the product of the base by the altitude, the area of the entire parabola is $-\frac{w\,l^3}{6}$. A portion of the parabola which has a base of length x has an area of $-\frac{w\,x^3}{6}$. Solving for the slope from these areas without integrating:

$$E\,I\,\theta = E\,I\,\theta_1 - \frac{w\,x^3}{6}; \qquad (1)$$

$$0 = E\,I\,\theta_1 - \frac{w\,l^3}{6}; \qquad (2)$$

$$\theta_1 = \frac{w\,l^3}{6\,E\,I}.$$

$$E\,I\,\frac{dy}{dx} = \frac{w\,l^3}{6} - \frac{w\,x^3}{6}; \qquad (3)$$

$$E\,I\,y = E\,I\,y_1 + \left[\frac{w\,l^3x}{6} - \frac{w\,x^4}{24}\right]_0^x; \qquad (4)$$

$$E\,I\,y = E\,I\,y_1 + \frac{w\,l^3x}{6} - \frac{w\,x^4}{24}. \qquad (5)$$

When l is taken as the upper limit in Equation (4) or substituted for x in Equation (5),

$$0 = E\,I\,y_1 + \frac{w\,l^4}{8}; \tag{6}$$

$$E\,I\,y_1 = E\,I\,y_{\max} = -\frac{w\,l^4}{8};$$

$$E\,I\,y = -\frac{w\,l^4}{8} + \frac{w\,l^3x}{6} - \frac{w\,x^4}{24}; \tag{7}$$

$$y = -\frac{w}{24\,E\,I}(3\,l^4 - 4\,l^3x + x^4). \tag{8}$$

$$y_{\max} = -\frac{w\,l^4}{8\,E\,I} = -\frac{W\,l^3}{8\,E\,I}, \qquad \text{Formula XIX}$$

in which $W = w\,l =$ total load.

Problems

1. A 4-in. by 6-in. cantilever, 10 ft. long, carries a uniformly distributed load which makes the maximum stress 1,200 lb. per sq. in. Find the deflection at the end and at 40 in. from the free end if $E = 1{,}500{,}000$.
2. At what point on a uniformly loaded cantilever is the slope one-half the slope at the free end?
Ans. $x = 0.5^{1/3}\,l = 0.794\,l$.

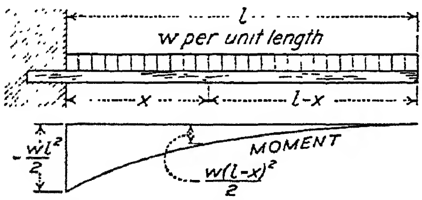


FIG. 163.—Uniformly loaded cantilever fixed at left end.

(The remainder of this article may be omitted.)

Figure 163 shows a uniformly loaded cantilever which is fixed at the left end. If a portion of length $l - x$, measured from the free end, is taken as the free body, the moment at a distance x from the fixed end is found to be

$$M = -\frac{w(l - x)^2}{2}; \tag{9}$$

$$E\,I\,\theta = \left[\frac{w\,(l - x)^3}{6}\right]_0^x; \tag{10}$$

$$E\,I\,\theta = \frac{w(l - x)^3}{6} - \frac{w\,l^3}{6}; \tag{11}$$

$$E\,I\,y = \left[-\frac{w(l - x)^4}{24} - \frac{w\,l^3x}{6}\right]_0^x; \tag{12}$$

$$E\,I\,y = -\frac{w(l - x)^4}{24} + \frac{w\,l^4}{24} - \frac{w\,l^3x}{6}. \tag{13}$$

When l is taken as the upper limit in Eq. (12) or substituted for x in Eq. (13),

$$E\,I\,y_{\max} = \frac{w\,l^4}{24} - \frac{w\,l^4}{6} = -\frac{w\,l^4}{8} = -\frac{W\,l^3}{8}.$$

From the general moment equation or from the expansion of Eq. (9)

$$M = -\frac{w\,l^2}{2} + w\,l\,x - \frac{w\,x^2}{2}, \tag{14}$$

which may be represented by a negative rectangle, a positive triangle, and a negative parabola.

Problems

3. Sketch the separate figures of Eq. (14) with the positive triangle above the rectangle and the negative parabola below. Find the expression for the slope at a distance x from the origin by means of the areas of these diagrams. Integrate this expression between limits to get the deflection at the free end and the equation of the elastic line.
4. By Eq. (8) and by Eq. (13) find the deflection at the middle and at one-third the length from the free end.

Ans. $-\dfrac{17\,W\,l^3}{384\,E\,I}$; $-\dfrac{17\,W\,l^3}{243\,E\,I}$.
5. What is the ratio of the deflection at the middle to the deflection at one-third the length from the free end? *Ans.* 81:128.

100. Simply-supported Beam, Uniformly Loaded.—Figure 164 shows a uniformly loaded beam which is supported at the ends. The reaction of each support is $\frac{w\,l}{2}$.

$$M = \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}. \tag{1}$$

The moment diagram consists of a positive triangle and a negative parabola. The resultant moment diagram is a parabola

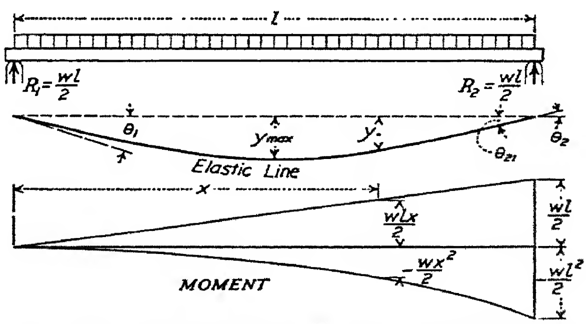


FIG. 164.—Simply-supported beam uniformly loaded.

which is convex upward and has its maximum altitude at the middle of the beam.

Using the area of the diagrams instead of integrating:

$$E\,I\,\theta = E\,I\,\theta_1 + \frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6}. \tag{2}$$

It is evident from symmetry that the beam is horizontal at the middle where $x = \frac{l}{2}$.

$$0 = E\,I\,\theta_1 + \frac{w\,l^3}{16} - \frac{w\,l^3}{48};$$

$$0 = E\,I\,\theta_1 + \frac{w\,l^3}{24}; \tag{3}$$

$$E\,I\,\theta_1 = -\frac{w\,l^3}{24} = -\frac{W\,l^2}{24}. \tag{4}$$

$$E\,I\,y = \left[-\frac{w\,l^3x}{24} + \frac{w\,l\,x^3}{12} - \frac{w\,x^4}{24}\right]_0^x; \tag{5}$$

$$y = -\frac{w\,x}{24\,E\,I}(l^3 - 2\,l\,x^2 + x^3). \tag{6}$$

$$y_{\max} = -\frac{5\,w\,l^4}{384\,E\,I} = -\frac{5\,W\,l^3}{384\,E\,I}. \qquad \text{Formula XX}$$

The slope at the right end may be found from the slope at the left end and the area of the moment diagram.

$$E\,I\,\theta_2 = -\frac{w\,l^3}{24} + \frac{w\,l^3}{4} - \frac{w\,l^3}{6} = \frac{w\,l^3}{24}, \tag{7}$$

in which θ_2 is the slope at the right support with x in the positive direction. In the negative direction, from the support toward the span, $E\,I\,\theta_{21} = -\frac{w\,l^3}{24}$. In this book, toward the right slope will be designated by the subscript of the support. Toward the left, the slope will be designated by the subscript of the support followed by the subscript of the preceding support.

Problems

1. A 12-in. by 10-in. 64-lb. wide-flange beam rests on supports which are 15 ft. apart and carries a uniform load of 3,000 lb. per ft. over the entire span. Find the deflection at the middle caused by this load if E is 29,400,000. Find the maximum unit stress.

Ans. $y_{\max} = 0.220$ in.; $S = 11{,}800$ lb./in.^{2}

2. Find the deflection of the beam of Problem 1 at 5 ft. from the left support.

3. The beam of Problem 1 overhangs the left support 5 ft. How much is the free end deflected upward when the load is applied to the span?

Ans. 0.235 in.

4. A horizontal push of 3,000 lb. is applied at the free end of the beam of Problem 3. Find the maximum stress caused by this force.

5. A 6-in. by 10-in. beam of select dry oak, 20 ft. long between supports, carries a uniformly distributed load which makes the unit stress the maximum allowable in bending. Find the deflection at the middle and at 5 ft. from each support. Find the slope at the left support. Use the handbooks for all other data.)

101. Simply-supported Beam Loaded at Any Point.—Figure 165 shows a beam which is supported at the ends and loaded at a

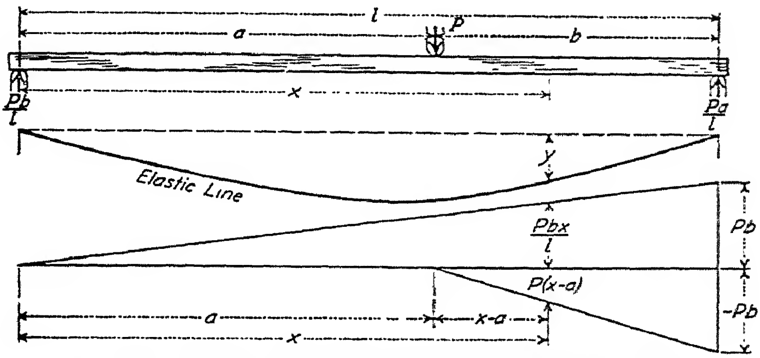


FIG. 165.—Simply-supported beam with load at any point.

point which is at a distance a from the left support and a distance b from the right support. The left reaction is $\frac{P\,b}{l}$.

$$M = \overset{0\,-\,-\,l}{\frac{P\,b\,x}{l}} - \overset{a\,-\,-\,-\,l}{P\,(x - a)}. \tag{1}$$

The broken line from 0 to l over the first term indicates that this term applies the entire length of the span. The line from a to l over the second term indicates that this term is valid from the load to the right support.

$$E\,I\,\theta = E\,I\,\theta_1 + \left[\frac{P\,b\,x^2}{2\,l}\right]_0^x - \left[\frac{P(x - a)^2}{2}\right]_a^x; \tag{2}$$

$$E\,I\,\frac{dy}{dx} = E\,I\,\theta = E\,I\,\theta_1 + \overset{0\,-\,-\,l}{\frac{P\,b\,x^2}{2\,l}} - \overset{a\,-\,-\,-\,-\,l}{\frac{P(x - a)^2}{2}}; \tag{3}$$

$$E\,I\,y = E\,I\theta\,x + \left[\frac{P\,b\,x^3}{6\,l}\right]_0^x - \left[\frac{P(x - a)^3}{6}\right]_a^x. \tag{4}$$

When l is used as the upper limit in Equation (4),

$$0 = E\,I\,\theta_1 l + \frac{P\,b\,l^2}{6} - \frac{P\,b^3}{6}; \tag{5}$$

$$0 = E\,I\,\theta_1 + \frac{P\,b}{6\,l}\,(l^2 - b^2); \tag{6}$$

$$E\,I\,\theta_1 = -\frac{P\,b}{6\,l}\,(l^2 - b^2); \tag{7}$$

$$0 - - - - - - - - - - l \qquad a - - - - l$$

$$E\,I\,y = -\frac{P\,b\,x}{6\,l}\,(l^2 - b^2 - x^2) - \frac{P}{6}\,(x - a)^3 \tag{8}$$

The terms of Equation (8) under the first line apply to the entire span. These terms give *all* the deflection between the left end and the load, and part of the deflection between the load and the right end. The last term must be added to get the entire deflection on the right of the load.

Problems

1. If the load is to the right of the middle of the beam, derive an expression for the position of maximum deflection. Use the slope expression of Eq. (3).

   Ans. $x^2 = \frac{l^2 - b^2}{3}$; $x = \sqrt{\frac{l^2 - b^2}{3}}$.

2. A beam 100 in. long carries a load 40 in. from the right end. Find the location of the lowest point.

   Ans. $x = \sqrt{2800} = 52.915$ in. from the left end

3. A 5-in. by 6-in. timber beam rests on supports 100 in. apart and carries a load of 1,200 lb. 30 in. from the right support. If $E = 1{,}000{,}000$ lb. per sq. in., find the deflection at 20 in., 40 in., 50 in., 60 in., and 70 in. from the left support.

   Ans. 0.116 in.; 0.200 in.; 0.220 in.; 0.220 in.; 0.196 in.

4. Find the deflection of the beam of Problem 3 at 80 in. from the left support by Eq. (9). Check by Eq. (8) starting from the right end.

   Ans. 0.1462 in.

5. Find θ_1 for the beam of Problem 3. Add the area of the moment diagram to find θ_2. Calculate θ_{21} with the origin of coördinates at the right support.

6. Find the maximum deflection for the beam of Problem 3.

102. Deflection in Terms of Left Reaction.—It is desirable, sometimes, to find the equation of the elastic line of a beam in terms of the reaction at one end. When the beam is supported at the left end, the equations are the same in terms of R_1 no matter what may be the conditions at the right support. In the follow-

ing illustrations, the left reaction is R_1 and the moment is zero at the left end.

Load Uniformly Distributed.

$$E\,I\,\frac{d\theta}{dx} = R_1x - \frac{w\,x^2}{2}; \tag{1}$$

$$E\,I\,\theta = E\,I\,\theta_1 + \frac{R_1x^2}{2} - \frac{w\,x^3}{6}. \tag{2}$$

Substitution of the limits 0 and x in Equation (2) makes no change in the form of the expression.

$$E\,I\,y = E\,I\,\theta_1x + \frac{R_1\,x^3}{6} - \frac{w\,x^4}{24}. \tag{3}$$

Since $y = 0$ when $x = l$, substitution of the limits 0 and l in Equation (3) gives

$$E\,I\,\theta_1 = -\frac{R_1l^2}{6} + \frac{w\,l^3}{24}. \tag{4}$$

which substituted in Equations (2) and (3) gives

$$E\,I\,\theta = -\frac{R_1l^2}{6} + \frac{w\,l^3}{24} + \frac{R_1x^2}{2} - \frac{w\,x^3}{6}; \tag{5}$$

$$E\,I\,y = -\frac{R_1l^2x}{6} + \frac{w\,l^3x}{24} + \frac{R_1x^3}{6} - \frac{w\,x^4}{24}; \tag{6}$$

$$E\,I\,y = -\frac{R_1x}{6}(l^2 - x^2) + \frac{w\,x}{24}(l^3 - x^3). \tag{7}$$

Load Concentrated.—For a beam supported at the left end with a load P at a distance a from the left support and at a distance b from the right end of the span, the equation is

$$\begin{array}{l} \quad\quad 0 - l \quad\; a - - - - l \\ E\,I\,\dfrac{d\theta}{dx} = R\,x - P(x - a); \end{array} \tag{8}$$

$$E\,I\,\theta = E\,I\,\frac{dy}{dx} = E\,I\,\theta_1 + \left[\frac{R\,x^2}{2}\right]_0^x - \left[\frac{P(x - a)^2}{2}\right]_a^x; \tag{9}$$

$$E\,I\,\frac{dy}{dx} = E\,I\,\theta_1 + \frac{R\,x^2}{2} - \frac{P(x - a)^2}{2}; \tag{10}$$

$$E\,I\,y = \left[E\,I\,\theta_1x + \frac{R\,x^3}{6}\right]_0^x - \left[\frac{P(x - a)^3}{6}\right]_a^x.$$

When l is used for the upper limit in Equation (9),

$$0 = E\,I\,\theta_1 + \frac{R\,l^2}{6} - \frac{P\,b^3}{6\,l}; \tag{11}$$

0 - - - - - - - - - - - - l a - - - - - - - l

$$E\,I\,y = -\frac{R\,x}{6}(l^2 - x^2) + \frac{P\,b^3x}{6\,l} - \frac{P\,b}{6}(x - a)^3 \tag{12}$$

Problems

1. A beam is supported at the left end and at a distance l from the left end. It carries a load of w per unit length between the supports. What is the left reaction if the beam is horizontal over the second support? Write the slope equation and the equation of the elastic line.

$$Ans.\ R_1 = \frac{3\,w\,l}{8};\ E\,I\frac{dy}{dx} = -\frac{w}{48}(8\,x^3 - 9\,l\,x^2 + l^3);$$

$$E\,I\,y = \frac{w}{48}(2\,x^4 - 3\,l\,x^3 + l^3x)$$

2. Find the position of maximum deflection for the beam of Problem 1 from the slope equation. Since the slope of the beam is zero over the second support, $x = l$ is a solution and $x - l$ is a factor. Divide by this factor and solve the quadratic. *Ans.* $x = 0.4215\,l$.
3. A uniformly loaded beam is supported at the left end and at a distance l from the left end. It overhangs the right support $0.4\,l$ and carries a concentrated load of $0.3\,w\,l$ on the right end. Find the slope at each support.

$$Ans.\ \frac{dy}{dx} = -\frac{w\,l^3}{120\,E\,I};\ -\frac{w\,l^3}{40\,E\,I}.$$

4. In Problem 3, what is the deflection at $x = 0.8\,l$?
5. A beam is supported at the left and at a distance l from the left end. It carries a load P at distance $k\,l$ from the left support. Find the left reaction if the beam is horizontal over the right support.

$$Ans.\ R_1 = \frac{P}{2}(2 - 3\,k + k^3).$$

103. Cantilever Partly Loaded.—Figure 166 shows a beam which is fixed at the left end and subjected to a load P at a distance b from the fixed end and a distance a from the free end. If the portion of the beam to the right of any section is taken as the free body,

$$M = -P(b - x) \tag{1}$$

from 0 to b. Beyond b the moment is 0.

$$E\,I\,\theta = \left[\frac{P(b - x)^2}{2}\right]_0^x; \tag{2}$$

0 - - - - - b 0 - l

$$E\,I\frac{dy}{dx} = \frac{P(b - x)^2}{2} - \frac{P\,b^2}{2}. \tag{3}$$

When $x = 0$, $E\,I\dfrac{dy}{dx} = 0$; when $x = b$, $E\,I\dfrac{dy}{dx} = -\dfrac{P\,b^2}{2}$

and remains unchanged throughout the remainder of the beam

$$E\,I\,y = -\left[\frac{P\,(b - x)^3}{6}\right]_0^x - \left[\frac{P\,b^2x}{2}\right]_0^x. \tag{4}$$

For the first integral, x cannot exceed b. For the second term x may extend to l. The general equation is

$$\begin{array}{cccc} & 0 - - - b & 0 - l & 0 - l \\ E\,I\,y = & -\dfrac{P(b - x)^3}{6} & + \dfrac{P\,b^3}{6} & - \dfrac{P\,b^2x}{2}. \end{array} \tag{5}$$

When $x = b$, $E\,I\,y = 0 + \dfrac{P\,b^3}{6} - \dfrac{P\,b^3}{2} = -\dfrac{P\,b^3}{3}$,

which is the deflection of a beam of length b which is loaded at the free end. When x is greater than b, only the last two terms of Equation (5) are used.

$$E\,I\,y_{\max} = \frac{P\,b^3}{6} - \frac{P\,b^2l}{2} = -\frac{P\,b^3}{3} - \frac{P\,b^2a}{2}. \tag{6}$$

The last form of Equation (6) represents the deflection under the load plus the slope at the load times the length of the portion which is not loaded.

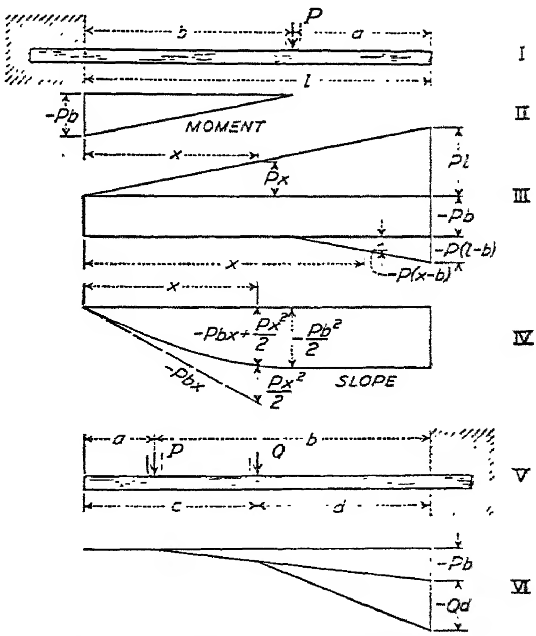


FIG. 166.—Cantilever variously loaded.

Figure 166, II, gives the moment diagram for the beam with the portion to the right of the section taken as the free body. Figure 166, III, gives the components of the diagram with the portion to the left of the section taken as the free body. By the general moment equation

$$M = -P\,b + P\,x - P(x - b), \tag{7}$$

which are represented by a negative rectangle, a positive triangle, and a negative triangle. The resultant of these, of course, is Fig. 166, II.

$$E\,I\,\theta = \overset{0\text{ - - - - - }l}{-P\,b\,x + \frac{P\,x^2}{2}} - \overset{b\text{ - - - - }l}{\frac{P(x-b)^2}{2}}; \tag{8}$$

$$E\,I\,y = \overset{0\text{ - - - - - - }l}{-\frac{P\,b\,x^2}{2} + \frac{P\,x^3}{6}} - \overset{b\text{ - - - - }l}{\frac{P(x-b)^3}{6}}. \tag{9}$$

Since inspection shows that substitution of the limits does not change the form of the expression, one step is omitted at each integration.

$$E\,I\,y_{\max} = -\frac{P\,b\,l^2}{2} + \frac{P\,l^3}{6} - \frac{P\,a^3}{6}. \tag{10}$$

When $b + a$ is substituted for l in Equation (10), the result is

$$E\,I\,y_{\max} = -\frac{P\,b^3}{3} - \frac{P\,b^2 a}{2}. \tag{6}$$

Figure 166, V, shows a cantilever fixed at the right end with two concentrated loads.

$$M = \overset{a\text{ - - - - }l}{-P\,(x-a)} - \overset{c\text{ - - - }l}{Q(x-c)} \tag{11}$$

Since substitution of limits makes no change in the expression, substitution is omitted in the derivation which follows.

$$E\,I\,\theta = E\,I\,\theta_1 - \overset{a\text{ - - - - }l}{\frac{P\,(x-a)^2}{2}} - \overset{c\text{ - - - - }l}{\frac{Q(x-c)^2}{2}}; \tag{12}$$

$$E\,I\,\theta_1 = \frac{P\,b^2}{2} + \frac{Q\,d^2}{2}; \tag{13}$$

$$E\,I\,\frac{dy}{dx} = \frac{P\,b^2}{2} + \frac{Q\,d^2}{2} - \overset{a\text{ - - - - }l}{\frac{P(x-a)^2}{2}} - \overset{c\text{ - - - - }l}{\frac{Q(x-c)^2}{2}}; \tag{14}$$

$$E\,I\,y = E\,I\,y_1 + \overset{0\text{ - - - - - - }l}{\frac{P\,b^2 x}{2} + \frac{Q\,d^2 x}{2}} - \overset{a\text{ - - - - }l}{\frac{P(x-a)^3}{6}} - \overset{c\text{ - - - - }l}{\frac{Q(x-c)^3}{6}}; \tag{15}$$

$$E\,I\,y_1 = -\frac{P\,b^2 l}{2} - \frac{Q\,d^2 l}{2} + \frac{P\,b^3}{6} + \frac{Q\,d^3}{6}; \tag{16}$$

$$E\,I\,y = \overset{0\text{ - }l}{-\frac{P\,b^2(l-x)}{2} - \frac{Q\,d^2(l-x)}{l} + \frac{P\,b^3}{6} + \frac{Q\,d^3}{6}} - \overset{a\text{ - - - }l}{\frac{P(x-a)^3}{6}} - \overset{c\text{ - - - }l}{\frac{Q(x-c)^3}{6}}. \tag{17}$$

Problems

1. A 4-in. by 6-in. timber cantilever is 10 ft. long and carries a load at 30 in. from the free end which makes the maximum unit stress 1,080 lb. per sq.

in. Find the deflection at the middle, under the load, 10 in. from the free end, and at the free end, if $E = 1{,}500{,}000$ lb. per sq. in.

Ans. -0.336 in.; -0.648 in.; -0.864 in.; -0.972 in.

2. A 3-in. 6.5-lb. I-beam, as a cantilever fixed at the right end, projects 10 ft. and carries 200 lb. 4 ft. from the left end and 300 lb. 2 ft. 6 in. from the fixed end. If $E = 29{,}400{,}000$, find the deflection at the free end which is caused by these loads. Find the maximum fiber stress including the weight of the beam. *Ans.* $y_{max} = 0.814$ in.

3. In Problem 2, find the deflection 20 in. from the free end and at the middle. *Ans.* 0.649 in.; 0.321 in.

Figure 167, I, shows a cantilever fixed at the left end, which carries a uniformly distributed load over a length b adjacent to the fixed end. When

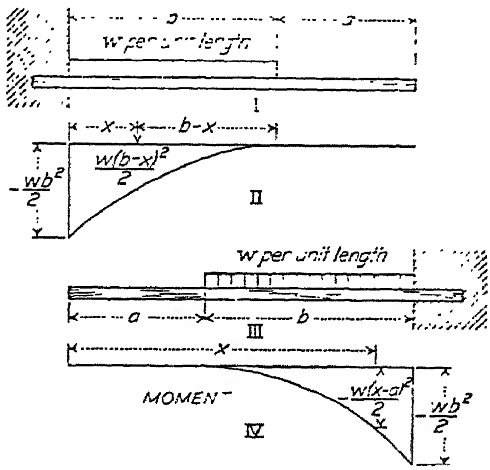


FIG. 167.—Cantilevers partly loaded.

a portion of length $l - x$, to the right of a section at a distance x from the origin, is taken as the free body, the moment is

$$M = -\frac{w(b-x)^2}{2} \tag{18}$$

from $x = 0$ to $x = b$.

$$E\,I\,\theta = \left[\frac{w(b-x)^3}{6}\right]_0^x; \tag{19}$$

0 - - - b 0 - l

$$E\,I\frac{dy}{dx} = \frac{w(b-x)^3}{6} - \frac{w\,b^3}{6}; \tag{20}$$

$$E\,I\,y = -\left[\frac{w(b-x)^4}{24}\right]_0^x - \left[\frac{w\,b^3x}{6}\right]_0^x; \tag{21}$$

0 - - - - b 0 - l 0 - - l

$$E\,I\,y = -\frac{w(b-x)^4}{24} + \frac{w\,b^4}{24} - \frac{w\,b^3x}{6}. \tag{22}$$

To get y_{max}, the first term of Equation (22) is not used, since this term applies only to $x = b$.

$$E\,I\,y_{max} = \frac{w\,b^4}{24} - \frac{w\,b^3 l}{6} = -\frac{w\,b^4}{8} - \frac{w\,b^3\,a}{6}. \tag{23}$$

Figure 167, III, shows a cantilever fixed at the right end which carries a uniformly distributed load over a length b adjacent to the fixed end.

$$M = -\frac{w(x-a)^2}{2}; \tag{24}$$

$$E\,I\,\theta = E\,I\,\theta_1 - \overset{a\,-\,-\,-\,-\,-\,l}{\frac{w(x-a)^3}{6}}; \tag{25}$$

$$E\,I\,\theta_1 = \frac{w\,b^3}{6}; \tag{26}$$

$$E\,I\,\frac{dy}{dx} = \frac{w\,b^3}{6} - \overset{a\,-\,-\,-\,-\,l}{\frac{w(x-a)^3}{6}}; \tag{27}$$

$$E\,I\,y = E\,I\,y_1 + \overset{0\,-\,-\,l}{\frac{w\,b^3\,x}{6}} - \overset{a\,-\,-\,-\,-\,l}{\frac{w(x-a)^4}{24}}; \tag{28}$$

$$0 = E\,I\,y_1 + \frac{w\,b^3\,l}{6} - \frac{w\,b^4}{24}; \tag{29}$$

$$E\,I\,y_{max} = E\,I\,y_1 = -\frac{w\,b^4}{8} - \frac{w\,b^3\,a}{6}; \tag{30}$$

$$E\,I\,y = -\frac{w\,b^4}{8} + \overset{0\,-\,-\,-\,-\,-\,l}{\frac{w\,b^3\,(x-a)}{6}} - \overset{a\,-\,-\,-\,-\,l}{\frac{w(x-a)^4}{24}} \tag{31}$$

The method of Fig. 167, II [Equations (18) to (23)], in which the cantilever is fixed at the left end, is not so good as that of Fig. 167, III [Equations (24) to (31)], in which the cantilever is fixed at the right end. The lower limit of the integral of Equation (19) gives a value of $-\frac{w\,b^3}{6}$ which somewhat complicates the problem. All the integrals in Equations (24) to (31) have zero as the value at the lower limit. For integration between limits it is best to set up the equations in such a way that each expression will have its upper limit at the right end of the span and will have its value zero at the lower limit. When a load does not extend to the end of the span, it may be assumed to go that far and to be balanced by an imaginary upward force.

Problems

4. A cantilever of length l carries a uniformly distributed load over 0.6 l adjacent to the fixed end. Find the maximum deflection.

Ans. $y_{max} = \frac{0.0306\,w\,l^4}{E\,I}$.

5. A cantilever of length l carries a uniformly distributed load over 0.4 l adjacent to the free end. Find the deflection at the free end by means of the answer to Problem 4.

6. In Problem 4, find the deflection $0.2\,l$ from the free end. Solve by one equation and check by another.

$$Ans.\ y = -\frac{0.0234\,w\,l^4}{E\,I}.$$

7. Find the deflection at the middle of the beam of Problem 4. Check.

$$Ans.\ y = -\frac{0.012604\,w\,l^4}{E\,I}.$$

8. A cantilever of length l carries a load of w per unit length over $0.5\,l$ which begins $0.2\,l$ from the fixed end. Find the deflection at the free end and at the middle. Check. Imagine that the load is extended to the fixed end.

104. Continued Integrations.—It has been shown in preceding articles that the derivative of the deflection is the slope, the derivative of the slope is the moment, and the derivative of the moment is the shear.

$$\frac{dy}{dx} = \text{slope};\quad \frac{d^2y}{dx^2} = \frac{\text{moment}}{E\,I};\quad \frac{d^3y}{dx^3} = \frac{\text{shear}}{E\,I}.$$

To these may be added $E\,I\,\dfrac{d^4y}{dx^4} = \dfrac{d^2M}{dx^2} = \dfrac{dV}{dx} = w$, in which w is the load per unit length.

In order to differentiate or integrate an expression, the function must be continuous. A concentrated load is, apparently, not

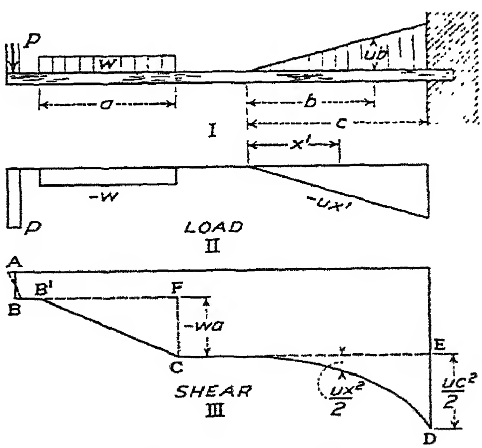


FIG. 168.—Types of loading.

continuous. It is customary to represent a concentrated load by a line. In reality, a concentrated load or reaction is distributed over an area. The concentrated load of Fig. 168 is represented on the load diagram by a rectangle (which, of course, it is not).

If it were a rectangle, the shear diagram would be the diagonal broken line of the figure. It is customary to use a vertical line on the shear diagram in line with the resultant "concentrated" load. At any rate, the shear after passing the load is equal to the total load or the area of the actual load diagram whatever its form may be. Figure 168 shows a uniformly distributed load,

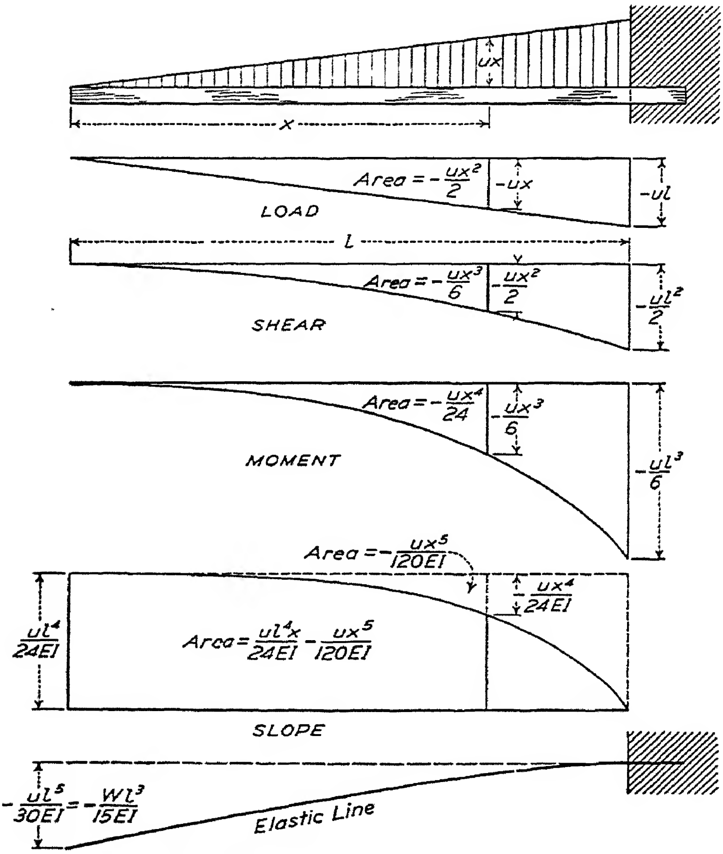


Fig. 169.—Uniformly increasing load.

for which the shear diagram $B'C$ is an inclined straight line. A uniformly increasing load is shown on the right end of the beam of Fig. 168. If $u\,b$ is the load per unit length at a distance b from the beginning, the equation of the load diagram is $-u\,x'$. The additional shear caused by this load at a distance x' from the beginning is the area of the load triangle of length x', which is $-\frac{u\,x'^2}{2}$. For most deflection problems, the moment equation

may be written from the loads and reactions. For some problems, it is convenient to begin with the shear, and for others it is necessary to integrate from the load.

Figure 169 shows a cantilever with uniformly increasing load.

$$V = -\int u\,x\,dx = -\left[\frac{u\,x^2}{2}\right]_0^x = -\frac{u\,x^2}{2}, \tag{1}$$

which might have been calculated by graphic integration from the area of the load triangle.

$$M = -\int \frac{u\,x^2}{2}dx = -\left[\frac{u\,x^3}{6}\right]_0^x = -\frac{u\,x^3}{6}, \tag{2}$$

which might have been calculated by graphic integration from the area of the load parabola.

$$E\,I\,\theta = E\,I\,\theta_1 - \left[\frac{u\,x^4}{24}\right]_0^x = E\,I\,\theta_1 - \frac{u\,x^4}{24}; \tag{3}$$

$$E\,I\,\theta_1 = \frac{u\,l^4}{24}; \quad \theta_1 = \frac{u\,l^4}{24\,E\,I}. \tag{4}$$

$$E\,I\frac{dy}{dx} = \frac{u\,l^4}{24} - \frac{u\,x^4}{24}; \tag{5}$$

$$E\,I\,y = E\,I\,y_1 + \left[\frac{u\,l^4x}{24} - \frac{u\,x^5}{120}\right]_0^x, \tag{6}$$

$$E\,I\,y = E\,I\,y_1 + \frac{u\,l^4x}{24} - \frac{u\,x^5}{120};$$

$$E\,I\,y_1 = -\frac{u\,l^5}{30}; \quad y_{max} = -\frac{u\,l^5}{30\,E\,I} = -\frac{W\,l^3}{15\,E\,I}, \tag{7}$$

in which $W = \frac{u\,l^2}{2}$, the total load on the beam.

$$E\,I\,y = -\frac{u\,l^5}{30} + \frac{u\,l^4x}{24} - \frac{u\,x^5}{120}. \tag{8}$$

Problems

1. A cantilever of length l carries a load which increases uniformly from the free end to the fixed end. Find the deflection at the middle.

$$Ans.\ y = -\frac{49\,u\,l^5}{3{,}840\,E\,I} = -\frac{49\,W\,l^3}{1{,}920\,E\,I}.$$

2. A cantilever of length l carries a load which increases uniformly from the fixed end to the free end. Find the deflection at the free end, using Eq. (7).

$$Ans.\ y_{max} = -\frac{11\,W\,l^3}{60\,E\,I}.$$

*

CHAPTER X

DEFLECTION BY AREA MOMENTS

105. Method of Area Moments.—When a beam is fixed at one end or at both ends, making the slope zero at these points, the calculation of deflection by "area moments"* has some advantages. Where there is more than one load on a span, the method is better than successive integration with arbitrary constants but is not superior to successive integration between limits, as given in Chapter IX. Most calculations of area moments may be made by "graphic integration." This is the determination of the areas and the moment of the areas of moment diagrams. Since moment diagrams are triangles, parabolas, or rectangles, or combinations of these figures, most of these calculations can be made without the use of calculus.

From the equation $M = E\,I\,\frac{d\theta}{dl} = E\,I\,\frac{d\theta}{dx}$, when θ is a small angle

$$d\theta = \frac{M}{E\,I}\,dx; \tag{1}$$

$$\theta = \int \frac{M}{E\,I}\,dx + C = \frac{1}{E\,I}\int M dx + C, \tag{2}$$

when I is constant.

Since $\int M\,dx$ is the area of the moment diagram, *the difference in slope between two points on a beam of uniform section is the area of the moment diagram between these points divided by $E\,I$.* If the moment of inertia varies, the difference of slope is the area of the $\frac{M}{I}$ diagram divided by E.

In Fig. 170, A_0B_0 represents a portion of a beam. The line $A_1B_1B_0$, which is straight from A_1 to B_1 and curved from B_1 to

* This method was devised by Mohr and independently in America by Prof. Charles E. Greene, who began to teach it in 1873. See paper by A. E. Greene in the *Michigan Technic* of June, 1910.

B_0, represents the same beam after the portion from B_1 to B_0 has been bent by a moment M. If the short portion between B_1 and B_2 is now bent by a moment M, the point A_1 moves to A_2. The angle between the tangent A_1B_1 and the tangent A_2B_2 (or the angle between the normals at B_1 and B_2) is $d\theta$. (It is assumed that the angles are so small that the angle θ is practically equivalent to tangent of θ, and the points $A_0A_1A_2$ lie on

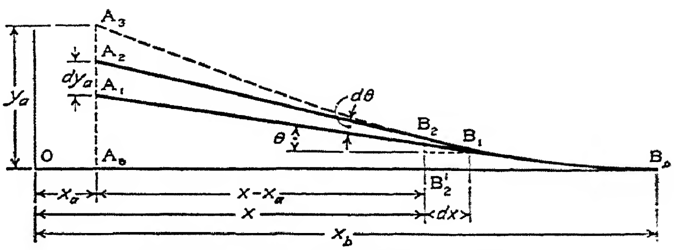


FIG. 170.—Derivation of area moments.

a straight line which is approximately vertical.) The deflection dy_a, from A_1 to A_2, when the infinitesimal length B_1B_2 is bent, is $A_2B_2\,d\theta$ (or $A_1B_1\,d\theta$, since B_1B_2 is infinitesimal). Since θ is small, A_2B_2 is practically equivalent to its horizontal projection A_0B_2', the length of which is $x - x_a$.

$$dy_a = (x - x_a)d\theta; \tag{3}$$

$$dy_a = \frac{M}{EI}(x - x_a)dx. \tag{4}$$

The total deflection of the point A is

$$y_a = \int \frac{M}{EI}(x - x_a)dx. \qquad \text{Formula XXIII}$$

For a uniform beam

$$E\,Iy_a = \int M(x - x_a)\,dx. \qquad \text{Formula XXIV}$$

When the origin of coördinates is taken at the point A_0,

$$E\,Iy_a = \int M\,x\,dx. \qquad \text{Formula XXV}$$

Since $M\,dx$ is an element of the moment diagram, $(x - x_a)$ $M\,dx$ is the moment of this element with respect to the line $A_0A_1A_2$. The integrals of Formulas XXIV and XXV give the moment with respect to A of the entire moment diagram from A to B_0. *The distance y_a is the deflection of the point A from the*

tangent at B_0. This is called the *Area-Moments Method, the Slope-Deflection Method,* or the *Integral of $M\ x\ dx$ Method.*

For a concentrated load or reaction, the moment diagram is a triangle; for a uniformly distributed load it is a parabola. Since the area and location of the center of gravity of these figures are known, the moment is usually computed geometrically without integration. This method of replacing $\int x\, f(x)\, dx$ by the moment of the area $\int f(x)\, dx$ is a form of graphic integration.

$$\int x\, f(x)\, dx = \bar{x} \int f(x)\, dx.$$

If $\int f(x)\, dx$ may be represented by a plane figure, of which the area and the location of the center of gravity are known, $\bar{x} \int f(x)\, dx$ may be evaluated arithmetically. When $\int f(x)\, dx$ cannot be represented by a simple plane figure, it is best to integrate.

106. Cantilever Loaded at Free End.—Figure 171 shows a cantilever with a load P on the free end. The figure shows also

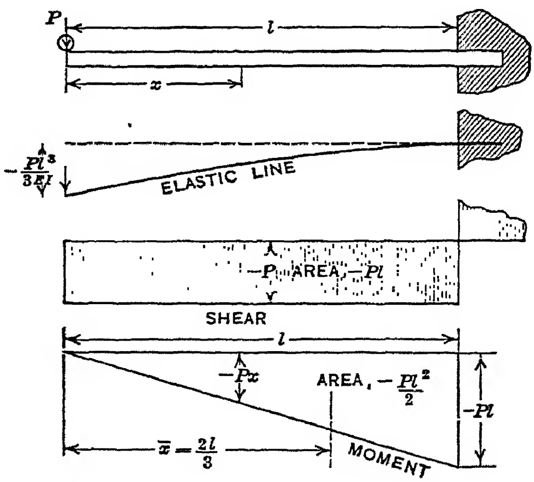


FIG. 171.—Area moments for cantilever.

the elastic line with deflections magnified, the shear diagram, and the moment diagram. The moment diagram is a negative triangle. The maximum ordinate is $-P\,l$, which is the area of the shear diagram. The area of the moment triangle is $-\dfrac{P\,l^2}{2}$

and its center of gravity is $\frac{2\,l}{3}$ from the left end. The deflection of the left end from the horizontal tangent at the right end is given by

$$E\,I\,y_{max} = -\frac{P\,l^2}{2} \times \frac{2\,l}{3} = -\frac{P\,l^3}{3}; \tag{1}$$

$$y_{max} = -\frac{P\,l^3}{3\,E\,I}. \qquad \text{Formula XVIII}$$

The change in slope from the free end to the fixed end is the area of the moment diagram divided by $E\,I$. Since the beam is horizontal at the fixed end, the slope at the free end is

$$\theta - \frac{P\,l^2}{2\,E\,I} = 0, \tag{2}$$

$$\theta = \frac{P\,l^2}{2\,E\,I}. \tag{3}$$

(Compare with Art. 89.)

Figure 172 is given to find the equation of the elastic line. The deflection is found for *any* point at a distance a from the free end.

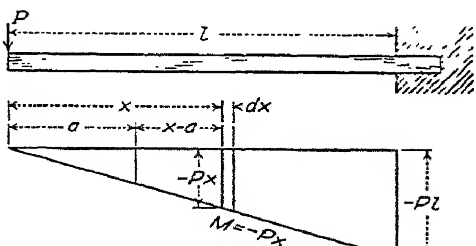


FIG. 172.—Moment diagram for equation of elastic line.

The coördinate is expressed by a instead of x since x is used for the element of the diagram. Formula XXIII is written

$$E\,I\,y_a = -P\int_a^l x(x - a)dx; \tag{4}$$

$$E\,I\,y_a = -P\left[\frac{x^3}{3} - \frac{a\,x^2}{2}\right]_a^l = -\frac{P}{6}(2\,l^3 - 3\,a\,l^2 + 5a^3). \tag{5}$$

When a is replaced by x in the parenthesis of Equation (5), the result is Equation (8) of Art. 89.

Figure 173 is the diagram for finding the deflection at a distance x from the free end by graphic integration. The trapezoid to the

right of the point B may be broken up into the rectangle of base $l - x$ and altitude $-P\,x$, and the lower triangle of base $l - x$ and altitude $-P(l - x)$. The moment arm of the rectangle is $\frac{l - x}{2}$ and that of the triangle is $\frac{2(l - x)}{3}$.

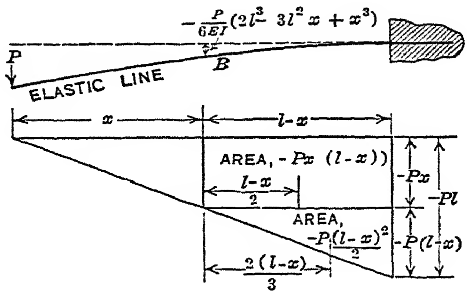


FIG. 173.—Area moments for any point of cantilever.

$$\text{Moment of rectangle} = -P\,x(l - x) \times \frac{l - x}{2} = -\frac{P\,x}{2}(l - x)^2. \qquad (6)$$

$$\text{Moment of triangle} = -\frac{P(l - x)^2}{2} \times \frac{2(l - x)}{3} = -\frac{P(l - x)}{3}(l - x)^2. \qquad (7)$$

$$\text{Total moment} = E\,I\,y = -\frac{P\,l^3}{3} + \frac{P\,l^2x}{2} - \frac{P\,x^3}{6}. \qquad (8)$$

$$y = -\frac{P}{6\,E\,I}(2\,l^3 - 3\,l^2x + x^3). \qquad (9)$$

Instead of the *sum* of the moments of the triangle and the rectangle to the right of B, the *difference* of the moments of the entire triangle and the small triangle to the left of B may be used. The area of the large triangle is $-\frac{P\,l^2}{2}$, and its center of gravity is $\frac{2\,l}{3} - x$ to the *right* of B. The area of the small triangle is $-\frac{P\,x^2}{2}$ and its center of gravity is $\frac{x}{3}$ to the *left* of B.

$$-\frac{P\,l^2}{2} \times \left(\frac{2\,l}{3} - x\right) = -\frac{P\,l^3}{3} + \frac{P\,l^2x}{2}; \qquad (10)$$

$$-\left(-\frac{P\,x^2}{2}\right) \times \left(-\frac{x}{3}\right) = -\frac{P\,x^3}{6}; \qquad (11)$$

$$y = -\frac{P}{6\,E\,I}(2\,l^3 - 3\,l^2x + x^3). \qquad (9)$$

Problems

1. Derive the equation of the elastic line for a cantilever by dividing the trapezoid of Fig. 173 into two triangles.
2. Find the slope of the elastic line at a distance x from the free end by means of the area of the trapezoid.
3. Differentiate Eq. (9) to find the slope of the tangent to the elastic line. Compare with answer of Problem 2.
4. A beam of length l carries a load P on the free end. Find the deflection at 0.4 l from the free end by the moment of the moment diagram. Divide trapezoid into a triangle and a rectangle. Also divide trapezoid into two triangles.
5. A 4-in. by 6-in. beam, 15 ft. long, is fixed at the right end. It carries a load on the free end which makes the maximum stress 1,080 lb. per sq. in. Find the deflection at the middle if E = 1,200,000 lb. per sq. in. Solve by area moments from the numerical moment diagram.

$$\textit{Ans. } y = \frac{17{,}496{,}000 + 69{,}984{,}000}{72 \times 1{,}200{,}000} = 1.0125 \text{ in.}$$

6. A cantilever 5 ft. long is deflected 2.4 in. at the free end. How much is it deflected 20 in. from the free end? Solve by moments of moment diagram without use of formulas.

107. Cantilever Loaded at Any Point.—Figure 174 shows a cantilever of length l, which carries a load P at a distance a from the free end. The area of the moment triangle is $-\frac{P(l-a)^2}{2}$. The moment arm for finding the deflection at the free end is $a + \frac{2(l-a)}{3}$, which reduces to $\frac{2\,l + a}{3}$.

$$E\,I\,y = -\frac{P(l-a)^2}{2} \times \frac{2\,l+a}{3} = -\frac{P}{6}(2\,l^3 - 3\,l^2a + a^3); \qquad (1)$$

$$y_{\max} = -\frac{P}{6\,E\,I}(2\,l^3 - 3\,l^2a + a^3). \qquad (2)$$

When x is substituted for a in Equation (2), the result is identical with Equation (9) of the preceding article. This identity is an illustration of Maxwell's theorem.

The slope under the load, or at any point to the left of the load, is equal to the area of the moment diagram with the positive sign divided by $E\,I$. This slope multiplied by a gives the vertical distance of the end of the beam below the load.

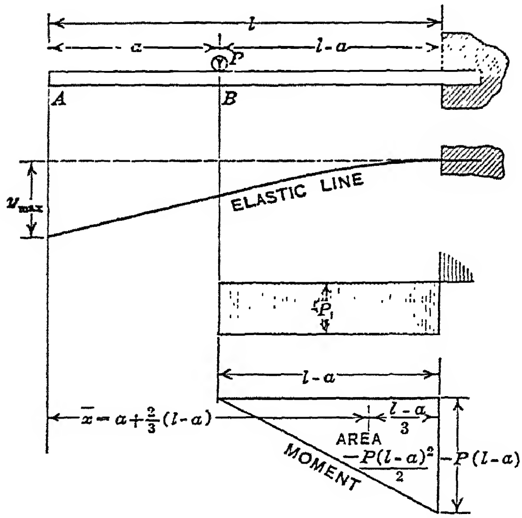


FIG. 174.—Cantilever with load at any point.

Problems

1. A 4-in. by 4-in. cantilever, 15 ft. long, carries a load of 40 lb. 5 ft. from the free end and a load of 60 lb. 10 ft. from the free end. Find the deflection at the free end if E is 1,350,000 lb. per sq. in. Solve from the two moment triangles without using Eq. (2). *Ans.* $y_{max} = 2$ in.
2. Derive Eq. (2) for the deflection at the end caused by a load P at a distance a from the end by integrating $M\,x\,dx$ between the proper limits.

108. Cantilever Uniformly Loaded.—Figure 175 shows a cantilever with uniformly distributed load of w per unit length. The shear diagram is a triangle of maximum altitude $-w\,l$ and area $-\frac{w\,l^2}{2}$. The moment diagram is a parabola, the equation of which is $M = -\frac{w\,x^2}{2}$. The maximum ordinate of the moment diagram is $-\frac{w\,l^2}{2}$. The area of a parabola which is convex toward the base is one-third the product of the base by the maximum height.

$$\text{Area of moment diagram} = -\frac{w\,l^2}{2} \times \frac{l}{3} = -\frac{w\,l^3}{6}. \qquad (1)$$

Since the slope at the fixed end is zero, the slope at the free end is given by $\theta_1 - \frac{w\,l^3}{6\,E\,I} = 0;$

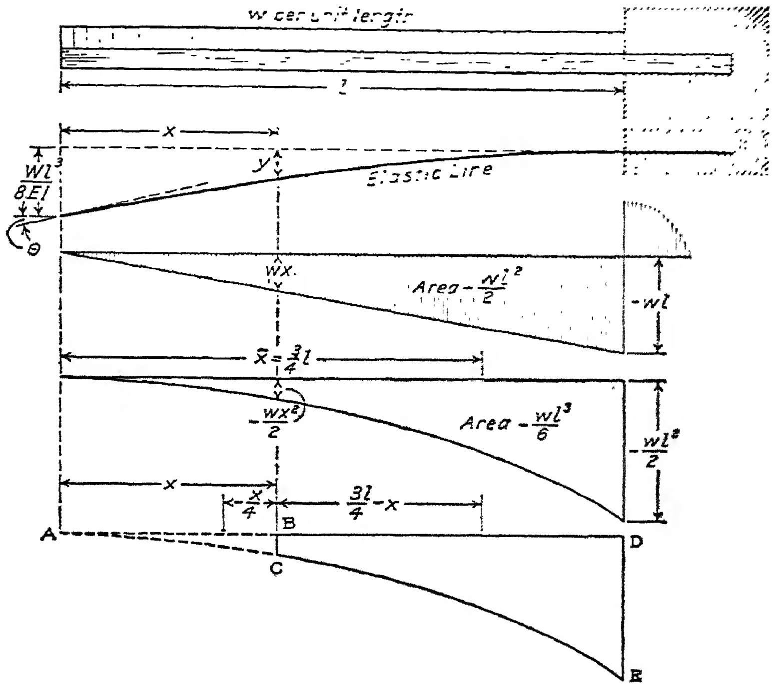


FIG. 175.—Cantilever with uniformly distributed load.

$$\theta_1 = \frac{w\, l^3}{6\, E\, I}. \tag{2}$$

The distance of the center of gravity of this parabola from the vertex is $\frac{3\, l}{4}$, which corresponds to the center of gravity of a pyramid or cone.

To find the deflection at the free end,

$$E\, I\, y_{max} = -\frac{w\, l^3}{6} \times \frac{3\, l}{4} = -\frac{w\, l^4}{8}, \tag{3}$$

$$y_{max} = -\frac{w\, l^4}{8\, E\, I} = -\frac{W\, l^3}{8\, E\, I}, \quad \text{Formula XIX}$$

in which $W = w\, l =$ total distributed load.

To derive the equation of the elastic line by Formula XXIII, moments are taken about any point at a distance a from the free end. The vertical element of area of the moment diagram

(not shown in Fig. 175) is $-\frac{w\,x^2}{2}$ high and dx wide. Its moment arm with respect to the point at a distance a from the free end is $x - a$.

$$E\,I\,y = -\frac{w}{2}\int_a^l x^2(x - a)dx = -\frac{w}{2}\left[\frac{x^4}{4} - \frac{a\,x^3}{3}\right]_a^l; \tag{4}$$

$$E\,I\,y = -\frac{w}{24}(3\,l^4 - 4\,l^3\,a + a^4). \tag{5}$$

Substituting x for a:

$$y = -\frac{w}{24\,E\,I}(3\,l^4 - 4\,l^3x + x^4). \tag{6}$$

To find the deflection at any point by graphic integration, the moment of the parabola to the left of B (Fig. 175) is subtracted from the moment of the entire parabola $A\ D\ E$. The area of the entire parabola is $-\frac{w\,l^3}{6}$, and its moment arm with respect to $B\ C$ is $\frac{3\,l}{4} - x$. The area of the small parabola $A\ B\ C$ is $-\frac{w\,x^3}{6}$, and its moment arm, measured from $B\ C$ toward the *left* is $-\frac{x}{4}$.*

$$\left(-\frac{w\,l^3}{6}\right)\left(\frac{3\,l}{4} - x\right) = -\frac{w\,l^4}{8} + \frac{w\,l^3x}{6}; \tag{7}$$

$$-\left(-\frac{w\,x^3}{6}\right)\left(-\frac{x}{4}\right) = -\frac{w\,x^4}{24}; \tag{8}$$

$$E\,I\,y = -\frac{w}{24}(3\,l^4 - 4\,l^3x + x^4); \tag{9}$$

$$y = -\frac{w}{24\,E\,I}(3\,l^4 - 4\,l^3x + x^4), \tag{10}$$

which is the equation of the elastic line (compare Art. 92).

In Fig. 173, the trapezoid is divided into a rectangle and a triangle. It might have been divided into two triangles. The truncated parabola of Fig. 175 may be divided into a rectangle, a

* The positive direction of moment arm is from the point about which moment is taken toward the point of tangency of the line from which deflections are measured. The direction from B toward D of Fig. 175 is positive.

triangle, and a parabola, which are not shown on the drawing. The base of each of these areas has the length $l - x$, which is equal to $B\ D$ of the figure. These terms may be derived from the general moment equation

$$M = M_0 + V_0 x - \frac{w\,x^2}{2}.$$

If the origin of coördinates is taken at the line $C\ B$,

$$M_0 = -\frac{w\,x^2}{2};\ V_0 = -w\,x;$$

and x of the general moment equation is replaced by $l - x$ to give the entire ordinate $E\ D$ at the end of the truncated parabola. The area of the rectangle is $-\frac{w\,x^2(l-x)}{2}$; the area of the triangle is $-\frac{w\,x(l-x)^2}{2}$; and the area of the parabola is $-\frac{w(l-x)^3}{6}$. To find the deflection at B from the fixed tangent at the fixed end,

$$E\,I\,y = -\frac{w\,x^2}{2}(l-x) \times \frac{l-x}{2} - w\,x(l-x)\frac{l-x}{2} \times \frac{2(l-x)}{3} - \frac{w(l-x)^2}{2}\frac{(l-x)}{3} \times \frac{3(l-x)}{4}; \quad (11)$$

$$E\,I\,y = -w(l-x)^2\left(\frac{x^2}{4} + \frac{x(l-x)}{3} + \frac{(l-x)^2}{8}\right); \quad (12)$$

$$y = -\frac{w(l-x)^2}{24\,E\,I}(6\,x^2 + 8\,l\,x - 8\,x^2 + 3\,l^2 - 6\,l\,x + 3\,x^2); \quad (13)$$

$$y = -\frac{w}{24\,E\,I}(x^4 - 4\,l^3x + 3\,l^4). \quad (10)$$

Problems

1. A cantilever 12 in. long carries a load of 6 lb. per in. Find the deflection 4 in. from the free end. Solve by means of moments of areas of moment diagram without using Eq. (6).

   Ans. $E\,I\,y = -1{,}728 \times 5 - (-64)(-1) = -8{,}704;\ y = -\frac{8{,}704}{E\,I}$.

2. An 8-in. cantilever carries a load of 24 lb., uniformly distributed. Find the deflection 3 in. from the free end from the diagram without using Eq. (6).

   Ans. $y = -\frac{778.125}{E\,I}$.

3. A 6-in. by 4-in. timber cantilever, 10 ft. long, carries a distributed load of 36 lb. per ft. If $E = 1{,}500{,}000$, find the deflection at the free end and find the maximum stress. *Ans.* $y = 1.62$ in.; $S = 1{,}350$ lb./in.^{2}
4. In Problem 3, find the slope at the left end and at the middle from the moment diagram.

Ans. $\frac{dy}{dx} = 0.018;\ 0.01575.$
5. A 3-in. 6-lb. standard channel, with web horizontal, projects 100 in. horizontally from a vertical wall. If $E = 29{,}400{,}000$ lb. per sq. in., how much is the free end deflected by its own weight? What is the slope at the end? Solve from the moment diagrams without using formulas. *Ans.* $y = 0.6857$ in.; slope $= 0.00914$.
6. Find the deflection at $0.4\,l$ from the free end of a uniformly loaded cantilever by subtracting the moment of the small moment parabola of length $0.4\,l$ from the moment of the large moment parabola of length l. Check by substitution in Eq. (10).
7. Solve Problem 6 by means of the rectangle, triangle, and parabola which make up the moment diagram from $x = 0.4\,l$ to the fixed end.

109. Cantilever Partly Loaded.—Figure 176 shows a cantilever which carries a distributed load over a portion of its length adja-

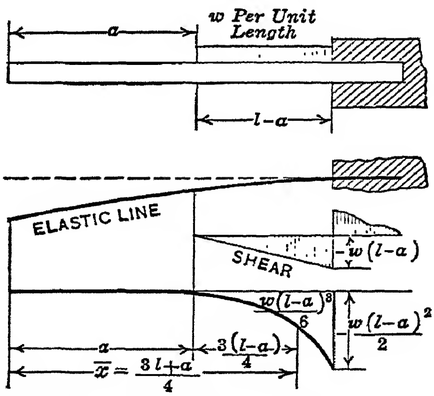


FIG. 176.—Cantilever with distributed load over part of length.

cent to the fixed end, while the remainder, of length a, is not loaded. To find the deflection at the free end, the area of the moment diagram is $-\frac{w(l-a)^3}{6}$, and the moment arm with respect to the end is

$$a + \frac{3}{4}(l - a) = \frac{3\,l + a}{4}. \tag{1}$$

$$E\,I\,y_{\max} = -\frac{w(l-a)^3}{6} \times \frac{3\,l + a}{4}, \tag{2}$$

$$y_{max} = -\frac{w}{24\,E\,I}(3\,l^4 - 8\,l^3a + 6\,l^2a^2 - a^4). \tag{3}$$

The slope of the straight-line portion of length a is $-\frac{w(l - a)^3}{6\,E\,I}$. To get the deflection in any part of the length a this slope is multiplied by $a - x$ and the product added with the proper sign to the deflection at $x = a$ or multiplied by x and added to y_{max} It is better, usually, to solve by area moments without use of formulas, as in Example I.

Example I

A cantilever of length l carries a load of w per unit length over 0.6 of its length adjoining the fixed end. Find the deflection at the free end, at $0.2\,l$ from the free end, and at $0.4\,l$ from the free end by the moment diagram without the use of formulas. Make a sketch of the beam, the moment diagram, and the elastic line similar to Fig. 176 for $a = 0.4\,l$.

$$\text{Maximum moment} = -0.18\,w\,l^2.$$
$$\text{Area of moment diagram} = -0.036\,w\,l^3.$$
$$y_{max} = -\frac{0.036\,w\,l^3\,(0.4\,l + 0.45\,l)}{E\,I} = -\frac{0.0306\,w\,l^4}{E\,I}.$$

At $0.2\,l$,

$$E\,I\,y = -0.036\,w\,l^3 \times 0.65\,l = 0.0234\,w\,l^4.$$

At $0.4\,l$,

$$E\,I\,y = -0.036\,w\,l^3 \times 0.45\,l = -0.0162\,w\,l^4.$$

Problems

1. Check the deflection at $0.4\,l$ by Formula XIX.
2. Check the deflection of the beam of Example I at $0.2\,l$ by adding the product of the slope times distance to the deflection at the free end.
3. A beam of length l, fixed at the right end, carries a uniformly distributed load over $0.8\,l$ adjacent to the fixed end. Find the deflection at the free end, at $0.2\,l$ from the free end, and at $0.4\,l$ from the free end. Solve each directly from the moment of the moment diagram.

Example II

A cantilever of length l carries a load of w per unit length over 0.6 the length adjacent to the free end and no load over the remainder. Find the deflection at the free end and at $0.6\,l$ from the free end.

Figure 177 shows the loaded beam; Fig. 177, II, is the moment diagram. The negative parabola from A to B has a maximum moment of $-0.18\,w\,l^2$ at $0.6\,l$ from the free end. The remainder of the diagram is a straight line which is the moment of the load $0.6\,w\,l$ concentrated at $0.3\,l$ from the end of the beam. The maximum moment at the wall is $-0.6\,w\,l \times 0.7\,l =$

$-0.42\ w\ l^2$. The trapezoid $B\ E\ G\ C$ is divided into two triangles. To find the deflection at the free end,

Parabola	$-\ 0.036\ w\ l^3 \times 0.45\ l$	$-0.0162\ w\ l^4$
Triangle $B\ C\ E$	$-\ 0.036\ w\ l^3 \times {}^{22}\!/_{30}\ l$	$-0.0264\ w\ l^4$
Triangle $C\ E\ G$	$-\ 0.084\ w\ l^3 \times {}^{26}\!/_{30}\ l$	$-0.0728\ w\ l^4$
	$E\ I\ y_{\max}$	$= -0.1154\ w\ l^4$

At 0.6 from the end,

$$E\ I\ y\ =\ -0.036\ w\ l^3 \times \frac{4\ l}{30} - 0.084\ w\ l^3 \times \frac{8\ l}{30} = -0.0272\ w\ l^4.$$

Figure 177, III, represents the cantilever loaded uniformly throughout its length, with an equal upward force on the last four-tenths of the length. Figure 177, IV, shows the corresponding moment diagrams.

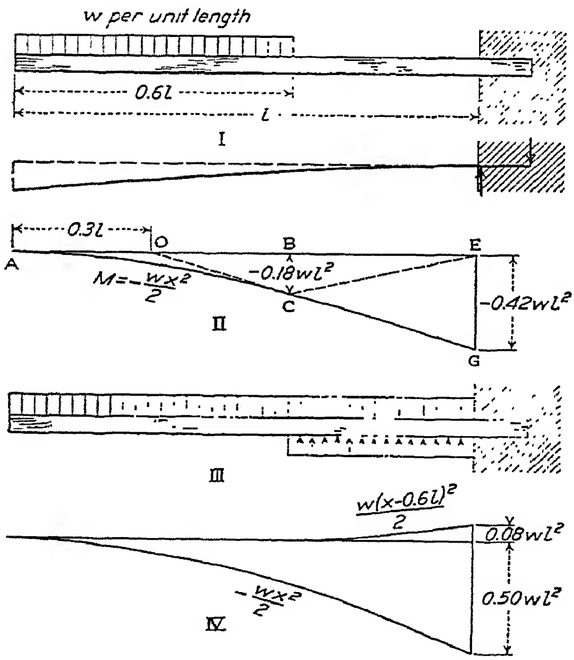


FIG. 177.—Partly loaded cantilever.

Problems

4. Find the deflection of the beam of Example II at 0.6 l from the left end by the moment diagram of Fig. 177, IV.
5. Solve Example II for the deflection at 0.2 l from the free end by both forms of the moment diagram. *Ans.* $E\ I\ y\ =\ -0.084267\ w\ l^4$.
6. A cantilever beam carries a uniformly distributed load over 0.3 l adjacent to the free end and 0.2 l adjacent to the fixed end. Find the deflection

at the free end. Consider the beam uniformly loaded over entire length, with equal upward force over 0.7 l to the fixed end, and an equal additional force over the last 0.2 l.

Ans. $E\,I\,y = (-0.1250 + 0.04716 - 0.00127)\,w\,l^4;\; y = -\dfrac{0.07911\,w\,l^4}{E\,I}.$

110. Simply-supported Beam, Uniformly Loaded.—The moment at a distance x from the left support is $\dfrac{w\,l\,x}{2} - \dfrac{w\,x^2}{2}.$ The diagrams for these terms are shown separately at the bottom of Fig. 178. The moment of the left reaction $\dfrac{w\,l}{2}$ is $\dfrac{w\,l\,x}{2}.$ The

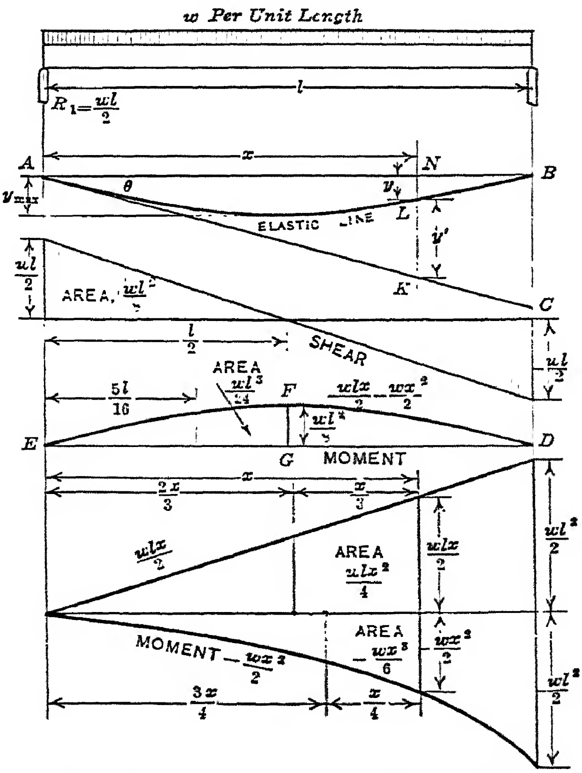


FIG. 178.—Beam supported at ends, distributed load.

moment diagram is the positive triangle. Its maximum ordinate is $\dfrac{w\,l^2}{2}$; its area is $\dfrac{w\,l^3}{4}$; and its center of gravity is at $\bar{x} = \dfrac{2\,l}{3}.$ Any smaller triangle cut off by an ordinate at x from the left vertex has a maximum ordinate of $\dfrac{w\,l\,x}{2}$, an area of $\dfrac{w\,l\,x^2}{4}$, and its

center of gravity at $\frac{2x}{3}$ from the origin of coördinates. The moment diagram for the uniformly distributed load is the negative parabola $M = -\frac{w x^2}{2}$. Its maximum ordinate is $-\frac{w l^2}{2}$. Since the area of a parabola convex toward the base is one-third the base times the altitude, its area is $-\frac{w l^3}{6}$. Its center of gravity is $\frac{3 l}{4}$ from the origin of coördinates. An ordinate at a distance x from the origin cuts off a parabola of altitude $-\frac{w x^2}{2}$. The area of this parabola is $-\frac{w x^3}{6}$ and its center of gravity is at $\frac{3x}{4}$ from the origin.

The combined moment diagram is $E F D$ of Fig. 178. The maximum ordinate at the middle is $\frac{w l^2}{8}$. Since the area of a parabola concave toward the base is *two-thirds* the product of the base by the altitude, its area is $\frac{w l^3}{12}$. Its center of gravity is at $\bar{x} = \frac{l}{2}$. One-half of this parabola has an area of $\frac{w l^3}{24}$, and its center of gravity is at $\frac{5 l}{16}$ from the origin. When part of the triangle or parabola of the *separate* diagrams is cut off by an ordinate at a distance x from the origin, this portion is another smaller triangle or parabola for which the area and the location of the center of gravity are known. For the *combined* diagrams, only the entire figure or one-half of it can be used conveniently. For this reason, separate diagrams are preferable when the equation of the elastic line is required. On the other hand, in the solution of indeterminate beams, which employ the moment of the entire span to find the reactions and moments at the supports, the combined diagram has many advantages.

From symmetry, it is evident that the beam supported at the ends and uniformly loaded (Fig. 178) is horizontal at the middle. It is possible, therefore, to find the deflection of any point from the tangent at the middle, and then find the equation of the elastic line by subtracting from this the deflection of the end

from the tangent at the middle. It is better, however, to begin with a general method, which is applicable to all beams supported at the ends of a span. This method consists of finding the slope of the tangent at the origin (generally the left end), multiplying this slope by x to find the negative deflection of the tangent ($N\ K$ of Fig. 178), finding the deflection upward of the elastic line from the tangent ($K\ L$ of Fig. 178), and finally adding these deflections algebraically to find the deflection (y) of the elastic line from the line $A\ B$ which connects the *fixed supports*. The slope of the tangent might be found from the area of one-half the combined diagram. The general method consists in finding the deflection of the right support from the tangent at the left support. This tangent makes an angle θ with the fixed line $A\ B$. To correspond to the positive $\frac{dy}{dx}$, θ of Fig. 178 is negative.

$$BC = -\theta\, l. \tag{1}$$

To find the deflection *upward* of the right end of the *elastic line* from the *straight line* which is tangent at its left end, the moment of the separate moment diagrams about the right end gives

$$E\,I\,CB = \frac{w\,l^3}{4} \times \frac{l}{3} - \frac{w\,l^3}{6} \times \frac{l}{4} = \frac{w\,l^4}{24}; \tag{2}$$

$$\theta_1 = -\frac{CB}{l} = -\frac{w\,l^3}{24\,E\,I}. \tag{3}$$

To find the equation of the elastic line,

$$y = NL = NK + KL = NK + y'. \tag{4}$$

$$NK = \theta\,x = -\frac{w\,l^3x}{24\,E\,I}. \tag{5}$$

The deflection of the elastic line upward from the tangent $A\ C$ at a distance x from the left end is

$$E\,I\,y' = \frac{w\,l\,x^2}{4} \times \frac{x}{3} - \frac{w\,x^3}{6} \times \frac{x}{4} = \frac{w\,l\,x^3}{12} - \frac{w\,x^4}{24}; \tag{6}$$

$$y' = \frac{w\,l\,x^3}{12\,E\,I} - \frac{w\,x^4}{24\,E\,I}. \tag{7}$$

The deflection y from the line of the supports is the algebraic sum of NK and y'

$$y = NK + y' = -\frac{w\,l^3x}{24\,E\,I} + \frac{w\,l\,x^3}{12\,E\,I} - \frac{w\,x^4}{24\,E\,I},$$

$$y = -\frac{w\,x}{24\,E\,I}(l^3 - 2\,l\,x^2 + x^3). \tag{8}$$

If $x = k\,l$, Equation (8) is

$$y = -\frac{w\,l^4 k}{24\,E\,I}(1 - 2\,k^2 + k^3). \tag{9}$$

$$y_{\max} = -\frac{5\,w\,l^4}{384\,E\,I} = \frac{5\,W\,l^3}{384\,E\,I} \qquad \text{Formula XX}$$

These deflection formulas have been derived from the tangent at the left support. With x positive toward the right, the areas from the point of tangency to the section are a triangle and a parabola, which are convenient for graphical integration. The moment arms of these areas from the section toward the point of tangency are positive toward the left. The deflections might have been calculated from the tangent at the right support. The moment areas would then be a trapezoid and a truncated parabola, which are not convenient for graphical integration. Algebraic integration with the tangent at the right support may be employed profitably for some problems.

Problems

1. Find the slope at a distance x from the left end by differentiating Eq. (8). Compare with Eq. (4) of Art. 93.
2. Find the deflection at one-fourth the length from the left end and at three-fourths the length from the left end from the value of θ and the moment of the moment diagrams without using any other equations from the text.
3. Since the beam is horizontal at the middle, find the slope at each end from the area of one-half the combined moment diagram. Check by means of the separate diagrams.
4. Find the deflection of the left end upward from the tangent at the middle by means of the moment of one-half the combined moment diagram. Check by means of the sum of the moments of the corresponding portions of the separate diagrams.
5. A 6-in. by 12-in. timber beam, 15 ft. long between supports, is deflected 0.375 in. at the middle by a uniformly distributed load of 6,912 lb. Find E and the maximum unit stress.
6. The deflection at the middle of a uniformly loaded beam is 0.2 in. What is the deflection 10 in. from the middle if the length is 5 ft.?
7. The deflection at the middle of a uniformly loaded beam is 0.6 in. The deflection at 2 ft. from the middle is 0.48 in. How long is the beam?
8. A 5-in. by 3-in. T-beam, 11.5 lb. per ft., is supported with flange horizontal. How much will it bend with its own weight when the supports are 30 ft. apart, if E = 29,400,000 lb. per sq. in.? What is the maximum fiber stress? *Ans.* $y_{\max}$ = 2.97 in.; S = 14,000 lb./in.^{2}

111. Simply-supported Beam, Load at Any Point.—In Fig. 179, the moment diagram consists of a positive triangle of base l and a

negative triangle of base b. The slope at the left end is found by dividing CB by the length.

$$E\,I \times CB = \frac{P\,b\,l}{2} \times \frac{l}{3} - \frac{P\,b^2}{2} \times \frac{b}{3} = \frac{P\,b}{6}\,(l^2 - b^2), \quad (1)$$

$$CB = \frac{P\,b}{6\,E\,I}\,(l^2 - b^2); \quad (2)$$

$$\theta_1 = \tan\theta = \frac{CB}{l} = -\frac{P\,b}{6\,E\,I\,l}\,(l^2 - b^2). \quad (3)$$

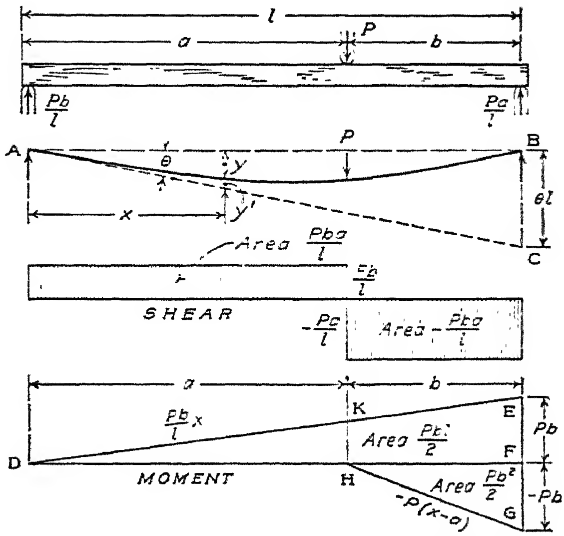


FIG. 179.—Simply-supported beam, load at any point.

The deflection y at any point is the algebraic sum of the deflection of the tangent at that point (which is negative) and deflection y' of the elastic line from the tangent.

Under the load,

$$E\,I\,y' = \frac{P\,b\,a^2}{2\,l} \times \frac{a}{3} = \frac{P\,b\,a^3}{6\,l}, \quad (4)$$

$$y = a\,\theta + y' = -\frac{P\,b\,a}{6\,E\,I\,l}\,(l^2 - b^2) + \frac{P\,b\,a^3}{6\,E\,I\,l}; \quad (5)$$

$$y = -\frac{P\,b\,a}{6\,E\,I\,l}\,(l^2 - b^2 - a^2) = -\frac{P\,a^2b^2}{3\,E\,I\,l}. \quad (6)$$

At a distance x from the left support (if x is less than a),

$$E\,I\,y' = \frac{P\,b\,x^2}{2\,l} \times \frac{x}{3} = \frac{P\,b\,x^3}{6\,l}; \quad (7)$$

$$y = -\frac{P\,b\,x}{6\,E\,I\,l}(l^2 - b^2) + \frac{P\,b\,x^3}{6\,E\,I\,l};\tag{8}$$

$$y = -\frac{P\,b\,x}{6\,E\,I\,l}(l^2 - b^2 - x^2).\tag{9}$$

Equation (9) applies from $x = 0$ to $x = a$. Beyond the load the moment of a small moment triangle, of length $x - a$ and altitude $-P(x - a)$, must be divided by $E\,I$ and the quotient added to Equation (9).

$$-\frac{P}{E\,I}\,\frac{(x-a)^2}{2} \times \frac{x-a}{3} = \frac{P(x-a)^3}{6\,E\,I}.\tag{10}$$

$$y = -\frac{P\,b\,x}{6\,E\,I\,l}(l^2 - b^2 - x^2) - \frac{P(x-a)^3}{6\,E\,I}.\tag{11}$$

At the point of maximum deflection the slope is zero. Since the area of the moment diagram divided by $E\,I$ measures the change in slope,

$$-\frac{P\,b}{6\,E\,I\,l}(l^2 - b^2) + \frac{P\,b\,x^2}{2\,E\,I\,l} = 0;\tag{12}$$

$$x^2 = \frac{l^2 - b^2}{3}; \qquad x = \sqrt{\frac{l^2 - b^2}{3}}\tag{13}$$

Equation (12) gives the abscissa of the point of maximum deflection, provided b is less than one-half the length. If b is greater than one-half the length, the point of maximum deflection falls to the right of the load, and another term $\frac{P}{2\,E\,I}(x - a)^2$ must be added to the slope equation.

The point of maximum deflection always lies between the load and the middle and is much closer to the middle than to the load.

When x from Equation (13) is substituted in Equation (9),

$$y_{max} = -\frac{P\,b(l^2 - b^2)\sqrt{3(l^2 - b^2)}}{27\,E\,I\,l} = -\frac{P\,b\,a(a + 2\,b)\sqrt{3\,a(a + 2\,b)}}{27\,E\,I\,l}\tag{14}$$

Problems

1. A beam is supported at points 100 in. apart and loaded 40 in. from the right support. Find the deflection under the load by Eq. (6) and by Eq. (9).

Ans. $y = \frac{19{,}200\,P}{E\,I}$.

2. Find the location of the maximum deflection of the beam of Problem 1, and calculate maximum deflection.

Ans. $x^2 = 2{,}800$, $x = 52.915$ in.; $y_{max} = \dfrac{19{,}755\,P}{E\,I}$.

3. Find the deflection at the middle of the beam of Problem 1.

Ans. $y = \dfrac{19{,}667\,P}{E\,I}$.

4. Find the deflection 30 in. from the right end of the beam of Problem 1. For a check start from the right support.

Ans. $y = -\dfrac{16{,}500\,P}{E\,I}$.

5. Find the slope of the beam of Problem 1 at 30 in. from each end. Check slope at 30 in. from the right end by working from that end.

Ans. $\theta = -\dfrac{380\,P}{E\,I}$; $\theta = \dfrac{370\,P}{E\,I}$.

6. A 6-in. by 5-in. timber beam rests on supports which are 6 ft. 2 in. apart and overhangs each support 4 ft. 2 in. If $E = 1{,}000{,}000$ lb. per sq. in., how much is each end elevated when a load of 1,776 lb. is applied 2 ft. 2 in. to the left of the right support?

Ans. $y_l = 0.3994$ in.; $y_r = 0.4872$ in.

7. Find the deflection of the beam of Problem 6 under the load and at the middle. Find the point of maximum deflection and calculate the deflection at that point.

Ans. $y = 0.19935$ in. under load; $y = 0.21124$ in. at the middle; max. $y = 0.21299$ in. at 40 in.

8. Find the slope of the beam of Problem 6 at 20 in. from each end. Check.

Ans. -0.005990 radian; 0.006058 radian.

112. Simply-supported Beam, Load at Middle.—A beam which is supported at the ends and loaded at the middle is a special case of Art. 111. From symmetry of Fig. 180, it is evident that the beam is horizontal under the load.

Example I

Since the beam is horizontal at the middle, show that the slope at the left end is

$$\theta_1 = -\frac{P\,l^2}{16\,E\,I} \qquad (1)$$

from the area of the moment diagram to the left of the middle. Solve also by the deflection of the right end from the tangent to the beam at the left end. Solve again by substitution in a suitable equation of Art. 111.

Example II

From Formula XVIII, find the deflection of one end upward from the tangent at the middle. Solve also by the moment of the moment diagram.

Example III

Derive the equation of the elastic line for the left half of the beam by the methods of the preceding articles. Solve also by substitution in some equation of Art. 111.

$$Ans.\ y = -\frac{P\,l^2x}{16\,E\,I} + \frac{P\,x^3}{12\,E\,I}. \tag{2}$$

$$y_{max} = -\frac{P\,l^3}{48\,E\,I}. \qquad \text{Formula XVIII}$$

This type of beam support and loading is the most used for testing. It is difficult to apply a uniformly distributed load

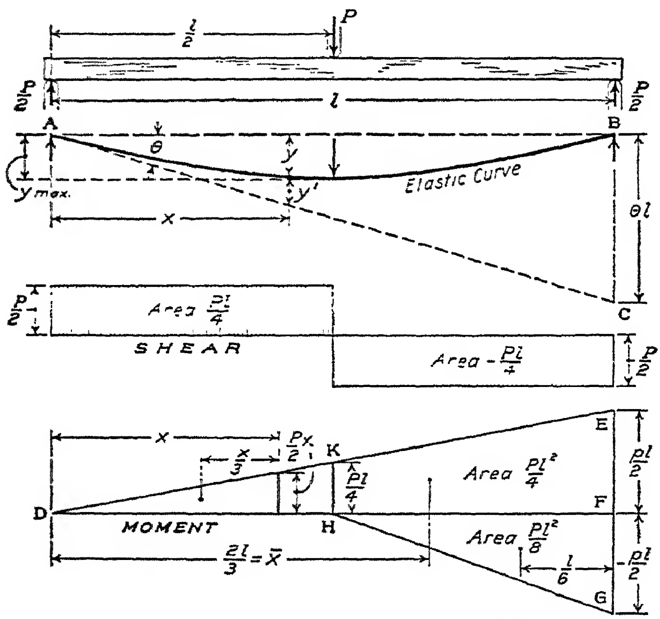


FIG. 180.—Simply-supported beam, load at middle.

except by actual weights. This is equivalent to two cantilevers which are fixed at the middle and bent upward by the end reactions.

Example IV

From the moment diagram show that the equation of the elastic line from $x = \frac{l}{2}$ to $x = l$ is

$$y = -\frac{P\,l^2x}{16\,E\,I} + \frac{P\,x^3}{12\,E\,I} - \frac{P\left(x - \frac{l}{2}\right)^3}{6\,E\,I}; \tag{3}$$

$$y = -\frac{P}{48\,E\,I}(-l^3 + 9\,l^2x - 12\,l\,x^2 + 4\,x^3).$$

Problems

1. A 6-in. by 6-in. beam, supported at points 80 in. apart, was deflected 0.252 in. when the load changed from 200 lb. to 4,200 lb. It failed under a load of 9,800 lb. Calculate E and the modulus of rupture. What would have been the deflection for the same change of load if the beam had been 20 ft. long between supports?
Ans. $E = 1,567,000$; $S = 5,444$ lb./in.^{2}; $y = ?$
2. A stick of longleaf yellow pine, 1.54 in. by 1.60 in., rested on supports 20 in. apart and carried a load at the middle. Part of the readings were

Load, pounds	Deflection, inches	Load, pounds	Deflection, inches
100	0.0195	1,100	0.1480
200	318	1,200	0.1605
300	435	1,300	0.1740
400	560	1,400	0.1880

The stick failed under a load of 10,750 lb. Calculate the modulus of elasticity for four 1,000-lb. intervals and calculate the modulus of rupture. What was the section modulus?
3. A second stick of longleaf yellow pine, 1.52 in. by 1.70 in., gave the following readings:

Load, pounds	Deflection, inches	Load, pounds	Deflection, inches
50	0.0761	850	0.1765
150	847	950	0.1904
250	980	1,050	0.2064
350	0.1098	1,150	0.2215

The stick failed under a load of 3,340 lb. Find E and the modulus of rupture.

In the solution of Problems 2 and 3, and all similar problems, compute the terms which are common once for all, and divide this result by the individual deflection differences to get the separate values of E.
4. In Problem 3, if an error of 0.01 in. were made in the breadth measurement, what would be the percentage of error in E? How much would this error change the average E? What would be the effect of an error of 0.01 in. in the depth measurement? What would be the effect of an error of 0.0001 in. in the deflection reading? Of an error of 1 lb. in the load readings?
Ans. $1 \div 152 = 0.0066 = 0.66$ per cent; $3 \div 170 = 0.0176 = 1.76$ per cent; $1 \div 1,000$, approximately; $1 \div 800$.
5. In Problems 3 and 4, what would be the percentage of error of E and of the modulus of rupture if there were an error of 0.1 in. in the reading

of the span? In Problem 4, what would be the percentage of error in modulus of rupture if there were an error of 0.01 in. in the depth?

6. What would be the percentage of error in the deflection of a beam supported at the ends and loaded at the middle if both the load and the deflection apparatus were displaced 5 per cent of the length from the middle? What would be the error if the load were at the middle and the deflection apparatus were displaced 5 per cent of the length?
Ans. 2 per cent; 1.45 per cent.

7. Solve Problem 6 for a displacement of 1 per cent of the length.
Ans. 0.08 per cent; 0.06 per cent.

8. Find the deflection at one-third the length and at two-thirds the length from the left end of a beam which is supported at the ends and loaded at the middle. Use Eqs. (2) and (3).
Ans. $y = -\dfrac{23\,P\,l^3}{1{,}296\,E\,I}$.

9. A 20-in. I-beam rests on supports 25 ft. apart and carries a load at the middle which makes the maximum stress 16,000 lb. per sq. in., neglecting the weight of the beam. What is the deflection at the middle and at 10 ft. from each load if E = 29,000,000 lb. per sq. in.?
Ans. y_{max} = 0.4138 in.; y = 0.3906 in.

10. A 20-in. 75-lb. standard I-beam with web horizontal is supported at points 25 ft. apart. Find the deflection at the middle caused by a load which makes the total stress, including that caused by the weight of the beam. 16,000 lb. per sq. in. E = 29,000,000. *Ans.* y_{max} = 0.6875 in.

113. Beam of Constant Moment.—If M is a constant moment, the moment diagram is a rectangle of height M. By using the method of Arts. 111 and 112, to find the slope at the left end by means of the zero deflection at the right end,

$$E\,I\,y = 0 = E\,I\,\theta_1\,l + M\,l \times \frac{l}{2}; \tag{1}$$

$$E\,I\,\theta_1 = -\frac{M\,l}{2}. \tag{2}$$

$$E\,I\,y = -\frac{M\,l\,x}{2} + M\,x \times \frac{x}{2};$$

$$E\,I\,y = -\frac{M\,x(l - x)}{2}. \tag{3}$$

Equation (3) shows that the curve is symmetrical. At the middle,

$$y_{max} = -\frac{M\,l^2}{8\,E\,I}. \tag{4}$$

Problems

1. An 8-in. 23-lb. standard I-beam rests on supports 20 ft. apart. It carries a load of 4,800 lb. 5 ft. from the left end, and an equal load 5 ft.

from the right end. Find the deflection of the middle below the loads.
Ans. $y = -0.2784$ in.

2. In Problem 1, find the deflection of the left support from the tangent at the left load, as a cantilever. Find the deflection of the tangent upward at 5 ft. from the load and get the deflection of the load downward below the line of the supports. Find the total deflection of the middle below the supports.
Ans. $-0.5568 - 0.1856$ in., support to load, $= -0.7424$ in.; $y_{max} = -1.0188$ in.

114. Beam Symmetrically Loaded.—Figure 181 shows a beam which is supported at the ends. It carries a load $\frac{P}{2}$ at a distance a from the left support and an equal load at the same distance

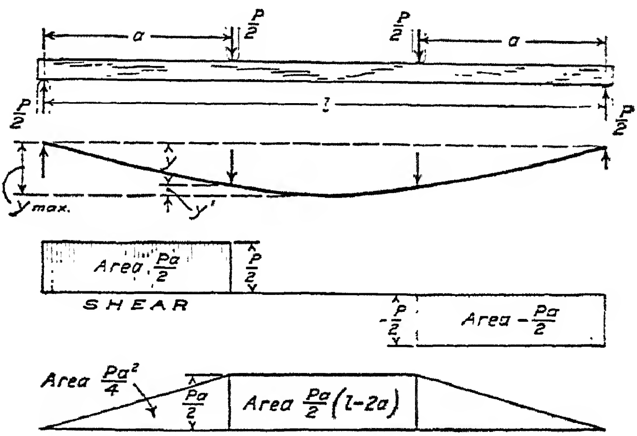


FIG. 181.—Moment constant between loads.

from the right support. If the weight of the beam is negligible, the moment is constant between these loads, which makes it possible for the beam to fail at any point throughout this range, instead of at a single section of maximum moment at the middle. For this reason symmetrical loading is frequently used in careful beam testing.

From the area of one-half the combined moment diagram of Fig. 181, the slope of the tangent is given by

$$E\,I\,\theta_1 + \frac{P\,a^2}{4} + \frac{P\,a}{4}\,(l - 2\,a) = 0;$$

$$E\,I\,\theta_1 = -\frac{P\,a(l-a)}{4}. \qquad (1)$$

The deflection of the point of application of the left load is

$$E\,I\,y = -\frac{P\,a(l - a)}{4} \times a + \frac{P\,a^2}{4} \times \frac{a}{3};$$

$$y = -\frac{P\,a^2}{12\,E\,I}(3\,l - 4\,a). \tag{2}$$

The deflection at the middle is

$$E\,I\,y_{max} = -\frac{P\,a(l - a)}{4} \times \frac{l}{2} + \frac{P\,a^2}{4} \times \left(\frac{l}{2} - \frac{2\,a}{3}\right) + \frac{P\,a}{4} \times \frac{(l - 2\,a)^2}{4};$$

$$y_{max} = -\frac{P\,a}{48\,E\,I}(3\,l^2 - 4\,a^2). \tag{3}$$

Problems

1. Find the deflection under the load and at the middle when the loads are at the third points.

   Ans. $y_a = -\frac{5\,P\,l^3}{324\,E\,I}$; $y_{max} = -\frac{23\,P\,l^3}{1{,}296\,E\,I}$.

2. In Problem 1, find the deflection of the middle below the concentrated loads by an equation of Art. 113. Check from the answers of Problem 1.
3. Shortleaf yellow-pine beam, C4, tested by Prof. A. N. Talbot, was 6.75 in. by 16 in. It was supported at points 13 ft. 6 in. apart and loaded at the third points. The modulus of elasticity was 1,585,000 lb. per sq. in. At the elastic limit the total load was 52,500 lb. What was the deflection at the middle and under each load at the elastic limit? What was the elongation of the middle 40 in. at the bottom? What was the radius of curvature of the middle third?

   Ans. $y_a = -0.9432$ in.; $y_{max} = -1.0847$ in.; elongation $= 0.1242$ in.; $\rho = 214.7$ ft.

4. Find the deflection of the left end of the beam of Fig. 181 from the tangent at the middle. Find the deflection of the left load from the tangent at the middle.
5. A beam 100 in. long carries 80 lb. 30 in. from the left support and 60 lb. 40 in. from the right support. Find the deflection at the 80 lb. load. Solve from a diagram similar to Fig. 181.

   Ans. $E\,I\,y = -81{,}200 \times 30 + 36{,}000 \times 10 = -2{,}076{,}000.$

6. Find the deflection under the 60 lb. load for Problem 5.

   Ans. $y = -\frac{2{,}352{,}000}{E\,I}$.

7. What would be the deflection of the beam of Problem 5 if the two loads were placed together at the middle?

115. Combined Moment Diagrams.—For a beam supported at the ends and uniformly loaded, the combined diagram is a parabola, which is concave downward. This parabola (Fig. 178)

is the sum of the positive triangle $\frac{w\,l\,x}{2}$ and the negative parabola $-\frac{w\,x^2}{2}$. The maximum ordinate of the combined parabola is $\frac{w\,l^2}{8}$, the area, which is two-thirds the base times the altitude, is $\frac{w\,l^3}{12}$, and the center of gravity is $\frac{l}{2}$ from each support. The center of gravity of each half of this parabola is three-sixteenths of the length from the middle, or five-sixteenth the length from the adjacent end.

For a beam supported at the ends and subjected to a single concentrated load at a distance a from the left support and a distance b from the right support, the combined diagram is a positive triangle. This is the sum of the positive triangle $\frac{P\,b\,x}{l}$ of Fig. 179, and the negative triangle $-P(x-a)$. The maximum ordinate is $\frac{P\,a\,b}{l}$ and the area is $\frac{P\,a\,b}{2}$.

The center of gravity of any triangular area is located at the center of gravity of three equal masses at the vertices of this triangle, since the center of gravity in each case is at the intersection of the medians. The center of gravity of the combined triangle is $\frac{l+a}{3}$ from the left support, or $\frac{l+b}{3}$ from the right support.

These combined diagrams for the reaction and load of a simply-supported beam may be called the *combined* simple-support diagrams. These diagrams are convenient for operations which involve the entire span. For example, to find the slope at the left support of a uniformly loaded, simply-supported beam,

$$E\,I\,\theta_1\,l + \frac{w\,l^3}{12} \times \frac{l}{2} = 0; \tag{1}$$

$$\theta_1 = -\frac{w\,l^3}{24\,E\,I}. \quad \text{[Compare Equation (3) of Art. 110.]} \tag{2}$$

For a beam supported at the ends with a load P at a distance a from the left end and b from the right end, the slope at the left end is obtained by

$$E\,I\,\theta_1\,l + \frac{P\,a\,b}{2} \times \frac{l+b}{3} = 0; \tag{3}$$

$$\theta_1 = -\frac{P\,b(l-b)\,(l+b)}{6\,E\,I\,l} = -\frac{P\,b(l^2-b^2)}{6\,E\,I\,l}. \tag{4}$$

116. Stiffness of Beams.—The stiffness of a beam is the reciprocal of the deflection. The stiffness of a beam may be defined as the load which will produce unit deflection. It is not customary to express stiffness in this way; it is generally used as a relative term.

In the expression for the maximum deflection of all the beams which have been considered, the terms E and I occur in the denominator. The stiffness of a beam varies directly as the modulus of elasticity and directly as the moment of inertia of its cross section. The moment of inertia of a rectangular section varies as the cube of the depth; consequently the stiffness of a rectangular section varies in the same ratio. All the expressions for the maximum deflection contained the cube of the length in the numerator. The stiffness of beams of the same cross section varies inversely as the cube of their length.

Problems

1. How does the stiffness of a 4-in. by 6-in. beam compare with that of a 4-in. by 4-in. beam of the same material?
2. How does the stiffness of a 4-in. by 6-in. beam with the 6-in. side vertical compare with that of the same beam with the 4-in. side vertical?
3. How does the stiffness of a 2-in. by 12-in. beam 15 ft. long compare with that of a 2-in. by 8-inch beam 10 ft. long? Which is the stronger?

CHAPTER XI

INDETERMINATE BEAMS

(To the teacher: Chapter XI is intended for a brief course or for a rather complete course. In Arts. 119 *to* 123, *inclusive, reactions and moments are first derived in the simplest manner by means of deflection formulas which the student has mastered in preceding chapters. These results are then applied in the first group of problems of each article. The remainder of each of these articles may be omitted or reserved for later study. Area moments and successive integration between limits are used in the advanced portion: either or both may be studied.*

A brief course will omit all of the chapter after Art. 128 *or Art.* 130. *The arrangement of the text is such that the entire subject of indeterminate beams may be deferred until after the completion of Chapter XVI or XVII.)*

117. Determinate and Indeterminate Beams.—A beam or structure is statically determinate when it is possible to compute external reactions and resisting moments by elementary statics without reference to deformations or elastic constants. A beam on two supports is *statically determinate.* A beam on three supports in the same plane is *statically indeterminate.* If the beam is *assumed* to be *absolutely rigid* and the supports *perfectly inelastic,* any infinitesimal displacement of one support will throw all the load on two supports. For an actual beam, which is always flexible, it is possible to calculate the reaction on any number of supports by means of deflection equations.

Much space is allotted in this book to the deflection of beams. Since stress is more important, it might seem that deflection has received undue prominence. However, for indeterminate beams, external reactions and moments, which are necessary for the calculation of stress, may be found only by means of deflection formulas.

118. Diagram of General Moment Equation.—The general moment equation for a uniformly loaded span between two points A and B

$$M = M_a + V_{ab}x - \frac{w\,x^2}{2}, \tag{1}$$

in which M_a is the moment at the left support, and V_{ab} is the shear

on the side of the left support toward B of Fig. 182. To find M_b at the second support,

$$M_b = M_a + V_{ab}l - \frac{w\,l^2}{2}; \tag{2}$$

$$V_{ab} = \frac{M_b - M_a}{l} + \frac{w\,l}{2}. \qquad \text{Formula XXVI}$$

When this value of V is substituted in Equation (1),

$$M = M_a + \frac{M_b - M_a}{l}x + \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}. \tag{3}$$

The terms $\frac{w\,l\,x}{2} - \frac{w\,x^2}{2}$ together give the moment of a simply-

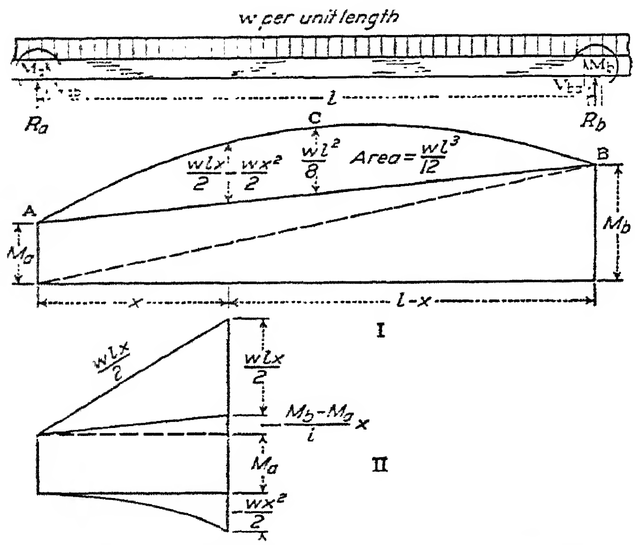


FIG. 182.—Separate and combined moment diagrams.

supported beam which is uniformly loaded. These may be called the *simple-support* moment expression. Graphically $\frac{w\,l\,x}{2}$ is represented by a positive triangle, and $-\frac{w\,x^2}{2}$ by a negative parabola. Together, they form the positive simple-support parabola $A\,C\,B$ of Fig. 182, I. The remaining terms may be represented by the trapezoid $A\,B$ of which the left end is M_a and the right end is M_b. This diagram may be called the *end-*

moment trapezoid. For graphic integration this trapezoid may be divided into two triangles.

Figure 182, II, shows a portion (of length x) of Fig. 182, I. The positive part of the simple-support diagram is drawn separately at the top and the negative part is placed at the bottom. The combined diagram is useful for calculations which involve the entire span. The separate diagrams are necessary for the deflection at intermediate points.

Figure 183, I, shows a span with a concentrated load at a distance a from the left support and at a distance b from the

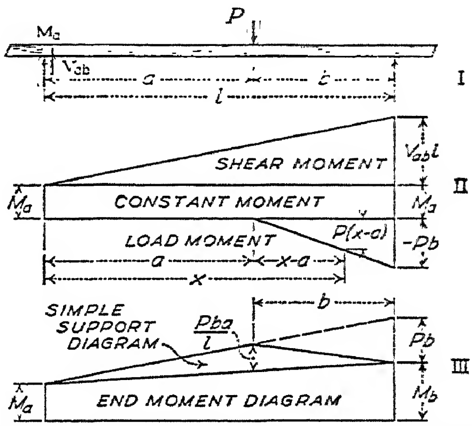


FIG. 183.—Separate and combined diagrams for concentrated load.

right support. From the general moment equation the moment at the right support is

$$M_b = M_a + V_{ab}l - Pb; \tag{4}$$

$$V_{ab} = \frac{M_b - M_a}{l} + \frac{Pb}{l}; \tag{5}$$

$$M = M_a + \frac{M_b - M_a}{l}x + \frac{Pbx}{l} \tag{6}$$

from 0 to a. Beyond the load the moment equation is

$$M = M_a + \frac{M_b - M_a}{l}x + \frac{Pbx}{l} - P(x - a) \tag{7}$$

Figure 183, II, shows the separate moment diagrams for M_a, $V_{ab}x$, and $-P(x - a)$. Figure 183, III, shows the combined end-moment trapezoid and the simple-support triangle. If the span were a simply-supported beam with the load P, the left

reaction would be $\frac{P\,b}{l}$ and the moment at each end would be zero. The area of this triangle is $\frac{P\,b\,a}{2}$. The center of gravity of a triangular area is located at the center of gravity of three equal loads at the three vertices (since the center of gravity of any pair of these loads is at the middle of the line which joins them). Measured from the left end,

$$\bar{x} = \frac{l + a}{3}. \tag{8}$$

Measured from the right end,

$$l - \bar{x} = \frac{l + b}{3}. \tag{9}$$

These expressions are useful for area-moment calculations.

119. Uniformly Loaded Span Fixed at One End.—Figure 184 shows a beam which is fixed at the right end and supported at the

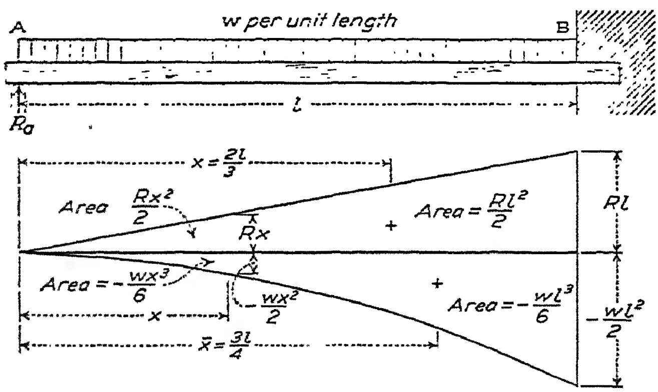


FIG. 184.—Separate diagrams for distributed load.

left end. The beam is supposed to touch the support under no load. (If the support has an initial reaction, the equations which follow give the additional reaction and moment caused by the additional load.) This beam may be regarded as a cantilever which is bent downward by the distributed load of w per unit length and bent upward by the concentrated reaction at the free end. Since these deflections are equal and opposite,

$$\frac{R\,l^3}{3\,E\,I} - \frac{w\,l^4}{8\,E\,I} = 0; \tag{1}$$

$$R = \frac{3\,w\,l}{8} = \frac{3\,W}{8}. \tag{2}$$

A uniformly loaded beam of two equal spans is horizontal over the middle support. It is equivalent, therefore, to two beams which are fixed at one end and supported at the other.

Problems

1. A 6-in. by 2-in. plank is fixed in a vertical wall at the right end and projects to a support which is 10 ft. from the wall. The plank carries a uniformly distributed load of 36 lb. per ft. Find the reaction at the support. With this reaction known, calculate the moment at the right end. Find the unit stress at the wall.
   Ans. $R = 135$ lb.; $M = -5{,}400$ in.-lb.; $S = 1{,}350$ lb./in.^{2}
2. Construct the shear diagram for the beam of Problem 1 to the scale of 1 in. equals 2 ft. of length and 100 lb. of shear. What is the shear at the fixed end? *Ans.* $V = -225$ lb.
3. Find the dangerous section between the support and the fixed end. What is the moment at this dangerous section?
   Ans. $M = 3{,}037.5$ in.-lb.
4. Write a moment equation for the beam of Problem 1 in terms of 135 lb. and x. Solve for the position of zero moment. Compare with the shear diagram.
5. Construct the moment triangle of the reaction and the moment parabola of the distributed load for the beam of Problem 1 to the scale of 1 in. equals 2 ft. of length and 5,000 in.-lb. of moment. Combine by measuring the negative ordinates of the parabola downward from the positive straight line of the triangle for each 1-ft. interval of the beam. Determine the position of maximum positive moment and the position of zero moment from this combined parabola.
6. For a uniformly loaded beam of span l, fixed at the right end and supported at the left end, find the shear at the fixed end, and the moment at each dangerous section in terms of w and l.
   Ans. $V = -\frac{5\,w\,l}{8} = -\frac{5\,W}{8}$; $M = \frac{9\,w\,l^2}{128}$ at $x = \frac{3\,l}{8}$; $M = -\frac{w\,l^2}{8} = -\frac{W\,l}{8}$ at the wall.
7. A 5-in. 14.75-lb. I-beam projects 10 ft. from a vertical wall and rests on a fixed support at the left end. What is the additional reaction of the support when a uniformly distributed load of 6,000 lb. is placed on the beam? What is the maximum unit stress at the two dangerous sections and at the middle? What is the shear next to the fixed end?
   Ans. $R = 2{,}250$ lb.; $S = 8{,}437$, $15{,}000$, and $7{,}500$ lb./in.^{2}; $V = -3{,}750$ lb.
8. In Problem 7, what is the maximum stress at 7 ft. from the left support and what is the vertical shear?
   Ans. $S = 2{,}100$ lb./in.^{2}; $V = -1{,}950$ lb.

9. If the support at the left end of the beam of Problem 7 is removed, what is the maximum stress caused by a uniformly distributed load of 1,500 lb.? What is the deflection at the end if $E = 30{,}000{,}000$ lb. per sq. in.? *Ans.* $S = 15{,}000$ lb./in.^{2}; $y_{max} = 0.72$ in.
10. If the support is removed from the left end of the beam of Problem 7, what concentrated load on the end will produce a deflection of 1 in. at the end? *Ans.* $P = 781.25$ lb.
11. If the support of the beam of Problem 7 settles 0.8 in. when the 6,000-lb. load is applied, what is the reaction? *Ans.* $R = 2{,}250 - 625 = 1{,}625$ lb.
12. What is the maximum stress in the beam of Problem 11? Where is the point of inflection? *Ans.* $S = 27{,}500$ lb./in.^{2} at wall; $x = 65$ in.
13. If the left support of the beam of Problem 7 is raised 0.8 in. above the tangent from the fixed end, what is the reaction when loaded? What and where is the maximum stress? *Ans.* $R = 2{,}875$ lb.; $S = 13{,}776$ lb./in.^{2} at $x = 57.5$ in.
14. A 4-in. by 6-in. timber beam, for which $E = 1{,}200{,}000$ lb. per sq. in., is fixed 10 ft. from the left end. It carries a uniformly distributed load of 48 lb. per ft. What is the deflection at the free end, and what is the maximum unit stress? What is the unit stress at 45 in. from the free end? *Ans.* $y_{max} = 1.2$ in.; $S_{max} = 1{,}200$ lb./in.^{2} at fixed end.
15. A support is placed under the free end of the beam of Problem 14. What is the reaction when the load is applied? What is the maximum unit stress and the unit stress at 45 in. from the left end? *Ans.* 180 lb.; 300 lb./in.^{2}; 168.75 lb./in.^{2}
16. How much would the beam of Problem 14 be deflected by a load of 120 lb. on the free end? *Ans.* $y_{max} = 0.80$ in.
17. If the support of Problem 15 settles 0.5 in. when the load is applied, what is the reaction at the support? *Ans.* $R = 180 - \dfrac{5 \times 120}{8} = 105$ lb.

AREA MOMENTS

(The remainder of this article may be omitted, area moments may be omitted and integration between limits studied, or vice versa.)

To find the unknown reaction at the support of the beam of Fig. 184 by area moments, the deflection at the support from the tangent at the fixed end is calculated from the reaction triangle and the load parabola of that figure.

$$0 = \frac{R\,l^2}{2} \times \frac{2\,l}{3} - \frac{w\,l^3}{6} \times \frac{3\,l}{4}; \tag{3}$$

$$\frac{R\,l^2}{3} = \frac{w\,l^3}{8}; R = \frac{3\,w\,l}{8} \tag{4}$$

Problems

18. Calculate M_b from the end reaction and the load.

19. Find the slope at the support by means of the area of the moment diagrams.

$$Ans.\ E\,I\,\theta_1 + \frac{3\,w\,l^3}{16} - \frac{w\,l^3}{6} = 0;\ \theta_1 = -\frac{w\,l^3}{48}.$$

Figure 185 represents a uniformly loaded beam that is fixed at the left end and supported at the right end. Figure 185, II, gives the simple-support and end-moment diagrams.

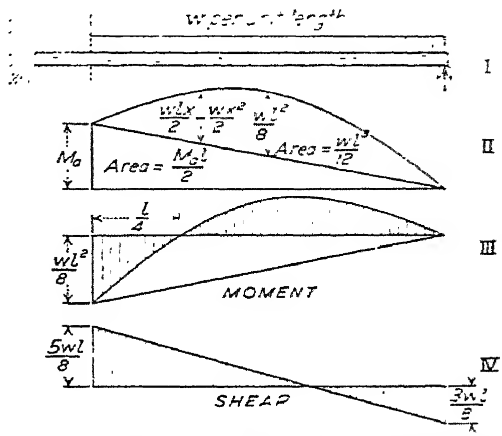


FIG. 185.—Combined diagrams for distributed load.

Since M_b is zero, the end-moment diagram is a triangle. The simple-support parabola is the same as in Fig. 182. To find M_a by means of the deflection at the right support from the tangent at the fixed end,

$$0 = \frac{M_a\,l}{2} \times \frac{2\,l}{3} + \frac{w\,l^3}{12} \times \frac{l}{2} \tag{5}$$

$$M_a = -\frac{w\,l^2}{8}.$$

The shaded area of Fig. 185 gives the combined moment diagram. The positive parabola is added to the negative end-moment triangle. The moment is zero at one-fourth the length from the fixed end.

INTEGRATION BETWEEN LIMITS

To find the moment at the fixed end of the beam of Fig. 185, since M_b is zero, the general moment equation is

$$M = M_a - \frac{M_a x}{l} + \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}; \tag{6}$$

$$E\,I\,\theta = 0 + M_a x - \frac{M_a x^2}{2\,l} + \frac{W\,l\,x^2}{4} - \frac{w\,x^3}{6}; \tag{7}$$

$$E\,I\,y = 0 + \frac{M_a x^2}{2} - \frac{M_a x^3}{6\,l} + \frac{w\,l\,x^3}{12} - \frac{w\,x^4}{24}. \tag{8}$$

At the right support where $x = l$

$$0 = \frac{M_a l^2}{2} - \frac{M_a l^2}{6} + \frac{w\,l^4}{12} - \frac{w\,l^4}{24}; \tag{9}$$

$$0 = \frac{M_a}{3} + \frac{w\,l^2}{24};\ M_a = -\frac{w\,l^2}{8};$$

$$E\,I\,y = -\frac{w\,l^2\,x^2}{16} + \frac{w\,l\,x^3}{48} + \frac{w\,l\,x^3}{12} - \frac{w\,x}{24}; \tag{10}$$

$$y = -\frac{w\,x^2}{48\,E\,I}(3\,l^2 - 5\,l\,x + 2\,x^2). \tag{11}$$

Problems

20. Derive the slope equation and find the position of maximum deflection.

Ans. $\theta = \frac{w\,x}{48\,E\,I}(6\,l^2 - 15\,l\,x + 8\,x^2);\ x = \frac{15 - \sqrt{33}}{16}\,l = 0.57846\,l.$

21. Find the maximum deflection of the beam of Problem 20. Find the deflection at the middle and at point of zero shear.

Ans. $y_{\max} = -\frac{(156 + 220\sqrt{33})\,w\,l^4}{512^2\,E\,I} = -\frac{0.00542\,w\,l^4}{E\,I};$

$y = -\frac{w\,l^4}{192\,E\,I}$ at middle; $y = -\frac{175\,w\,l^4}{32^3\,E\,I}$ at $x = \frac{5\,l}{8}$.

120. Two Equal Spans, Uniformly Loaded.—Figure 186 represents a continuous beam of two equal spans, each of length l. If the middle support were removed, it would become a beam of length $2\,l$, supported at the ends. The deflection at the middle would be $\frac{5\,w(2\,l)^4}{384\,E\,I}$. To bring the middle of this beam up to the plane of the end supports, the required upward force at the middle must be equal to the load at the middle which would produce this deflection.

$$\frac{R_2(2\,l)^3}{48} = \frac{5\,w(2\,l)^4}{384}; \tag{1}$$

$$R_2 = \frac{10\,w\,l}{8}. \tag{2}$$

Since the total load is $2\,w\,l$, $R_1 + R_3 = \frac{6\,w\,l}{8}$. It is evident from symmetry that $R_1 = R_3 = \frac{3\,w\,l}{8}$, and the beam is horizontal over the middle support. The left half is exactly the same as Fig. 184. The end reactions have already been shown to be the same as those of a beam which is fixed at one end and supported at the other. The equation of the elastic line and the shear and

moment diagrams also are the same. The maximum moment over the middle support is $-\frac{w\,l^2}{8}$. This is numerically the same as the positive moment at the middle of a beam of length l which is uniformly loaded and simply-supported at the ends. A beam which is continuous over three supports is no stronger than a beam of the same span which merely rests on the supports. If

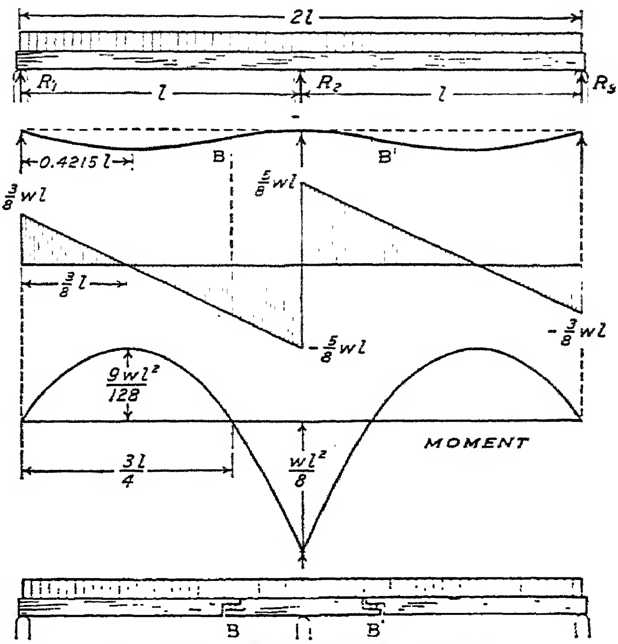


FIG. 186.—Beam with three supports.

the beam of Fig. 186 were cut in two at the middle, and each portion continued to rest at one end on the middle support, the shear diagram would pass through zero at the middle of each span. The moment curve would be highest at the middle of each span and would be positive throughout.

The shear diagram of Fig. 186 passes through zero at $\frac{3\,l}{8}$ from the outer supports. At $\frac{3\,l}{4}$ from each end the negative shear is equal to the positive shear and the moment is zero. The points B and B' are points of inflection. Between these points the beam is concave downward and the moment is nega-

tive. Between the left support and B and between B' and the right support, the beam is concave upward and the moment is positive. At each point of inflection the beam might be cut in two and one portion merely rest on the other. If the beam were divided as shown at the bottom of Fig. 186, the moment, shear, and deflection at every section would be the same as in the continuous beam at the top. For instance, three-eighths of the weight of the span rests on the middle portion at B. The moment at the middle support is

$$-\frac{3\,w\,l}{8} \times \frac{l}{4} - \frac{w\,l}{4} \times \frac{l}{8} = -\frac{w\,l^2}{8}. \tag{3}$$

Wherever there is a point of inflection in a beam, the beam may be divided and the portions connected by a pin or a slight projection which will resist the shear.

Problems

1. A 6-in. 12.5-lb. I-beam rests on two supports 15 ft. apart and carries a uniformly distributed load, including its own weight, of 5,400 lb. What is the maximum fiber stress? If E = 29,000,000 lb. per sq. in., what is the deflection at the middle and the slope at the left end?
Ans. S = 16,644 lb./in.²; y_{max} = 0.6486 in.; $\frac{dy}{dx} = \theta_1 = -0.01153$.

2. A support is placed under the middle of the beam of Problem 1 and raised until the tops of the three supports are at the same level. Calculate the load on each support. What and where is the maximum stress? What is the slope at the left end?
Ans. Reactions are 1,012.5 lb., 3,375 lb., and 1,012.5 lb.; maximum stress = 4,161 lb./in.² tension at top over middle support; $\theta_1 = -0.0007207$.

3. What load at the middle of the beam of Problem 1 will bend the beam 0.4 in.? If the middle support settles 0.4 in. below the line of the end supports, what is each end reaction? *Ans.* 2,081 lb.; 2,053 lb.

4. A 4-in. by 6-in. beam, 20 ft. long, carries a distributed load which makes the maximum stress 1,000 lb. per sq. in. What is the load? What is the deflection at the middle and at 20 in. to the left of the middle if E = 1,000,000 lb. per sq. in.? *Ans.* 2 in.; 1.9337 in.

5. What force at the middle of the beam of Problem 4 will lift the middle 2 in.? What will be the reaction of each support if a support is placed at the middle and the top of all three supports are at the same level?
Ans. Reactions are 150 lb.; 500 lb.; and 150 lb.

6. How much must the middle support in Problem 5 settle below the level of the outer supports in order to make all reactions equal?

7. What force 20 in. to the left of the middle of the beam of Problem 4 will lift that point to the level of the other two supports? *Ans.* 511.4 lb.

8. Using the answer of Problem 7, find the reaction of each of the other supports?

121. Span Fixed at One End, Load Concentrated.—Figure 187 shows a beam which is fixed at the right end, supported at the left end, and subjected to a load P at a distance a from the supported end and at a distance b from the fixed end. This beam may be regarded as bent downward at the left end by the load P applied at a distance a from the end and bent upward an equal amount by the reaction R. By Maxwell's theorem, the deflection

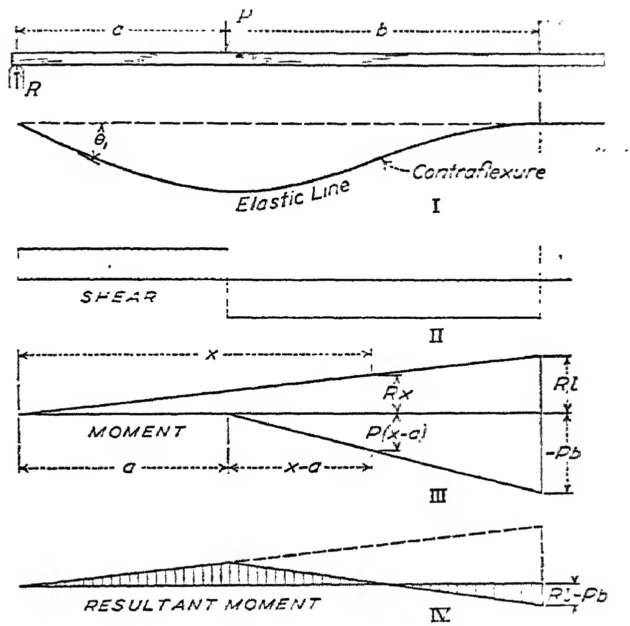


FIG. 187.—Separate and combined diagrams for concentrated load.

at the end caused by a load P at a distance a from the end is equal to the deflection at a distance a from the end which would be caused by a load P at the end. From the equation of the elastic line for a cantilever with a load on the free end [Equation (8), Art. 89; Equation (8), Art. 98; or Equation (9), Art. 106], this deflection is

$$y = -\frac{P}{6\,E\,I}(2\,l^3 - 3\,l^2a + a^3); \qquad (1)$$

$$0 = \frac{R\,l^3}{3\,E\,I} - \frac{P}{6\,E\,I}(2\,l^3 - 3\,l^2a + a^3); \qquad (2)$$

$$R = \frac{P}{2}(2 - 3\,k + k^3), \qquad (3)$$

in which $k = \frac{a}{l}$.

Problems

1. Find the reaction when the load P is at $0.4\,l$ from the supported end, as shown in Fig. 187. Find the moment under the load and at the fixed end by definition of moment. Check by area of shear diagram.
   Ans. $M_a = 0.1728\,P\,l$; $M_2 = -0.168\,P\,l$.
2. Calculate the reaction when the load is at the middle. Find the moment under the load and at the fixed end. Check as in Problem 1.
   Ans. $R = \frac{5\,P}{16}$; $M_a = \frac{5\,P\,l}{32}$; $M_1 = -\frac{3\,P\,l}{16}$.
3. A 3-in. by 4-in. timber beam is supported at the left end and fixed 100 in. from the left end. It carries a load of 400 lb. 40 in. from the support. Find the unit stress in the outer fibers under the load and at the fixed end.
   Ans. 864 lb./in.^{2}; 840 lb./in.^{2}
4. Solve Problem 3 if the load is 40 in. from the fixed end of the span.
5. A 4-in. 7.7-lb. standard I-beam, 13 ft. 4 in. long, is supported at the ends and at the middle. A load of 3,072 lb. is placed 30 in. from the left support and an equal load is placed 30 in. from the right support. Find the reaction of each support and the stress over the middle support caused by the concentrated loads.
   Ans. 1,425 lb.; 3,294 lb.; and 1,425 lb.; 13,200 lb./in.^{2}
6. If the middle support of the beam of Problem 5 is lowered 0.12 in., and $E = 30{,}000{,}000$, what is the reaction of each support?
7. Derive an expression for the moment at the fixed end for a beam which is fixed at one end, supported at the other, and subjected to a load at a distance $k\,l$ from the support?
   Ans. $M_1 = -\frac{P\,l\,k}{2}(1 - k^2)$.

AREA MOMENTS

(This may be omitted)

Figure 187, III, shows the positive reaction triangle and the negative load triangle for a beam which is fixed at the right end, supported at the left end, and loaded at a distance b from the right end of the span. To find the deflection of the left end from the tangent at the fixed end,

$$0 = \frac{R\,l^2}{2} \times \frac{2\,l}{3} - \frac{P\,b^2}{2}\left(l - \frac{b}{3}\right); \tag{4}$$

$$\frac{R\,l^3}{3} = \frac{P\,b^3}{2}\left(l - \frac{b}{3}\right); \tag{5}$$

$$R = \frac{P\,b^2}{2\,l^3}(3\,l - b); \tag{6}$$

$$R = \frac{P(l-a)^2}{2\,l^3}(2\,l + a); \tag{7}$$

$$R = \frac{P}{2}(l - k)^2(2 + k) = \frac{P}{2}(2 - 3\,k + k^3), \tag{8}$$

in which $k = \frac{a}{l}$.

The slope at the left support is calculated from the area of the moment diagram. If θ_1 is this slope,

$$E\,I\,\theta_1 + \frac{R\,l^2}{2} - \frac{P\,b^2}{2} = 0. \tag{9}$$

When the value of R from Equation (6) is substituted in Equation (9),

$$E\,I\,\theta_1 = -\frac{P\,b^2}{4\,l}\,(3\,l - b - 2\,l) = -\frac{P\,b^2\,a}{4\,l}. \tag{10}$$

To find the equation of the elastic line by means of deflection from the tangent at the support,

$$E\,I\,y = E\,I\,\theta_1\,x + \frac{R\,x^2}{2} \times \frac{x}{3} - \frac{P(x-a)^2}{2} \times \frac{x-a}{3}; \tag{11}$$

$$E\,I\,y = -\frac{P}{12}\left(\frac{3\,b^2\,a\,x}{l} - \frac{3\,b^2\,x^3}{l^2} + \frac{b^3x^3}{l^3} + 2(x-a)^3\right). \tag{12}$$

The first three terms in the parenthesis of Equation (12) give the deflection when x is not greater than a. All four terms are required for the deflection between the load and the fixed end.

The deflection under the load is given by

$$E\,I\,y_a = -\frac{P\,b^2\,a^2}{12\,l}\left(3 - \frac{3\,a}{l} + \frac{(l-a)a}{l^2}\right); \tag{13}$$

$$E\,I\,y_a = -\frac{P\,b^2\,a^2}{12\,l}\left(3 - \frac{2a}{l} - \frac{a^2}{l^2}\right); \tag{14}$$

$$y_a = -\frac{P\,l^3\,k^2\,(1-k)^2}{12\,E\,I}(3 - 2k - k^2). \tag{15}$$

Problems

8. Check Eq. (12) by substituting $x = l$.
9. Find the deflection under the load when the load is at the middle. Solve first by the moment diagram to get the deflection from the tangent at the fixed end. Divide the positive trapezoid into two triangles. Check by Eq. (12) and again by Eq. (15).

   Ans. $y_a = -\dfrac{7\,P\,l^3}{768\,E\,I}$.
10. How does the deflection at the middle of a beam which is supported at one end, fixed at the other end, and loaded at the middle compare with the deflection of a simply-supported beam which is loaded at the middle? How does it compare with the deflection of a simply-supported beam which is uniformly loaded? *Ans.* $\frac{7}{16}$; $\frac{7}{10}$.
11. Locate the point of maximum deflection for the beam of Problem 9, and calculate the maximum deflection.

   Ans. $y_{max} = -\dfrac{\sqrt{5}\,P\,l^3}{240\,E\,I} = -\dfrac{0.009317\,P\,l^3}{E\,I}$ at $x = \dfrac{l}{\sqrt{5}}$.
12. Find the reaction at the support when the load is 0.6 l from it. Find the deflection at the middle and under the load. Find the moment at the fixed end. *Ans.* $R = 0.208\,P$; $M = -0.192\,P\,l$.

13. Find the reaction at the support and the slope over the support when the load is at 0.4 l from it.

Ans. $\theta_1 = -\dfrac{0.036\,P\,l^2}{E\,I}$.

14. Find the deflection under the load, and at $x = 0.2\,l$, $x = 0.4\,l$, $x = 0.6\,l$, and $x = 0.8\,l$ for the beam of Problem 13. Locate the point of contraflexure. Compare with Fig. 187.

Ans. Contraflexure at $0.7042\,l$; $E\,I\,y = -0.009792\,P\,l^3$ at $0.4\,l$, $-0.007381\,P\,l^3$ at $x = 0.6\,l$, and $-0.002602\,P\,l^3$ at $x = 0.8\,l$.

15. Calculate the slope at the left support for the beam of Problem 9. Get the deflection for Problems 9 and 11 by means of the deflection from the tangent at the left end.

INTEGRATION BETWEEN LIMITS

(This may be omitted)

Figure 188 is Fig. 187 reversed with the left end fixed. Both slope and deflection are zero at the origin. Formula XXVI (Art. 118) gives

$$M = M_a - \frac{M_a x}{l} + \frac{P\,a\,x}{l} - P(x - b). \tag{16}$$

In Fig. 188, M_a is drawn positive although it turns out to be negative. The first three terms of the moment expression apply to the entire span; the last term applies from the load to the right end.

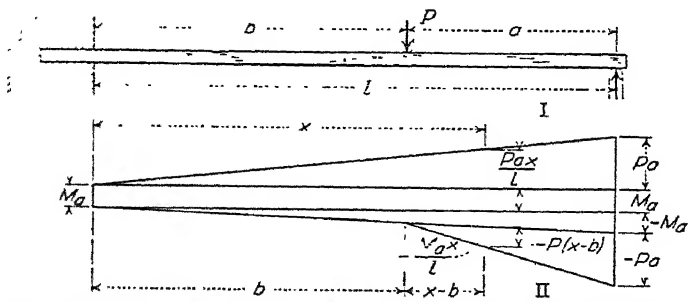


FIG. 188.—General moment diagram for beam fixed at left end.

$$E\,I\,\theta = 0 + M_a x - \frac{M_a x^2}{2\,l} + \frac{P\,a\,x^2}{2\,l} - \frac{P(x - b)^2}{2}; \tag{17}$$

$$E\,I\,y = \frac{M_a x^2}{2} - \frac{M_a x^3}{6\,l} + \frac{P\,a\,x^3}{6\,l} - \frac{P(x - b)^3}{6}; \tag{18}$$

$$0 = \frac{M_a l^2}{2} - \frac{M_a l^2}{6} + \frac{P\,a\,l^3}{6} - \frac{P\,a^3}{6}; \tag{19}$$

$$M_a = \frac{P\,a}{2\,l^2}(l^2 - a^2) = -\frac{P\,l\,k}{2}(1 - k^2). \tag{20}$$

Problems

16. Find the deflection under the load when the load is at the middle of the span. Find the slope at the support. Find the moment at the fixed end.

$$\textit{Ans. } \theta = \frac{P\,l^2}{32\,E\,I};\ y = -\frac{7\,P\,l^3}{768\,E\,I};\ M_a = -\frac{3\,P\,l}{16}.$$

17. Locate the position of maximum deflection when the load is 0.4 l from the support.

$$\textit{Ans. } \frac{42\,l}{71} = 0.5916\,l \text{ from fixed end.}$$

122. Uniformly Loaded Span, Fixed at Both Ends.—From the symmetry of Fig. 189, it is evident that the moment is the same at both ends of the span. Since the slope is zero at both ends,

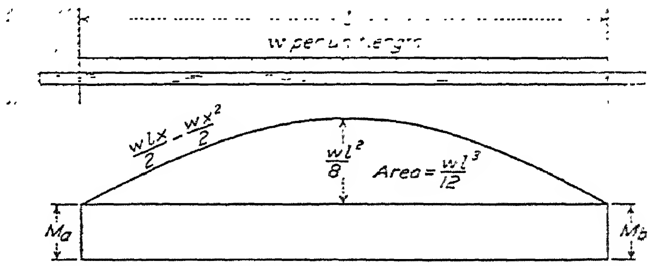


FIG. 189.—End-moment and simple-support diagrams.

the area of the resultant moment diagram must be zero. The end-moment diagram is a rectangle of which the area is $M_a l$. The simple-support diagram is a parabola of maximum altitude $\frac{w\,l^2}{8}$ and area $\frac{w\,l^3}{12}$.

$$M_a l + \frac{w\,l^3}{12} = 0; \tag{1}$$

$$M_a = -\frac{w\,l^2}{12} = -\frac{W\,l}{12}. \tag{2}$$

The moment equation is

$$M = -\frac{w\,l^2}{12} + \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}. \tag{3}$$

Problems

1. Find the moment at the middle of a uniformly loaded beam which is fixed at both ends.

$$\textit{Ans. } M = \frac{w\,l^2}{24} = \frac{W\,l}{24}.$$

2. Find the position of contraflexure.

$$Ans.\ x = \frac{3 \pm \sqrt{3}}{6}\, l = 0.21133\, l \text{ or } 0.78867\, l.$$

3. A 6-in. by 8-in. beam is 20 ft. long between fixed ends. It carries a load of 160 lb. per ft. What is the maximum unit stress? What would be the maximum unit stress if the beam were simply supported at the ends?

Ans. 1,000 lb./in.^{2}; 1,500 lb./in.^{2}

4. What is the unit stress at the top of the beam of Problem 3 at a section 3 ft. from the left end of the span and at a section 8 ft. from the end of the span.

Ans. S = 235 lb./in.^{2} tension; 440 lb./in.^{2} compression.

The portion of the span between the points of inflection may be regarded as a simply-supported beam. Each portion between a fixed end and a point of inflection is a cantilever with a uniform load and a concentrated load on the end.

Problems

5. Find the deflection of the points of contraflexure below the supports. Regard the portion between that point and the end as a cantilever which carries a uniformly distributed load and a load on the free end. Use the radical answer of Problem 2.

$$Ans.\ y = -\frac{w\, l^4}{864\, E\, I}.$$

6. Find the deflection of the middle of the beam below the points of inflect tion, and the deflection below the supports.

$$Ans.\ -\frac{5\, w\, l^4}{3456\, E\, I};\ -\frac{w\, l^4}{384\, E\, I}.$$

7. How does the maximum deflection of a uniformly loaded beam which is fixed at both ends compare with that of a uniformly loaded beam which is simply-supported?

8. The ends of a uniformly loaded beam are attached to columns which bend sufficiently to make the moment at each end equal to $-\frac{w\, l^2}{10}$ when the load is applied. Calculate the moment at the middle of the beam and locate the points of inflection.

$$Ans.\ M = \frac{w\, l^2}{40};\ x = \frac{5 \pm \sqrt{5}}{10}\, l = 0.2764\, l \text{ or } 0.7236\, l.$$

When a uniformly loaded span is absolutely fixed at the end, the moment at the ends is $-\frac{w\, l^2}{12}$ and the maximum positive moment at the middle is $\frac{w\, l^2}{24}$. A uniformly loaded beam which is simply-supported has the maximum moment of $\frac{w\, l^2}{8}$. For continuous reinforced-concrete beams it is customary to regard

the ends as partially fixed and use $-\frac{w\,l^2}{10}$ as the maximum moment at the ends.

AREA MOMENTS

(This may be omitted.)

If it is not assumed that M_a and M_b of Fig. 189 are equal, the end-moment diagram becomes a trapezoid, which may be divided into two triangles. To find the deflection at the left end from the tangent at the right end,

$$\frac{M_a l^2}{2} \times \frac{l}{3} + \frac{M_b l^2}{2} \times \frac{2\,l}{3} + \frac{w\,l^3}{12} \times \frac{l}{2} = 0; \tag{4}$$

$$\frac{M_a l^3}{6} + \frac{M_b l^3}{3} + \frac{w\,l^4}{24} = 0. \tag{5}$$

Similarly, to find the deflection of the right end from the tangent at the left end

$$\frac{M_a l^3}{3} + \frac{M_b l^3}{6} + \frac{w\,l^4}{24} = 0. \tag{6}$$

From Equations (5) and (6),

$$M_a = M_b = -\frac{w\,l^2}{12}.$$

Figure 190, II, gives separately the diagrams of the three terms of the moment equation. To derive the equation of the elastic line from these separate areas,

$$E\,I\,y = -\frac{w\,l^2\,x}{12} \times \frac{x}{2} + \frac{w\,l\,x^2}{4} \times \frac{x}{3} - \frac{w\,x^3}{6} \times \frac{x}{4};$$

$$y = -\frac{w\,x^2}{24\,E\,I}(l - x)^2. \tag{7}$$

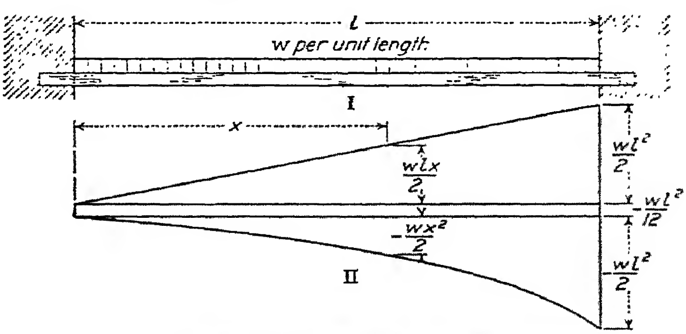


FIG. 190.—Diagrams for general moment equation.

Figure 191, II, shows the elastic line. Figure 191, IV, shows the resultant moment. The ordinates are on a larger scale than those of Fig. 190. The

scale of Fig. 191 would make the separate diagrams of Fig. 190 excessively high.

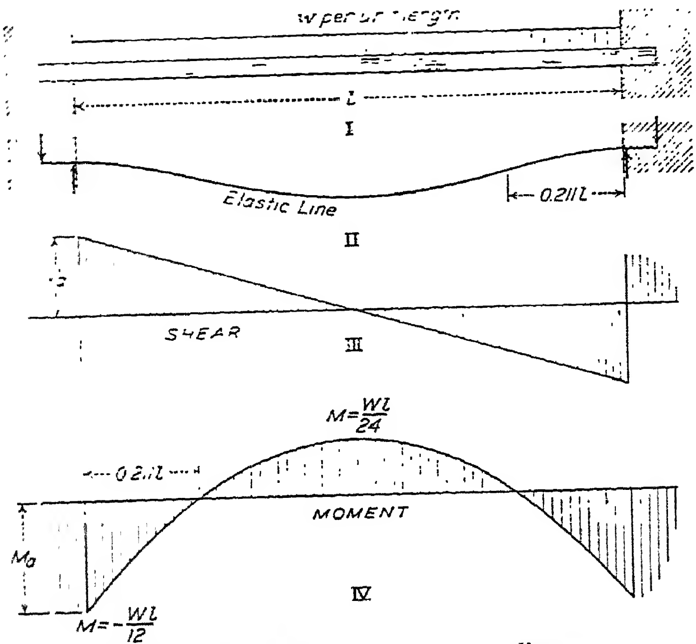


FIG. 191.—Shear and resultant moment diagrams.

Problems

9. Find the deflection at the middle, at one-fourth l, and at one-third l.

Ans. $y_{max} = -\frac{w\,l^4}{384\,E\,I}$; $y = -\frac{3\,w\,l^4}{2048\,E\,I}$ and $-\frac{w\,l^4}{486\,E\,I}$.

10. Find y_{max} by means of the deflection of the left end upward from the tangent at the middle. Use the diagrams of Fig. 190.

INTEGRATION BETWEEN LIMITS

(This may be omitted.)

If it is not assumed that $M_b = M_a$, the moment equation is

$$M = M_a + \frac{(M_b - M_a)\,x}{l} + \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}; \tag{8}$$

$$E\,I\,\theta = M_a x + \frac{(M_b - M_a)x^2}{2\,l} + \frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6}; \tag{9}$$

$$0 = M_a + \frac{M_b - M_a}{2} + \frac{w\,l^2}{4} - \frac{w\,l^2}{6}; \tag{10}$$

$$M_a + M_b = -\frac{w\,l^2}{6}. \tag{11}$$

$$E\,I\,y = \frac{M_a x^2}{2} + \frac{(M_b - M_a)x^3}{6\,l} + \frac{w\,l\,x^3}{12} - \frac{w\,x^4}{24}; \tag{12}$$

$$0 = \frac{M_a}{2} + \frac{M_b - M_a}{6} + \frac{w\,l^2}{12} - \frac{w\,l^2}{24}; \tag{13}$$

$$0 = \frac{M_a}{3} + \frac{M_b}{6} + \frac{w\,l^2}{24}. \tag{14}$$

From Equations (11) and (14), $M_a = M_b = -\frac{w\,l^2}{12}$;

$$E\,I\,\theta = -\frac{w\,l^2x}{12} + \frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6}; \tag{15}$$

$$E\,I\,y = -\frac{w\,l^2x^2}{24} + \frac{w\,l\,x^3}{12} - \frac{w\,x^4}{24} = -\frac{w\,x^2(l-x)^2}{24}. \tag{16}$$

Problem

11. Find the position at which the deflection is $-\frac{w\,l^4}{600\,E\,I}$.

Ans. $x = \frac{l}{2}\left(1 \pm \frac{\sqrt{5}}{10}\right)$.

123. Span Fixed at Both Ends, Load Concentrated.—For a beam fixed at both ends with a load concentrated at the middle, the moment is the same at both ends. Since the slope is zero at both ends, the area of the resultant moment diagram is zero. The simple-support diagram is a triangle of maximum altitude $\frac{P\,l}{4}$ and area $\frac{P\,l^2}{8}$. If M_1 is the moment at either end,

$$M_1 l + \frac{P\,l^2}{8} = 0; \quad M_1 = -\frac{P\,l}{8}. \tag{1}$$

Points of contraflexure are located at one-fourth the length from each end. One-half of the beam may be regarded as two cantilevers, each $\frac{l}{4}$ in length. The cantilever at the fixed end is bent downward by a load of $\frac{P}{2}$ at the point of contraflexure, while the other is bent upward from the tangent at the middle by an equal force. The deflection at one-fourth the length from the fixed end is

$$E\,I\,y = -\frac{P}{6}\left(\frac{l}{4}\right)^3; \quad y = -\frac{P\,l^3}{384\,E\,I} \tag{2}$$

$$y_{max} = -\frac{P\,l^3}{192\,E\,I}. \tag{3}$$

Problems

1. What is the moment at one-sixth and at one-third the length from the end of a beam which is fixed at the ends and loaded at the middle. Find the deflection at these points by means of the cantilever equations of preceding articles.

$$Ans.\ M = -\frac{P\,l}{24} \text{ and } \frac{P\,l}{24};\ y = -\frac{14\,P\,l^3}{10{,}368\,E\,I} \text{ and } -\frac{40\,P\,l^3}{10{,}368\,E\,I}.$$

2. What is the maximum slope for a beam which is fixed at the ends and loaded at the middle?

$$Ans.\ -\frac{P\,l^2}{64\,E\,I}.$$

3. How does the maximum deflection of a beam fixed at both ends and loaded at the middle compare with the deflection of a similar beam which is fixed at both ends and loaded uniformly (*a*) when the loads are equal, (*b*) when the maximum stresses are equal?

AREA MOMENTS

(This may be omitted.)

Figure 192 shows a beam which is fixed at both ends and subjected to a load P at a distance a from the left end and b from the right end. The

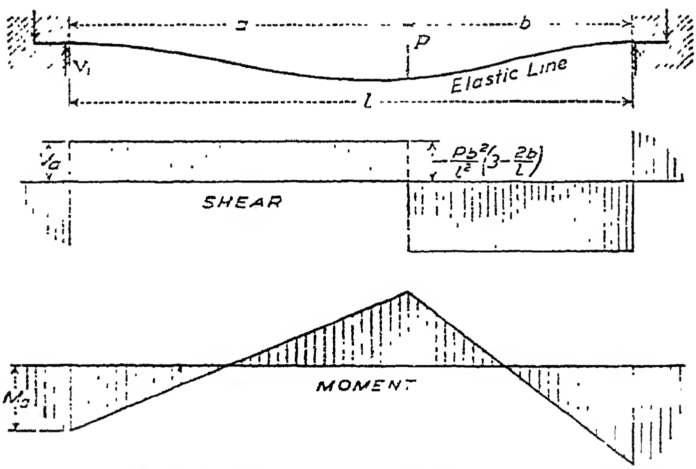


FIG. 192.—Shear and resultant moment diagrams for concentrated load.

end-moment diagram, which is not shown, is a trapezoid of altitude M_a at the left end and M_b at the right end. The simple-support diagram is a triangle of altitude $\frac{P\,a\,b}{l}$ and area $\frac{P\,a\,b}{2}$.

Since the slope is zero at each end,

$$\frac{(M_a + M_b)l}{2} + \frac{P\,a\,b}{2} = 0. \qquad (4)$$

The deflection of the right end from the tangent at the left end is zero.

$$0 = \frac{M_a l}{2} \times \frac{2\,l}{3} + \frac{M_b l}{2} \times \frac{l}{3} + \frac{P\,a\,b}{2} \times \frac{l + b}{3}; \tag{5}$$

$$2\,M_a + M_b + \frac{P\,a\,b}{l^2}(l + b) = 0. \tag{6}$$

From Equation (4), $M_a + M_b + \dfrac{P\,a\,b\,l}{l^2} = 0;$ (7)

$$M_a = -\frac{P\,a\,b^2}{l^2}; \quad M_b = -\frac{P\,a^2\,b}{l^2} \tag{8}$$

Problems

4. Find the moment at each end, the maximum positive moment, and the location of points of contraflexure for a beam which is fixed at the ends and loaded at six-tenths the length from the left end of the span.
Ans. $M_a = -0.096\,P\,l$; $M_b = -0.144\,P\,l$; M at load $= 0.1152\,P\,l$; contraflexure at $\dfrac{3\,l}{11}$ from left end, and $\dfrac{2\,l}{9}$ from right end.

5. Calculate the shear at the left end for Problem 4. Write the general moment equation and calculate the deflection under the load by means of the deflection from the tangent at the left end.
Ans. $y = -\dfrac{0.004608\,P\,l^3}{E\,I}$.

6. Find the point of maximum deflection for the beam of Problem 4 and calculate the maximum deflection.
Ans. $y_{max} = -\dfrac{6.336\,P\,l^3}{1{,}331\,E\,I} = -\dfrac{0.00476\,P\,l^3}{E\,I}$.

INTEGRATION BETWEEN LIMITS

(This may be omitted.)

Figure 193 shows the moment diagram drawn from the general moment equation (Formula XI of Art. 70) instead of the end-moment simple-support form of Equation (3) of Art. 118. Instead of integrating, the slope may be calculated geometrically from the areas of the separate moment diagrams, and the deflections may be calculated from the areas of the slope diagrams.

$$M = M_a + V_a x - P(x - a); \tag{9}$$

$$E\,I\,\theta = M_a x + \frac{V_a x^2}{2} - \frac{P(x - a)^2}{2}. \tag{10}$$

At the right end,

$$0 = M_a l + \frac{V_a l^2}{2} - \frac{P\,b^2}{2}; \tag{11}$$

$$2\,M_a + V_a l - \frac{P\,b^2}{l} = 0. \tag{12}$$

From the slope diagram, the deflection at any point x is given by

$$E\,I\,y = \frac{M_a x^2}{2} + \frac{V_a x^3}{6} - \frac{P(x-a)^3}{6}. \tag{13}$$

At this time, it should not be necessary to state that the last term of Equations (10) and (13) is valid to the right of the load only.

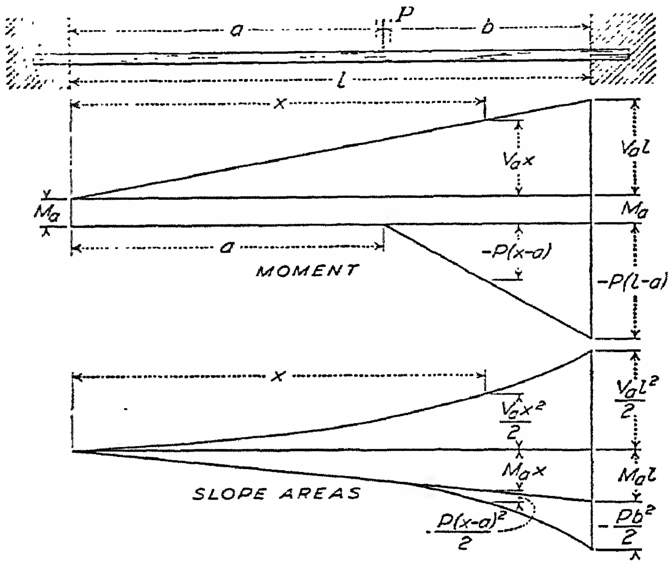


FIG. 193.—General moment and separate slope diagrams.

From Equation (13) at the right end,

$$0 = 3\,M_a + V_a l - \frac{P\,b^3}{l^2}. \tag{14}$$

$$0 = 2\,M_a + V_a l - \frac{P\,b^2 l}{l^2}. \tag{12}$$

$$M_a = -\frac{P\,b^2(l-b)}{l^2} = -\frac{P\,b^2 a}{l^2}. \tag{8}$$

Problems

7. Find the moment and the shear at the left end of a beam which is fixed at both ends and loaded at 0.4 l from the left end.
8. Find the deflection under the load for the beam of Problem 7. Find the maximum deflection. Find the maximum negative slope.

124. Two Moments.—Figure 182 of Art. 118 represents any uniformly loaded span. The moment at the left end is M_a. The arrow over the left support represents the moment which acts on the span from the portion of the beam at the left of the support. The direction is clockwise and the moment is positive. The arrow at the right support represents the moment which

acts on the span from the portion of the beam at the right of the support. This arrow is counterclockwise from right to left; consequently, the moment from left to right is clockwise and M_b also is positive. It is desirable to derive an equation which will give the slope at the ends of the span in terms of the end moments and the loading. This will be done by area moments and by successive integration between limits. Either method may be used. Area moments is often briefer. On the other hand, integration between limits gives the equation of the elastic line in addition to the required equation of two moments.

AREA MOMENTS

By using Fig. 182 to find the deflection of the right end of the span from the horizontal line through the left end,

$$0 = E\,I\,\theta_1 l + \frac{M_a l}{2} \times \frac{2\,l}{3} + \frac{M_b l}{2} \times \frac{l}{3} + \frac{w\,l^3}{12} \times \frac{l}{2}; \tag{1}$$

$$0 = E\,I\,\theta_1 l + \frac{M_a l^2}{3} + \frac{M_b l^2}{6} + \frac{w\,l^4}{24}; \tag{2}$$

$$6\,E\,I\,\theta_1 + 2\,M_a l + M_b l + \frac{w\,l^3}{4} = 0. \tag{3}$$

Equation (2) is multiplied by $\frac{6}{l}$ to get Equation (3).

INTEGRATION BETWEEN LIMITS

Equation (3) of Art. 118 gives the moment in terms of the end moments and the loads on the span. Since all integrals are zero at the lower limit, it is not necessary to write limiting expressions.

$$M = M_a + \frac{(M_b - M_a)x}{l} + \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}; \tag{4}$$

$$E\,I\,\theta = E\,I\,\theta_1 + M_a x + \frac{(M_b - M_a)x^2}{2\,l} + \frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6}; \tag{5}$$

$$E\,I\,y = E\,I\,\theta_1 x + \frac{M_a x^2}{2} + \frac{(M_b - M_a)x^3}{6\,l} + \frac{w\,l\,x^3}{12} - \frac{w\,x^4}{24}; \tag{6}$$

$$0 = E\,I\,\theta_1 l + \frac{M_a l^2}{2} + \frac{(M_b - M_a)l^2}{6} + \frac{w\,l^4}{12} - \frac{w\,l^4}{24}; \tag{7}$$

$$0 = E\,I\,\theta_1 + \frac{M_a l}{3} + \frac{M_b l}{6} + \frac{w\,l^3}{24}; \tag{2}$$

$$6\,E\,I\,\theta_1 + 2\,M_a l + M_b l + \frac{w\,l^3}{4} = 0. \tag{3}$$

The first three terms of Equation (3) are derived from the end-moment trapezoid. They are the same, in terms of the end moments, no matter what the loading may be. The remaining term (or terms) are derived from the simple-support moment diagrams.

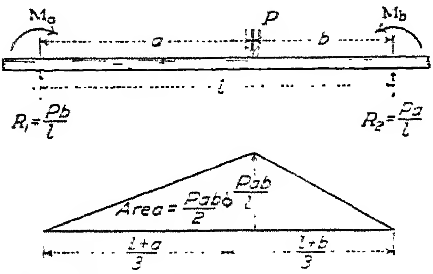


FIG. 194.—Simple-support diagram for concentrated load.

AREA MOMENTS, LOAD CONCENTRATED

Figure 194 shows the simple-support triangle for a beam which is loaded at a distance a from the left support of the span. The maximum moment is $\frac{P\,a\,b}{l}$ and the area of the triangle is $\frac{P\,a\,b}{2}$. The center of gravity is $\frac{l+a}{3}$ from the left end and $\frac{l+b}{3}$ from the right end. The moment of the area with respect to the right end is

$$\frac{P\,a\,b}{2} \times \frac{l+b}{3} = \frac{P\,a\,b(l+b)}{6} = \frac{P\,b(l^2 - b^2)}{6}. \tag{8}$$

Combining this area moment with the first three terms of Equation (2):

$$E\,I\,\theta_1 l + \frac{M_a l^2}{3} + \frac{M_b l^2}{6} + \frac{P\,a\,b(l+b)}{6} = 0; \tag{9}$$

$$6\,E\,I\,\theta_1 + 2\,M_a l + M_b l + \frac{P\,a\,b(l+b)}{l} = 0; \tag{10}$$

$$6\,E\,I\,\theta_1 + 2\,M_a l + M_b l + \frac{P\,b(l^2 - b^2)}{l} = 0. \tag{11}$$

INTEGRATION BETWEEN LIMITS, LOAD CONCENTRATED

To find the deflection of the simply-supported beam from the tangent at the left support, the moment is

$$\frac{P\,b\,x}{l} - P(x - a).$$

Slope change is $\frac{P\,b\,x^2}{2\,l} - \frac{P(x-a)^2}{2}.$

Deflection from tangent is given by

$$E\,I\,y = \frac{P\,b\,x^3}{6\,l} - \frac{P(x-a)^3}{6}.$$

Deflection of right end from tangent at left end for a simply-supported beam is

$$\frac{P\,b\,l^2}{6} - \frac{P\,b^3}{6} = \frac{P\,b(l^2 - b^2)}{6} = \frac{P\,a\,b(l+b)}{6}. \qquad (12)$$

When this deflection is added to the first three terms of Equation (2), the result is

$$E\,I\,\theta_1 l + \frac{M_a l^2}{3} + \frac{M_b l^2}{6} + \frac{P\,b(l^2 - b^2)}{6} = 0; \qquad (13)$$

$$6\,E\,I\,\theta_1 + 2\,M_a l + M_b l + \frac{P\,b(l^2 - b^2)}{l} = 0. \qquad (11)$$

The last term of Equation (11) may be written

$$\frac{P\,a\,b(l+b)}{l}.$$

Equation (3) may be called the *equation of two moments* for a uniformly loaded span which is supported or fixed at the ends. Equation (10) is the equation of two moments for a span which carries a single concentrated load. The first term of these equations, which includes the slope, may be called the *end-slope term.* The next two, which include the end moments, may be called the *end-moment terms.* The last term, which is calculated from the moment diagram of the span as a simply-supported beam, may be called the *simple-support term.*

In the next paragraph it is assumed that the two-moment equation begins at the left end of the span.

To obtain the simple-support term, the moment of the simple-support moment diagram about the right end of the span is multiplied by $\frac{6}{l}$. Expressed in another way, the deflection of the right end of the simply-supported beam from the tangent at the left end is multiplied by $\frac{6\,E\,I}{l}$ to give the simple-support term.

If the two-moment equation begins at the right end, right and left change places. If the slope at the right end in the negative

direction toward the left is θ_{21}, which means the slope at 2 in the direction toward 1, the two-moment equation for a uniformly loaded span is

$$6\,E\,I\,\theta_{21} + M_a l + 2\,M_b l + \frac{w\,l^3}{4} = 0. \tag{14}$$

Since the combined simple-support moment diagram for a uniformly loaded span is symmetrical with respect to the middle ordinate, the simple-support term is the same in either direction. The combined simple-support diagram for a concentrated load is not symmetrical, except when the load is at the middle. From right toward left, Equation (11) becomes

$$6\,E\,I\,\theta_{21} + M_a l + 2\,M_b l + \frac{P\,a(l^2 - a^2)}{l} = 0. \tag{15}$$

To get Equation (15) from Equation (11), a and b change places.

Example I

A uniformly loaded span of length l is fixed at the left end and supported at the right end. Find the moment at the fixed end (Fig. 185).

$$0 + 2\,M_a l + 0 + \frac{w\,l^3}{4} = 0;\; M_a = -\frac{w\,l^2}{8}.$$

Example II

A span fixed at the right end and supported at the left end (Fig. 187) carries a load P at a distance a from the left end and at a distance b from the fixed end. Find the moment at the right end. Compare Eq. (20) of Art. 121.

$$0 = 2\,M_b\,l + \frac{P\,a(l^2 - a^2)}{l};\; M_b = -\frac{P\,a(l^2 - a^2)}{2\,l^2} = -\frac{P\,a\,b(l + b)}{2\,l^2}.$$

Example III

A span fixed at both ends (Fig. 191) carries a uniformly distributed load. Find the moment at both ends without assuming that they are equal.

$$2\,M_a l + M_b l + \frac{w\,l^3}{4} = 0; \quad M_a l + 2\,M_b l + \frac{w\,l^3}{4} = 0.$$

$$M_a = M_b = -\frac{w\,l^2}{12}.$$

Example IV

A span fixed at both ends carries a load P at a distance a from the left end and at a distance b from the right end. Find the moment at each end.

$$2\,M_a l + M_b l + \frac{P\,a\,b(l + b)}{l} = 0;$$

$$M_a l + 2 M_b l + \frac{P a b (l + a)}{l} = 0;$$

$$4 M_a l + 2 M_b l + P a b (2 l + 2 a) = 0;$$

$$3 M_a l + \frac{P a b (l + 2 a - b)}{l} = 0;$$

$$M_a = -\frac{P a^2 b}{l^2}.$$

These four examples solve the indeterminate portion of Arts. 119 to 123 and illustrate the convenience of the theorem of two moments for indeterminate problems. The theorem might be used in place of the derivations which are given in case it is necessary to abbreviate the course. The theorem is given here instead of earlier because many students would substitute instead of mastering first principles.

Problems

1. A uniformly loaded beam, 22 in. long, is supported 4 in. from the left end and 6 in. from the right end. Find the slope at the left support by the theorem of two moments. Find the slope at the right support toward the span. *Ans.* $E I \theta_1 = -4 w$; $E I \theta_{21} = 16 w$.
2. In Problem 1, find the slope at the right support in the direction toward the right end of the beam by adding the area of the moment diagram of the span to $E I \theta_1$. Use the end-moment trapezoid and the combined simple-support parabola as in Fig. 182.
3. With the origin of coördinates at the left support, derive the equation of the elastic line for the span of Problem 1. Use general moment Eq. (3) of Art. 118. Check the slope equation at the right support. Check the deflection equation at the right support.

   Ans. $E I \theta = -4 w - 8 w x - \frac{5 w x^2}{12} + 3 w x^2 - \frac{w x^3}{6}.$
4. Find the deflection at the middle of the span of Problem 1 by means of the deflection equation of Problem 3. Check by area moments without using the deflection equation.

   Ans. $y = -\frac{36 w}{E I}.$
5. If the length of the span of Fig. 195 is l and each end overhangs $0.1\,l$, find the slope at each end when a load of $0.4\,P$ is placed on the left end, a load of $0.16\,P$ is placed on the right end, and a load P is placed $0.6\,l$ from the left support. Check by means of the area of the end-moment diagram (not drawn) and the combined simple-support diagram.

   Ans. $E I \theta_{21} = -0.52\,P l^2$.
6. Derive the equation of the elastic line for the beam of Problem 5, preferably by integration between limits. Calculate the deflection at $0.4\,l$, $0.6\,l$, and $0.8\,l$. Check the last one by area moments from the combined end-moment diagram and the separate simple-support diagrams.

   Ans. $E I y = -0.01468\,P l^3$; $-0.01592\,P l^3$; $-0.00997\,P l^3$.
7. The uniformly loaded beam of Fig. 196 is fixed at the right end and supported 60 in. from the fixed end. The beam overhangs the left

support 20 in. Find the moment at the fixed end by the theorem of two moments. Then find the reaction of the support. Find the slope at

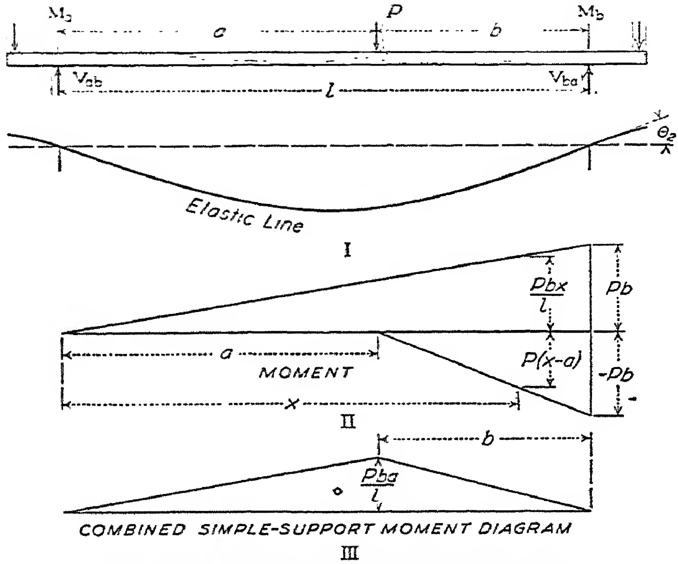


FIG. 195.—Elastic line and moment diagrams.

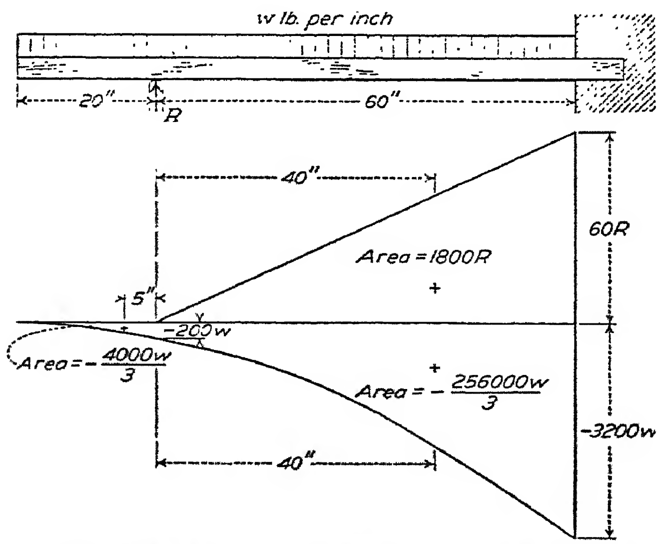


FIG. 196.—Separate moment diagrams for overhanging beam.

the support by means of the separate diagrams shown in the figure.

Ans. $R = 47.5\,w;\ \theta_1 = -\dfrac{1{,}500\,w}{EI}$.

8. In Problem 7, check the slope at the left support by means of the deflection of the right end from the horizontal through the left support. Solve by area moments or by integration between limits.
9. Solve for V_{ab} in Fig. 197 by the general moment equation as shown by the moment diagram, using area moments or integration between limits. Calculate the moment at the fixed end and the slope over the support.

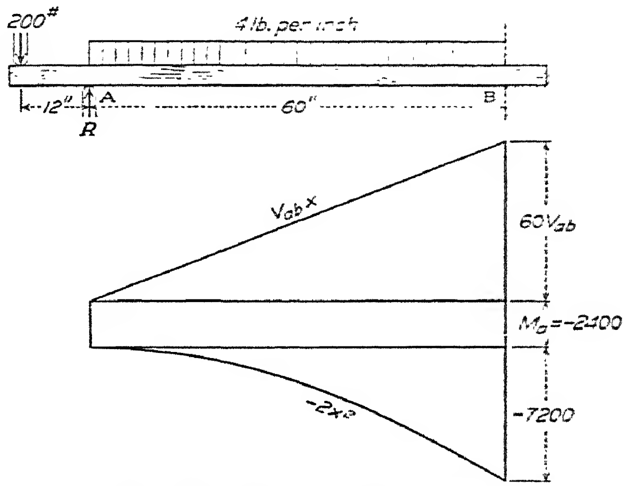


FIG. 197.—General moment diagrams.

10. Solve Problem 9 by the equation of two moments. Compare with results of Problem 9.
11. A beam 10 in. long between supports overhangs the left support 4 in. and the right support 2 in. There is a uniformly distributed load of 12 lb. per in. over the entire length, a load of 30 lb. on the right end, and a load of 60 lb. 7 in. from the left support. Find the slope at the left end of the span.

Ans. $\theta_1 = -\dfrac{313}{EI}$.

12. Using the answer of Problem 11, find the slope at the right end of the span by means of the area of the moment diagram. Check by the equation of two moments. Why is there a difference in sign?
13. Find the deflection of the beam of Problem 11 at the middle of the span. Find the deflection at the free ends by means of the slope at the supports and the cantilever formulas.

Ans. $-\dfrac{1{,}427.5}{EI}$; $+\dfrac{868}{EI}$; $+\dfrac{730}{EI}$

14. Find the points of inflection for the beam of Problem 11.

Ans. 1.35 in., and 9.12 in. from left support.

15. A beam fixed at the left end is supported at a distance l from the left end and overhangs the support 0.4 l. It carries a uniformly distributed load of w per unit length. Find the moment at the fixed end by the two-moment equation. *Ans.* $M_1 = -0.21\, w\, l^2$.

16. A beam weighing 12 lb. per ft. is fixed at the left end and supported 10 ft. from the fixed section. It overhangs the support 6 ft. and carries 36 lb. 1 ft. from the free end and 60 lb. 6 ft. from the fixed section. Find the moment at the fixed end. Find the shear at the fixed end.

Ans. $M_1 = -528$ ft.-lb.

125. Theorem of Three Moments.—The methods of the preceding articles may be applied to any number of spans and to any distribution of loads. When, as usually happens, it is desired to find the moments, reactions, and shears without calculating the deflections, the *theorem of three moments* is a valuable labor-saving device.

The theorem of three moments is an *algebraic equation* which expresses the relation of the moments at three successive supports

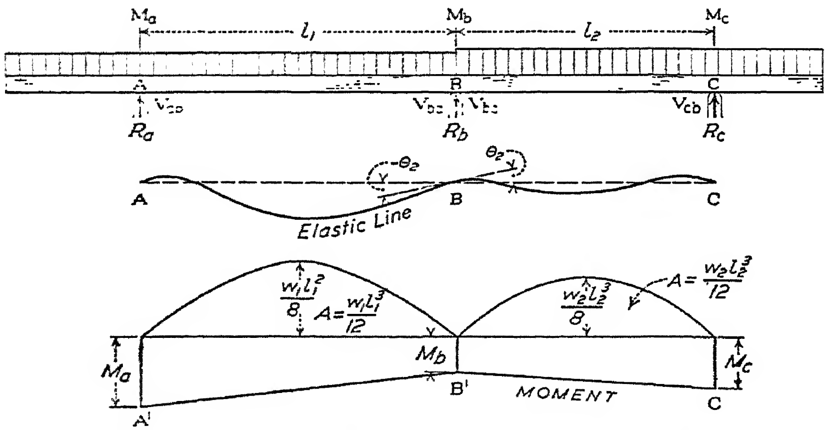


FIG. 198.—End-moment and simple-support diagrams for two spans.

of a continuous beam in terms of the lengths of the intervening spans and the loads which they carry. In Fig. 198, the moments over the supports are M_a, M_b, and M_c. The length of the span between support A and support B is l_1, and the length between support B and support C is l_2. The figure represents a uniformly distributed load of w_1 over the first span and w_2 over the second span.

The subscripts a, b, and c represent order from left to right and may be applied to *any three supports in succession.*

The shear adjacent to B on the right side toward C is V_{bc}. The shear on the side of B toward A is V_{ba}. In all cases shear is from the left side of the section to the right side.

When there are four supports, two equations are written. For the first equation, supports 1, 2, and 3 are represented by

A, B, and C. For the second equation, 2, 3, and 4 follow in the same order.

126. Three Moments, Load Uniformly Distributed.—When the equation of two moments is written for the second span of Fig. 198, the result is

$$6\,E\,I\,\theta + 2\,M_b l_2 + M_c l_2 + \frac{w_2 l_2^3}{4} = 0, \qquad (1)$$

in which θ is the slope at B in the direction toward C. The slope at B toward A is equal and opposite θ. For the first span, from B toward A,

$$-6\,E\,I\,\theta + 2\,M_b l_1 + M_a l_1 + \frac{w_1 l_1^3}{4} = 0. \qquad (2)$$

By addition of Equations (1) and (2),

$$M_a l_1 + 2\,M_b(l_1 + l_2) + M_c l_2 = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4}. \qquad (3)$$

Equation (3) is the theorem of three moments for uniformly distributed loads. It is understood, of course, that the three supports are at the same level. If they are not, the slopes in opposite directions at B would not have equal *magnitudes*. For equally loaded spans of equal length, $w_1 = w_2$, $l_1 = l_2$ and

$$M_a + 4\,M_b + M_c = -\frac{w\,l^2}{2} \quad \text{Formula XXVII}$$

Example I

A beam of uniform weight w per unit length rests on four supports to form three equal spans, each of length l, and overhangs the left support $0.4\,l$ and the right support $0.2\,l$. Find the moment over each support.

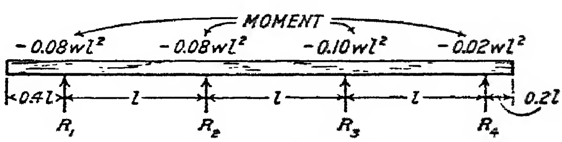


FIG. 199.—Three equal spans.

From the overhanging ends $M_1 = -0.4\,w\,l \times 0.2\,l = -0.08\,w\,l^2$; $M_4 = -0.2\,w\,l \times 0.1\,l = -0.02\,w\,l^2$. Writing Formula XXVII for the first three supports:

$$-0.08\, w\, l^2 + 4\, M_2 + M_3 = -0.50\, w\, l^2; \tag{4}$$
$$4\, M_2 + M_3 = -0.42\, w\, l^2. \tag{5}$$

For the second, third, and fourth supports,

$$M_2 + 4\, M_3 - 0.02\, w\, l^2 = -0.50\, w\, l^2; \tag{6}$$
$$M_2 + 4\, M_3 = -0.48\, w\, l^2. \tag{7}$$

Solving Eqs. (5) and (7) simultaneously, $M_2 = -0.08\, wl^2$;

$$M_3 = -0.10\, w\, l^2.$$

Example II

A uniformly loaded beam, 24 ft. long, is supported 6 ft. from the left end, 16 ft. from the left end, and at the right end. Find the moment over each support.

$$-18\, w \times 10 + 36\, M_2 + 0 = -\frac{w}{4}\,(1{,}000 + 512)$$
$$M_2 = -5.5\, w.$$

Problems

1. A beam 20 ft. long is supported 2 ft. from the left end, 10 ft. from the left end, and at the right end. It carries 60 lb. per ft. over the entire length and 30 lb. concentrated at the left end. Find the moment over each support. *Ans.* −180 ft.-lb.; −590 ft.-lb.; 0.
2. A uniformly loaded beam, 20 ft. long, is supported at the ends and 12 ft. from the left end. Find the moment over the intermediate support. *Ans.* $M_b = -14\, w$.
3. A uniformly loaded beam, 22 ft. long, is supported at the left end, 12 ft. from the left end, and 2 ft. from the right end. Find the moment over each support. *Ans.* $M_b = -13.60\, w$; $M_c = -2\, w$ ft.-lb.
4. A uniformly loaded beam, 26 ft. long, is supported 4 ft. from the left end, 16 ft. from the left end, and 2 ft. from the right end. Find the moment over each support. *Ans.* $M_a = -8\, w$; $M_b = -11.2\, w$; $M_c = -2\, w$.
5. A beam 16 ft. long is supported at the ends and 6 ft. from the right end. It carries 24 lb. per ft. over the 10-ft. span and 40 lb. per ft. over the 6-ft. span. Find the moment over the intermediate support. *Ans.* $M_b = -255$ ft.-lb.
6. A beam 26 ft. long is supported 4 ft. from the left end, 14 ft. from the left end, and 2 ft. from the right end. It carries 18 lb. per ft. over the left 14 ft. and 36 lb. per ft. over the remainder. Find the moment over each support. *Ans.* −144 ft.-lb.; −283.5 ft.-lb.; −72 ft.-lb.
7. Solve Problem 6 by the shortest method for a uniform load of 24 lb. per ft. over the entire length.
8. Solve Problem 6 for M_b if there is no load over the left 10-ft. span and the load over the remainder is not changed. *Ans.* $M_b = -171$ ft.-lb.

9. Find the moment over each support for four equal spans with uniform load and no overhang, as shown in Fig. 200.

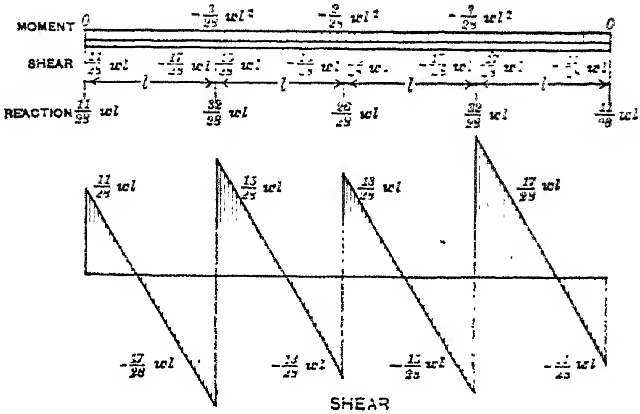


FIG. 200.—Beam of four spans.

10. Solve Problem 9 for an overhang of 0.4 l at the left end and a uniform load of w per unit length over the four spans and 1.5 w per unit length on the overhang.

Ans. $-0.12\ w\ l^2$; $-0.075\ w\ l^2$; $-0.08\ w\ l^2$; $-0.105\ w\ l^2$; 0,

11. The beam of Fig. 198 overhangs the left support 5 in. and the right support 4 in. The left span is 12 in. long and the right span is 10 in. long. The load is 16 lb. per in. over 17 in. from the left end to the intermediate support and 18 lb. per in. over the remaining 14 in. There is a concentrated load of 31 lb. 3 in. from the left end and another of 29 lb. 2 in. from the right end. Find the moment over each support. (*These data were used in calculating the elastic line of Fig.* 198.)

Ans. $M_a = -262$ lb.-in.; $M_b = -142$ lb.-in.; $M_c = -202$ lb.-in.

12. Find the moment over each support for the beam of Fig. 198 if the load is 16 lb. per in. over the left 5 in., 10 lb. per in. over the next 12 in., 12 lb. per in. over the remainder of the beam, and 22 lb. concentrated 2 in. from the right end. *Ans.* -200 lb.-in.; -80 lb.-in.; -140 lb.-in.

13. A uniformly loaded beam rests on three supports to form two equal spans with equal overhang on each end. What must be the ratio of the overhang to the length of a span if the moment is the same at each support? What is this moment? *Ans.* $a/l = 0.408$; $M = -w\ l^2/12$.

14. Solve Problem 13 for three spans.

127. Reactions by Moments.—After the moments over the supports have been calculated by the theorem, the reaction of any support may be determined by equating the known moment at an adjacent support to the moment about that support of the unknown reaction and of all other known forces which act on the portion of the beam which is taken as the free body.

Example

In Fig. 199, which applies to Example I of the preceding article, find the reaction of each support.

To find the left reaction, R_1, moments are taken about a section above the second support, at which the resisting moment is known to be $-0.08\,w\,l^2$. The portion of the beam to the left of the second support is taken as the free body.

$$R_1 l - 1.4\,w\,l \times 0.7\,l = -0.08\,w\,l^2; \tag{1}$$
$$R_1 l = 0.90\,w\,l^2; \quad R_1 = 0.90\,w\,l. \tag{2}$$

To find R_2 moments are taken about a section over the third support.

$$0.90\,w\,l \times 2\,l + R_2 l - 2.4\,w\,l \times 1.2\,l = -0.10\,w\,l^2; \tag{3}$$
$$R_2 l = 0.98\,w\,l^2; \quad R_2 = 0.98\,w\,l. \tag{4}$$

A similar moment equation about a section over the fourth support gives $R_3 = 1.10\,w\,l$. To find R_4, the portion of the beam to the right of the third support is used as the free body. $R_4 = 0.62\,w\,l$.

Problems

1. In the example above, check R_4 and R_3 by moments about the section over the second support. Check all reactions by vertical resolution. Check all by moments about the right end.
2. Find the reactions over the supports for the beam of Example I of Art. 126. *Ans.* $R_1 = 0.88\,w\,l$; $R_2 = 1.10\,w\,l$; $R_3 = 0.62\,w\,l$.
3. Find the reactions of the supports for the beam of Problem 11 of the preceding article. Check. *Ans.* 217 lb., 170 lb., and 197 lb.
4. Find the reaction of each support for the beam of Problem 13 of the preceding article. *Ans.* 150 lb., 104 lb., and 156 lb.

128. Reactions by Shear.—When there are more than three supports, the calculation of reactions by the method of Art. 127 becomes laborious. Moreover, the method cannot be used when the ends are fixed. A more general method, applicable to any number of spans, is based on the difference between the shear on opposite sides of the support.

Infinitely close to the right of support A of Fig. 198, the shear as given by Formula XXVI of Art. 118 is

$$V_{ab} = \frac{M_b - M_a}{l} + \frac{w\,l}{2}. \tag{1}$$

A similar expression applies at the right of the second support. When the shear is known at the left of any span from Equation (1), the shear at the right end of the span is found by subtracting the intervening load. The reaction of any support is found by subtracting the shear at the left from the shear at the right.

Usually, the shear is negative at the left side and positive at the right, which makes the algebraic difference the sum of the two shears.

Example

To find the shear at the right side of support A of Fig. 198 from the data of Problem 11 of Art. 126,

$$V_{ab} = \frac{-142 + 262}{12} + 96 = 106 \text{ lb.}$$

At the left side of support A by definition of shear,

$$V_{ao} = -80 - 31 = -111;$$
$$R_a = 106 - (-111) = 217 \text{ lb.}$$

Problems

1. Solve for the remaining reactions for the beam of Problem 13 of Art. 126.
2. Find the reactions for Problem 14 of Art. 126.
3. Find the shears and reactions for the beam of Problem 10 (Art. 126). Compare with Fig. 200.
4. The beam of Fig. 200 carries a load w per unit length over the first, third, and fourth span, and a load of 2 w over the second span. Find the moment over each support. Find the reactions.

Ans. Moments $= \left(-\frac{w\,l^2}{32}\right) \times (0,\ 5,\ 4,\ 3,\ 0)$; reactions $= \left(\frac{w\,l}{32}\right) \times (11, 54, 48, 34, 13)$.

5. A beam of four equal spans carries a load of w per unit length over the second span and no load on the others. If the supports are hinged to the beam so that they may exert downward as well as upward reactions, find the moment at each. Check by adding these moments to the moments of Fig. 200. Find the reactions and check with Problem 4 and Fig. 200.

Ans. Moments $= 0,\ -\frac{11\,w\,l^2}{224},\ -\frac{12\,w\,l^2}{224},\ +\frac{3\,w\,l^2}{224},\ 0.$

6. Find the moments and reactions for a uniformly loaded beam of five equal spans.
7. Solve Problem 6 if the beam overhangs the left support four-tenths the length of a span.

129. Three Moments, Loads Concentrated.—Equation (11) of Art. 124 is the equation of two moments for a single concentrated load:

$$6\,E\,I\,\theta_1 + 2\,M_a l + M_b l + \frac{P\,b(l^2 - b^2)}{l} = 0, \qquad (1)$$

in which M_a is the moment at any support, θ_1 is the slope at that support toward the span, M_b is the moment at the other support of the span, and b is the distance of the load P from this second support. When this equation is applied to the second

span of Fig. 201 to express the slope θ at the middle support B,

$$6\,E\,I\,\theta + 2\,M_b l_2 + M_c l_2 + \frac{Q\,c(l_2^2 - c^2)}{l_2} = 0. \tag{2}$$

In the opposite direction from B,

$$-6\,E\,I\,\theta + 2\,M_b l_1 + M_a l_1 + \frac{P\,a(l_1^2 - a^2)}{l_1} = 0. \tag{3}$$

Addition of these two equations gives the theorem of three moments for a single concentrated load on each span.

$$M_a l_1 + 2\,M_b(l_1 + l_2) + M_c l_2 = -\frac{P\,a(l_1^2 - a^2)}{l_1} - \frac{Q\,c(l_2^2 - c^2)}{l_2}. \tag{4}$$

The moment terms of Equations (4) and (3) of Art. 126 for a uniformly distributed load are the same. The equations for any

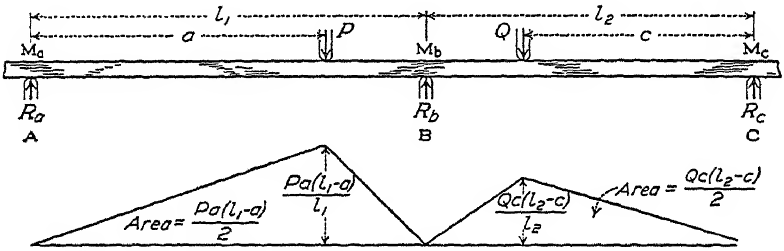


FIG. 201.—Simple-support diagrams for concentrated loads.

loading differ only in the simple-support terms, which depend upon the loading.

Example

A beam 23 ft. long is supported 2 ft. from the left end, 12 ft. from the left end, and 3 ft. from the right end. It carries 90 lb. on the left end, 96 lb. on the right end, 120 lb. 6 ft. from the left end, and 240 lb. 5 ft. from the right end. Find the moments and reactions. Draw the shear diagram and find the maximum moment on each span.

$$-180 \times 10 + 36\,M_2 - 288 \times 8 = -\frac{120 \times 4 \times 84}{10} - \frac{240 \times 2 \times 60}{8}; \tag{5}$$

$$\begin{aligned} 36\,M_2 &= 1{,}800 - 4{,}032 \\ &\ 2{,}304 - 3{,}600 \\ 36\,M_2 &= 4{,}104 - 7{,}632 = -3{,}528 \\ M_2 &= -\ \ 98 \end{aligned}$$

$$V_{ab} = \frac{-98 + 180}{10} + \frac{6}{10} \times 120 = 8.2 + 72 = 80.2 \text{ lb.}$$

$V_{ba} = 80.2 - 120 = -39.8$; $V_{bc} = 36.25$; $V_{cb} = -203.25$.
$R_a = 80.2 - (-90) = 170.2$; $R_b = 76.05$; $R_c = 299.75$.
$M_{max} = 140.8$ ft.-lb. in first span; $M_{max} = 119.5$ ft.-lb. in second span.

Problems

1. A beam 35 ft. long is supported 3 ft. from the left end, 15 ft. from the left end, 23 ft. from the left end, and 2 ft. from the right end. It carries 80 lb. on the left end, 50 lb. on the right end, 120 lb. 7 ft. from the left support, 160 lb. in the second span 5 ft. from the second support, and 200 lb. 6 ft. from the fourth support. Solve for the moments and reactions. *Ans.* $M_3 = -266.45$ ft.-lb.; $R_1 = 139.71$; $R_2 = 102.43$.
2. A beam 30 ft. long is supported 4 ft. from the left end, 14 ft. from the left end, and 2 ft. from the right end. It carries 120 lb. per ft., 600 lb. 3 ft. to the right of the left support, and 840 lb. 9 ft. to the right of the intermediate support. Find the moment over each support and find the reactions. *Ans.* $M_2 = -3{,}480$ ft.-lb. $R_1 = 1{,}248$ lb.

130. Continuous Beams, Ends Fixed.—When an end of a continuous beam is fixed, as in Fig. 202, the equation of two moments may be written for the span adjacent to the fixed end, and the equation of three moments may be written for this span and the one next to it. For the 8-foot span of Fig. 202 the

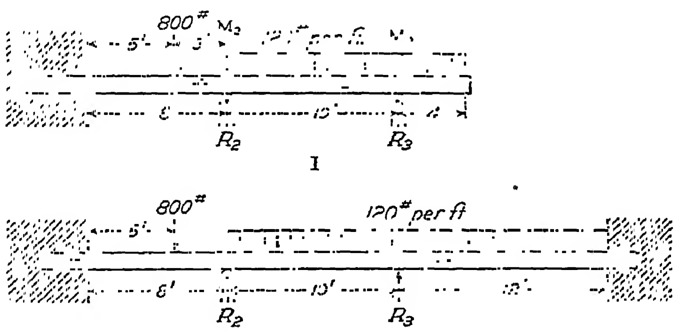


FIG. 202.—Continuous beams with one or both ends fixed.

equation of two moments, starting from the left end, at which the slope is zero, gives

$$8 \times 2\,M_1 + 8\,M_2 + \frac{800 \times 3 \times 55}{8} = 0; \qquad (1)$$

$$16\,M_1 + 8\,M_2 = -16{,}500. \qquad (2)$$

For the first two spans of Fig. 202, I, the theorem of three moments gives

$$8\,M_1 + 36\,M_2 - 960 \times 10 = -\frac{800 \times 5 \times 39}{8} - \frac{120 \times 1{,}000}{4}; \qquad (3)$$

$$8\,M_1 + 36\,M_2 = -39{,}900; \qquad (4)$$

$$M_1 = -536.72; \quad M_2 = -989.06.$$

Example

The beam of Fig. 202, I, is extended to form a third span (Fig. 202, II) which is 12 ft. long, is fixed at the right end, and is uniformly loaded with 120 lb. per ft. Find the moment at the fixed ends and at the supports.

Equation (4) is now changed to read

$$8\,M_1 + 36\,M_2 + 10\,M_3 = -\frac{800 \times 5 \times 39}{8} - \frac{120 \times 1{,}000}{4}. \tag{5}$$

For the second and third spans,

$$10\,M_2 + 44\,M_3 + 12\,M_4 = -81{,}840. \tag{6}$$

For the third span alone,

$$12\,M_3 + 24\,M_4 + \frac{120 \times 1{,}728}{4} = 0. \tag{7}$$

$$M_1 = -579.5; \quad M_2 = -903.6; \quad M_3 = -1{,}233.8; \quad M_4 = -1{,}543.1$$

Problems

1. A beam fixed at both ends has an intermediate support. The left span is 8 ft. long and carries 240 lb. per ft. The right span is 12 ft. long and carries 90 lb. per ft. Find the moments at the ends of the spans.
Ans. $M_1 = -1{,}340$ ft.-lb.; $M_2 = -1{,}160$ ft.-lb.; $M_3 = -1{,}040$ ft.-lb.
2. In Problem 1, find the shear at the fixed ends and the reaction of the support. *Ans.* $V_{12} = 982.5$ lb.; $R = 1{,}387.5$ lb.
3. Find the maximum positive moments for Problem 1.
Ans. 671.08 ft.-lb. at $4\frac{3}{32}$ ft. from left end; ?
4. A beam fixed at both ends has an intermediate support, which makes the left span 10 ft. and the right span 8 ft. There is a load of 400 lb. 3 ft. to the left of the support and a load of 640 lb. 3 ft. to the right of the support. Find the moments at the five dangerous sections.
Ans. -216; 313.2; -660; 601.9; -495.
5. A beam fixed at both ends has two intermediate supports. The left span is 8 ft. long and carries 120 lb. per ft. The second span is 12 ft. and carries 180 lb. per ft. The third span is 10 ft. long and carries 100 lb. per ft. Find the moment at each of the seven dangerous sections.

131. Two Moments, Spans Partially Loaded.—The first three terms of the general equation of two moments are the same for any loading. Only the simple-support terms change with load. When it is difficult to calculate the simple-support term for one direction, it may be derived from the simple-support term in the opposite direction by the relation *The sum of the simple-support terms for any span is equal to six times the area of the simple-support moment diagram for that span.*

$$E\,I\,\theta_1 + \text{area} = E\,I\,\theta_2 = -E\,I\,\theta_{21} \tag{1}$$

$$-6\,E\,I\,\theta_1 - 6\,E\,I\,\theta_{21} = 6 \text{ area}. \tag{2}$$

For a simply-supported beam the two moment equations are

$$-6\,E\,I\,\theta_1 = \text{simple-support term, left to right;} \qquad (3)$$

$$-6\,E\,I\,\theta_{21} = \text{simple-support term, right to left.} \qquad (4)$$

The sum of Equations (3) and (4) combined with Equation (2) proves the proposition above.

Example I

Derive the two-moment equations for a simply-supported-beam which carries a load of w per unit length over a length b adjacent to the right support (Fig. 203).

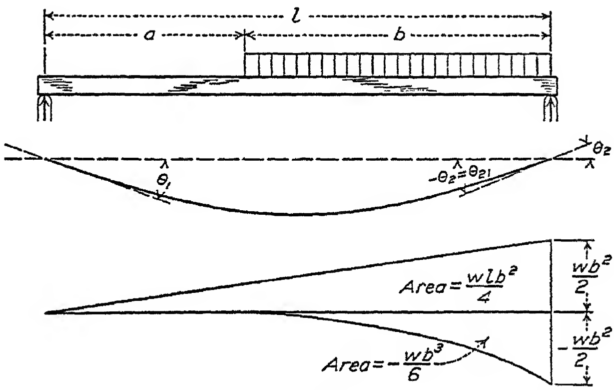


FIG. 203.—Load uniformly distributed over part of span.

By Area Moments.—From left to right,

$$E\,I\,\theta_1 l + \frac{w\,b^2}{2} \times \frac{l}{2} \times \frac{l}{3} - \frac{w\,b^2}{2} \times \frac{b}{3} \times \frac{b}{4} = 0; \qquad (5)$$

$$6\,E\,I\,\theta_1 + \frac{w\,b^2}{4\,l}(2\,l^2 - b^2) = 0. \qquad (6)$$

From right to left,

$$E\,I\,\theta_{21} l + \frac{w\,b^2 l}{4} \times \frac{2\,l}{3} - \frac{w\,b^3}{6}\left(l - \frac{b}{4}\right) = 0; \qquad (7)$$

$$6\,E\,I\,\theta_{21} + w\,b^2 l - w\,b^3 + \frac{w\,b^4}{4\,l} = 0; \qquad (8)$$

$$6\,E\,I\,\theta_{21} + \frac{w\,b^2}{l}\left(l - \frac{b}{2}\right)^2 = 0. \qquad (9)$$

Integration between Limits.

$$M = \frac{w\,b^2 x}{2} - \frac{w(x-a)^2}{2};$$

$$E\,I\,\theta = E\,I\,\theta_1 + \frac{w\,b^2 x^2}{4\,l} - \frac{w(x-a)^3}{6}; \qquad (10)$$

$$0 = E\,I\,\theta_1\,l + \frac{w\,b^2 l^2}{12} - \frac{w\,b^4}{24}; \qquad (11)$$

$$6\,E\,I\,\theta_1 + \frac{w\,b^2}{4\,l}\,(2\,l^2 - b^2) = 0. \tag{6}$$

$$\text{Area} = \frac{w\,b^2l}{4} - \frac{w\,b^3}{6};$$

$$6 \times \text{area} = \frac{3\,w\,b^2l}{2} - w\,b^3. \tag{12}$$

$$\text{Simple-support term for } \theta_1 = \frac{w\,b^2l}{2} - \frac{w\,b^4}{4\,l}. \tag{13}$$

Subtract Eq. (13) from Eq. (12);

$$\text{simple-support term for } \theta_{21} = w\,b^2l - w\,b^3 + \frac{w\,b^4}{4\,l};$$

$$6\,E\,I\,\theta_{21} + w\,b^2l - w\,b^3 + \frac{w\,b^4}{4\,l} = 0. \tag{8}$$

Example II

Derive the two-moment equation, left to right, for a simply-supported beam which is uniformly loaded over a length a adjacent to the left end.

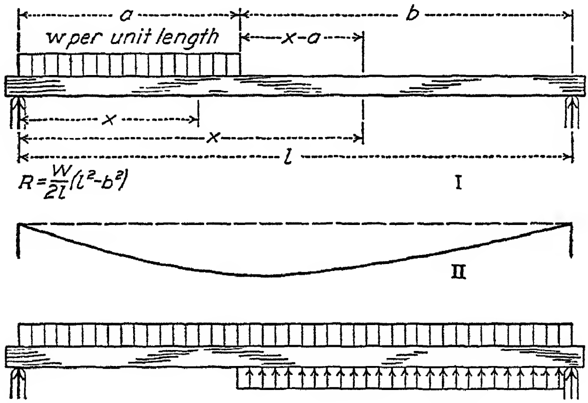


FIG. 204.—Imaginary load with equal upward reactions.

Figure 204, III, shows the beam loaded over the entire length, with an equal uniformly distributed reaction over the length b. For successive integrations between limits,

$$R_1 = \frac{w\,l}{2} - \frac{w\,b^2}{2\,l};\quad M = \frac{w\,l\,x}{2} - \frac{w\,b^2x}{2\,l} - \frac{w\,x^2}{2} + \frac{w(x-a)^2}{2}.$$

$$E\,I\,\theta = E\,I\,\theta_1 + \frac{w\,l\,x^2}{4} - \frac{w\,b^2x^2}{4\,l} - \frac{w\,x^3}{6} + \frac{w(x-a)^3}{6}; \tag{14}$$

$$E\,I\,y = E\,I\,\theta_1 x + \frac{w\,l\,x^3}{12} - \frac{w\,b^2x^3}{12\,l} - \frac{w\,x^4}{24} + \frac{w(x-a)^4}{24}; \tag{15}$$

$$0 = E\,I\,\theta_1 + \frac{w\,l^3}{12} - \frac{w\,b^2l}{12} - \frac{w\,l^3}{24} + \frac{w\,b^4}{24\,l}; \tag{16}$$

$$6\,E\,I\,\theta_1 + \frac{w\,l^3}{4} - \frac{w\,b^2l}{2} + \frac{w\,b^4}{4\,l} = 0. \tag{17}$$

When $l - a$ is substituted for b in Equation (17), the two-moment equation becomes

$$6\,E\,I\,\theta_1 + w\,l\,a^2 - w\,a^3 + \frac{w\,a^4}{4\,l} = 0. \tag{18}$$

Compare with Eq. (8).

Problems

1. A simply-supported beam of length l is uniformly loaded over $0.3\,l$ which begins at $0.4\,l$ from the left end. Derive the two-moment equations. From left to right use Eq. (6) with $b = 0.6\,l$ and subtract same equation with $b = 0.3\,l$.
   Ans. $6\,E\,I\,\theta_1 + 0.104625\,w\,l^3 = 0$; $6\,E\,I\,\theta_{21} + 0.111375\,w\,l^3 = 0$.
2. A beam of length l is fixed at the right end and supported at the left end. It carries w per unit length over $0.4\,l$ adjacent to the right end and $2\,w$ per unit length over $0.3\,l$ adjacent to the left end. Find the moment at the fixed end, the reaction of the support, and the maximum positive moment.
   Ans. $R = 0.4958\,w\,l$; $M = -0.09417\,w\,l^2$; $M = 0.06146\,w\,l^2$.

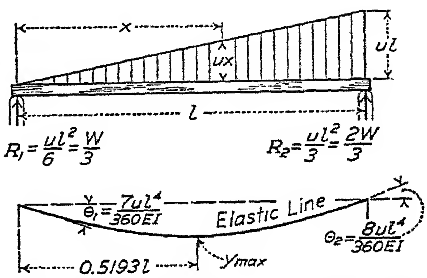


FIG. 205.—Uniformly increasing load.

132. Uniformly Increasing Load.—Figure 205 shows a simply-supported beam with a load which increases uniformly from left to right. The load per unit length is $u\,x$. The load diagram is a triangle. The shear diagram for the load is a negative parabola (Fig. 169). The moment diagram for the load is a cubical parabola $M = -\dfrac{u\,x^3}{6}$. The area of this cubical parabola is one-fourth the product of the base times the altitude, which is $-\dfrac{u\,x^4}{24}$. The center of gravity is one-fifth the base from the large end.

AREA MOMENTS

When the load of Fig. 205 is increasing from left to right, the left reaction is $\dfrac{u\,l^2}{6}$,

$$M = \frac{u\,l^2 x}{6} - \frac{u\,x^3}{6}. \tag{1}$$

From the moment triangle and cubical parabola, the deflection at the right end is

$$0 = E\,I\,\theta_1 l + \frac{u\,l^4}{12} \times \frac{l}{3} - \frac{u\,l^4}{24} \times \frac{l}{5}; \tag{2}$$

$$E\,I\,\theta_1 = -\frac{7\,u\,l^4}{360}. \tag{3}$$

The deflection at the left end is

$$0 = E\,I\,\theta_{21} + \frac{u\,l^4}{12} \times \frac{2\,l}{3} - \frac{u\,l^4}{24} \times \frac{4\,l}{5}; \tag{4}$$

$$E\,I\,\theta_{21} = -\frac{8\,u\,l^4}{360}. \tag{5}$$

The two-moment equations for a simply-supported beam, with load increasing uniformly from 0 to $u\,l$, are

$$6\,E\,I\,\theta_1 + \frac{7\,u\,l^4}{60} = 0; \quad 6\,E\,I\,\theta_{21} + \frac{8\,u\,l^4}{60} = 0. \tag{6}$$

For a uniformly increasing load over a length b adjacent to the right end, as shown in Fig. 206, the left reaction is $\frac{u\,b^3}{6\,l}$. The area

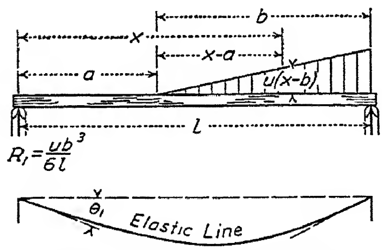


FIG. 206.—Uniformly increasing load over part of span.

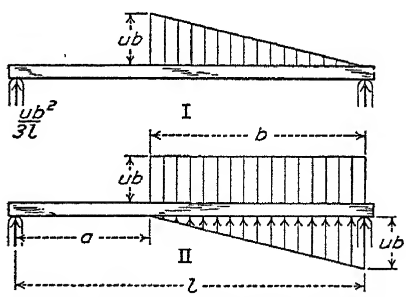


FIG. 207.—Uniformly decreasing load with imaginary upward reactions.

of the positive moment triangle is $\frac{u\,b^3\,l}{12}$, and the area of the negative cubical parabola is $\frac{u\,b^4}{24}$.

$$0 = E\,I\,\theta_1 l + \frac{u\,b^3 l}{12} \times \frac{l}{3} - \frac{u\,b^4}{24} \times \frac{b}{5}; \tag{7}$$

$$6\,E\,I\,\theta_1 + \frac{u\,b^3 l}{6} - \frac{u\,b^5}{20\,l} = 0. \tag{8}$$

$$0 = E\,I\,\theta_{21}l + \frac{u\,b^3l}{12} \times \frac{2\,l}{3} - \frac{u\,b^4}{24} \times \left(l - \frac{b}{5}\right); \tag{9}$$

$$6\,E\,I\,\theta_{21} + \frac{u\,b^3l}{3} - \frac{u\,b^4}{4} + \frac{u\,b^5}{20\,l} = 0. \tag{10}$$

Figure 207, I, shows a load which increases uniformly from the right end over a length b. Figure 207, II, shows a uniformly distributed load of $u\,b$ per unit length over the length b, and an upward reaction which increases from left to right. From Equation (6) of Art. 131, the simple-support term for the uniform load is $\frac{u\,b^3l}{2} - \frac{u\,b^5}{4\,l}$. When the simple-support term of Equation (8) for a uniformly increasing load is subtracted, the result is the simple-support term for Fig. 207, I.

$$\frac{u\,b^3l}{2} - \frac{u\,b^5}{4\,l} - \left(\frac{u\,b^3l}{6} - \frac{u\,b^5}{20\,l}\right) = \frac{u\,b^3l}{3} - \frac{u\,b^5}{5\,l}; \tag{11}$$

$$6\,E\,I\,\theta_1 + \frac{u\,b^3l}{3} - \frac{u\,b^5}{5\,l} = 0. \tag{12}$$

From Equation (8) of Art. 131, and Equation (10),

$$u\,b^3l - u\,b^4 + \frac{u\,b^5}{4\,l} - \left(\frac{u\,b^3l}{3} - \frac{u\,b^4}{4} + \frac{u\,b^5}{20\,l}\right) = \frac{2\,u\,b^3l}{3} - \frac{3\,u\,b^4}{4} + \frac{u\,b^5}{5\,l}; \tag{13}$$

$$6\,E\,I\,\theta_{21} + \frac{2\,u\,b^3l}{3} - \frac{3\,u\,b^4}{4} + \frac{u\,b^5}{5\,l} = 0. \tag{14}$$

For a simply-supported beam carrying a load which increases uniformly from zero at the left end to $u\,l$ at the right end, the slope at the left end is $-\frac{7\,u\,l^4}{360\,E\,I}$ and the reaction is $\frac{u\,l^2}{6}$. The equation of the elastic line is given by $E\,I\,y = -\frac{7\,u\,l^4x}{360} + \frac{u\,l^2x^3}{36} - \frac{u\,x^5}{120}$.

$$y = -\frac{u}{360\,E\,I}(7\,l^4x - 10\,l^2x^3 + 3\,x^5);$$

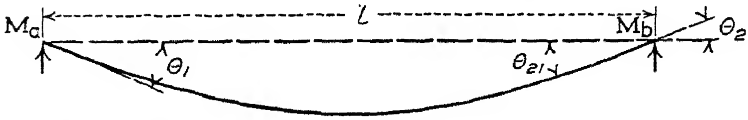
$$\theta = \frac{dy}{dx} = -\frac{u}{360\,E\,I}\,(7\,l^4 - 30\,l^2x^2 + 15\,x^4).$$

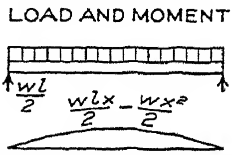
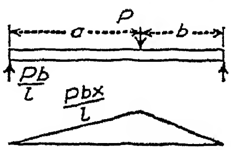
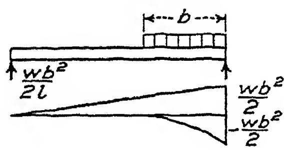
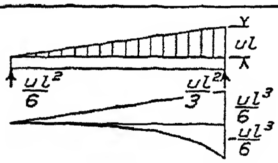
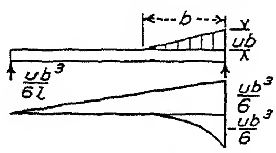
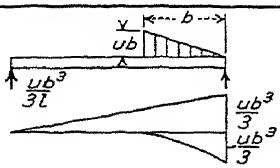
The position of maximum deflection is given by

$$x^2 = l\left(1 - \frac{4}{\sqrt{30}}\right); \quad x = 0.51933\,l;$$

TABLE XXI.—SIMPLE-SUPPORT TERMS FOR TWO-MOMENT EQUATIONS

$$6\,E\,I\,\theta_1 + 2\,M_a l + M_b l + (\text{simple-support term, } A \text{ to } B) = 0.$$
$$6\,E\,I\,\theta_{21} + M_a l + 2\,M_b l + (\text{simple-support term, } B \text{ to } A) = 0$$



Moment diagram and load	Simple-support term: A to B	Simple-support term: B to A
	$\frac{w\,l^3}{4}$ I	$\frac{w\,l^3}{4}$ II
	$\frac{P\,b(l^2 - b^2)}{l}$ or $\frac{P\,a\,b(l+b)}{l}$ III	$\frac{P\,a(l^2 - a^2)}{l}$ or $\frac{P\,a\,b(l+a)}{l}$ IV
	$\frac{w\,b^2(2\,l^2 - b^2)}{4\,l}$ V	$\frac{w\,b^2}{l}\left(l - \frac{b}{2}\right)^2$ or $w\,b^2 l - w\,b^3 + \frac{w\,b^4}{4\,l}$ VI
	$\frac{7\,u\,l^4}{60}$ VII	$\frac{8\,u\,l^4}{60}$ VIII
	$\frac{u\,b^3 l}{6} - \frac{u\,b^5}{20\,l}$ IX	$\frac{u\,b^3 l}{3} - \frac{u\,b^4}{4} + \frac{u\,b^5}{20\,l}$ X
	$\frac{u\,b^3 l}{3} - \frac{u\,b^5}{5\,l}$ XI	$\frac{2\,u\,b^3 l}{3} - \frac{3\,u\,b^4}{4} + \frac{u\,b^5}{5\,l}$ XII

$$y_{max} = -\frac{0.013044\, u\, l^5}{2\, E\, I} = -\frac{0.013044\, W\, l^3}{E\, I},$$

in which W is the total load on the span (see "Carnegie Pocket Companion," p. 165).

Problems

1. A span of length l is fixed at the right end and supported at the left end. It carries a load which increases uniformly from zero at the left end to $u\, l$ per unit length at the right end. Find the moment at the fixed end and the reaction of the support.

Ans. $M = -\frac{u\, l^3}{15} = -\frac{2\, W\, l}{15}$; $R = \frac{W}{5}$.

2. Find the slope at the support for the beam of Problem 1.

Ans. $\theta_1 = -\frac{u\, l^4}{120\, E\, I}$.

3. Solve Problems 1 and 2 for a load which increases uniformly from zero at the fixed end to $u\, l$ at the support.

Ans. $M = -\frac{7\, u\, l^3}{120} = -\frac{7\, W\, l}{60}$; $R = \frac{11\, W}{20}$; $\theta = -\frac{u\, l^4}{80\, E\, I}$

4. Check Problems 1, 2, and 3 by comparison with a uniformly loaded beam.

5. A span of length l is fixed at the right end and supported at the left end. It carries a load which increases uniformly from zero at 0.4 l from the left end to 0.6 $u\, l$ at the right end. Find the reaction at the support and the moment at the fixed end. *Ans.* $M = -0.021744\, u\, l^3$; $R = 0.014256\, u\, l^2$.

133. Miscellaneous Problems.—Table XXI presents a summary of the simple-support terms for two-moment equations. Combinations of these terms will solve the usual problems of indeterminate beams.

Problems

1. A 10-ft. span is fixed at the left end and supported at the right end. The load increases uniformly from 20 lb. per ft. at the left end to 80 lb. per ft. at the support. Find the moment and shear at the fixed end and the reaction of the support.

Ans. $M = -600$ ft.-lb.; $V = 260$ lb.; $R = 240$ lb.

2. Find the maximum positive moment of the beam of Problem 1. Solve first from the fixed end. Solve again from the right end, assuming a uniform load of 80 lb. per ft. and an upward reaction (or negative load) which increases uniformly from right to left. *Ans.* $M = 393.0$ ft.-lb.

3. A span 10 ft. long is fixed at the left end and supported at the right end. At 4 ft. from the left end the load is 20 lb. per ft. and increases uniformly from that point to 68 lb. per ft. at each end. Find the moment at the fixed end, the reaction at the support, and the maximum positive moment. Solve for the moment by a uniform load of 20 lb. per ft. and two uniformly increasing loads. Check by a uniform load of 68 lb. per ft. and two negative loads or positive upward reactions.

Ans. $M = -471.12$ ft.-lb.; $R = 180.888$ lb.; $M_{max} = 274.6$ ft.-lb.

The theorem of three moments was derived in Art. 126 by applying the equation of two moments to the left span from *right* to *left,* then applying the equation to the right span from *left* to *right,* and adding the two expressions. Since the slope from left to right at the intermediate support is equal and opposite the slope from right to left, the slope terms vanish when the expressions are added. For any loading, the first three terms of the theorem of three moments are

$$M_a l_1 + 2\,M_b(l_1 + l_2) + M_c l_2 + \text{etc.} = 0$$

To these terms are added, on the left of the equality sign, the two-moment expression for the left span, from right to left, and that of the right span from left to right. *Each of these expressions begins at the intermediate support.* Table XXI gives most of the two-moment expressions which are likely to occur.

Example

The beam of Fig. 208 is fixed at the left end, is supported at three points, and overhangs the right support 4 ft. The left span is 8 ft. long and uniformly loaded with 96 lb. per ft. for 5 ft. adjacent to the support. The

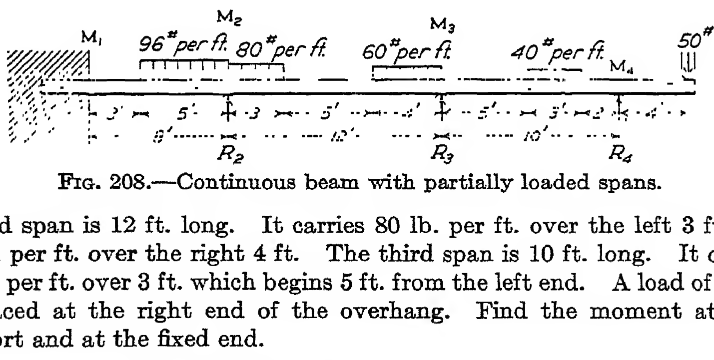


FIG. 208.—Continuous beam with partially loaded spans.

second span is 12 ft. long. It carries 80 lb. per ft. over the left 3 ft. and 60 lb. per ft. over the right 4 ft. The third span is 10 ft. long. It carries 40 lb. per ft. over 3 ft. which begins 5 ft. from the left end. A load of 50 lb. is placed at the right end of the overhang. Find the moment at each support and at the fixed end.

$$16\,M_1 + 8\,M_2 + \frac{96 \times 25(128 - 25)}{32} = 0;$$

$$16\,M_1 + 8\,M_2 + 7{,}725 \text{ ft.-lb.} = 0.$$

$$8\,M_1 + 40\,M_2 + 12\,M_3 + \frac{96 \times 25(8 - 2.5)^2}{8} + \frac{80 \times 9(12 - 1.5)^2}{12} + \frac{60 \times 16(288 - 16)}{48} = 0;$$

$$8\,M_1 + 40\,M_2 + 12\,M_3 + 21{,}130 = 0.$$

$$12\,M_2 + 44\,M_3 - 2{,}000 + \frac{80 \times 9(288 - 9)}{48} + \frac{60 \times 16 \times 100}{12} + \frac{40 \times 25 \times 175}{40} - \frac{40 \times 4 \times 196}{40} = 0;$$

$12\,M_2 + 44\,M_3 + 13{,}776 = 0.$
$M_1 = -276.4;\ M_2 = -297{,}229 \div 720 = -412.8;$
$M_3 = -48{,}121 \div 240 = -200.5.$

Problem

4. Find the shear at the left end of the beam of above example and the reaction at each support.

134. Deflection of Indeterminate Beams.—After the moments have been found by the theorem of three moments, the slope at the ends may be calculated by the two-moment equation, and the deflection may be found by area moments or by integration between limits.

Example

Derive the equation of the elastic line for the left span of the beam of Problem 11 (Art. 126).

$$6\,E\,I\,\theta_1 - 24 \times 262 - 12 \times 142 + 4 \times 1{,}728 = 0; \qquad (1)$$
$$E\,I\,\theta_1 = 180.$$

The shear to the right of the left support is 106.

$$M = -262 + 106\,x - 8\,x^2; \qquad 2)$$
$$E\,I\,\theta = 180 - 262\,x + 53\,x^2 - \frac{8\,x^3}{3};$$
$$E\,I\,y = 180\,x - 131\,x^2 + \frac{53\,x^3}{3} - \frac{2\,x^4}{3}. \qquad (3)$$

Problems

1. Find the deflection at $x = 1$, 4, 6, and 11 for the foregoing example.
Ans. $E\,I\,y = 66$, -416, -684, and -117.33.
2. Find the slope at the second support for Fig. 198. Derive the equation of the elastic line. Find the deflection at 1 in., 4 in., 8 in., and 9 in. from the support. *Ans.* $E\,I\,y = 2.25$, -192, 32 and 74.25.
3. Solve Problem 2 from right to left. Check the deflections above by the derived equation.
4. Derive the equation of the elastic line for the beam of Problem 1 (Art. 133).
5. Derive the equation of the elastic line for the 12-ft. span of the beam of the example of Art. 133.

135. Deflection from the Tangent.—Figure 209 shows the elastic line for any span between supports. A tangent is drawn through any point C on this curve. The vertical distance $E\,A$ upward from this tangent to the left end of the span is y_1 and the corresponding distance $D\,B$ at the right end is y_2. The

distance $C F$ upward to the horizontal line which connects A and B is given by

$$C F = y_1 + \frac{a}{l}(y_2 - y_1) = y_1 + \frac{a(y_2 - y_1)}{a + b}; \tag{1}$$

$$C F = \frac{a\,y_1 + b\,y_1 + a\,y_2 - a\,y_1}{a + b} = \frac{b\,y_1 + a\,y_2}{a + b} = -y, \tag{2}$$

in which y is the deflection of C from the line of the supports, a is the horizontal distance from the left support to C, and b is the horizontal distance from the right support to C. [Equation (2) may be written geometrically from Fig. 209 by means of the broken line $E\,B$.]

Figure 209, II, is the moment diagram for a uniformly loaded, simply-supported beam. The moment is calculated from each

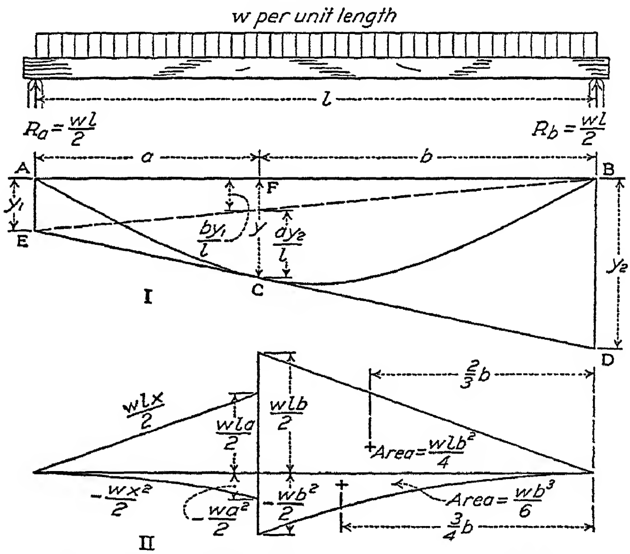


FIG. 209.—Deflection of supports from tangent.

end to any point C. Each portion of the beam is a cantilever, which is bent upward by the reaction and bent downward by the uniformly distributed load. By using the equations for the deflection at the ends,

$$E\,I\,y_1 = \frac{w\,l\,a^3}{6} - \frac{w\,a^4}{8}; \quad E\,I\,y_2 = \frac{w\,l\,b^3}{6} - \frac{w\,b^4}{8}, \tag{3}$$

in which y_1 and y_2 are measured upward. By substitution in Equation (2), with $a + b = l$, Equation (3) becomes

$$E\,I\,y = -\frac{w\,a\,b}{6}(a^2 + b^2) + \frac{w\,a\,b}{8\,l}(a^3 + b^3). \qquad (4)$$

Equation (4) is the form of the elastic-line expression which is most convenient to derive by the methods of elastic energy. If x is substituted for a and $l - x$ for b, Equation (4) may be reduced to Equation (7) of Art. 93.

Example

Find the deflection of a uniformly loaded, simply-supported beam at one-third the length from the left end. Calculate y_1 and y_2 by the equations for the deflection at the end of a cantilever and substitute in Eq. (2).

$$E\,I\,y_1 = \frac{1}{3} \times \frac{w\,l}{2}\left(\frac{l}{3}\right)^3 - \frac{w}{8} \times \frac{l^4}{3} = \frac{3\,w\,l^4}{648};$$

$$E\,I\,y_2 = \frac{1}{3} \times \frac{w\,l}{2} \times \left(\frac{2\,l}{3}\right)^3 - \frac{w}{8} \times \left(\frac{2\,l}{3}\right)^4 = \frac{16\,w\,l^4}{648};$$

$$-y = \frac{w\,l^4}{648\,E\,I} \times \frac{3 \times \frac{2\,l}{3} + 16 \times \frac{l}{3}}{l};$$

$$y = -\frac{11\,w\,l^4}{972\,E\,I}.$$

Problems

1. Check example above by Eq. (4).

$$Ans.\ y = -\frac{w\,l^4}{E\,I}\left(\frac{5}{243} - \frac{1}{108}\right) = -\frac{11\,w\,l^4}{972\,E\,I}.$$

2. Solve example for the deflection at one-fourth the length from each end.

$$Ans.\ y = -\frac{19\,w\,l^4}{2{,}048\,E\,I}.$$

3. A simply-supported beam carries a load P at a distance a from one end and b from the other. Find the deflection under the load by Eq. (2) using the method of the foregoing example.

136. Three Moments with Deflected Supports.—The methods of the preceding article may be applied to a continuous beam. If the support B of Fig. 198 is deflected a distance y_b from the straight line which joins A to C, Equation (2) of Art. 135 may be used with l_1 in place of a and l_2 in place of b. Each end-moment trapezoid is divided into two triangles for area moments. The value of y_1 at A is multiplied by $\frac{l_2}{l_1 + l_2}$ and y_2 is multiplied by $\frac{l_1}{l_1 + l_2}$.

$$\left(\frac{M_a l_1^2}{6} + \frac{M_b l_1^2}{3} + \frac{w_1 l_1^4}{24}\right)\frac{l_2}{l_1 + l_2} + \left(\frac{M_b l_2^2}{3} + \frac{M_c l_2^2}{6} + \frac{w_2 l_2^4}{24}\right)\frac{l_1}{l_1 + l_2} + E\,I\,y_b = 0; \quad (1)$$

$$M_a l_1 + 2\,M_b(l_1 + l_2) + M_c l_2 + \frac{w_1 l_1^3}{4} + \frac{w_2 l_2^3}{4} + 6\,E\,I\,y_b\,\frac{l_1 + l_2}{l_1 l_2} = 0. \quad (2)$$

For other loadings the only changes required apply to the simple-support terms.

When one end is fixed, the two-moment equation applies to the adjacent span. If M_1 is the moment at the fixed end and M_2 is the moment at the next support,

$$2\,M_1 l + M_2\,l + \text{simple-support term} = \frac{6\,E\,I\,y}{l},$$

in which y is the deflection at the second support.

Example I

A 6-in. by 5-in. beam is 100 in. long and has a modulus of elasticity of 1,600,000 lb. per sq. in. It is supported at the ends and has a third support 40 in. from the left end which is 0.06 in. below the line of the end supports. Find the moment and the reaction at each support when the total load, uniformly distributed, is 2,400 lb. Check by deflection equations.

$$E\,I = 100{,}000{,}000,\; w = 24 \text{ lb. per in.,}$$

$$200\,M_2 + 6(64{,}000 + 216{,}000) - 6 \times 10^8 \times 0.06 \times \frac{100}{60 \times 40} = 0, \quad (1)$$

$$200\,M_2 = -180{,}000;\quad M_2 = -900;$$

$$R_1 = 457.5,\; R_3 = 705,\; R_2 = 2{,}400 - 1{,}162.5 = 1{,}237.5 \text{ lb.}$$

From the equation of the elastic line for a beam 100 in. long which is supported at the ends, the deflection at 40 in. from the left end would be 0.2916 in. From the deflection of a beam supported at the ends with a concentrated load, the deflection at the reaction is

$$\frac{1{,}600 \times 3{,}600 \times R_2}{3 \times 10^8 \times 100} = 0.2976 - 0.06 = 0.2376;$$

$$192\,R_2 = 237{,}600,\; R_2 = 1{,}237.5 \text{ lb.}$$

Problem

1. At what deflection will the moment at the second support of the beam of the foregoing example be zero? What will be the reactions?

Ans. $y = -\dfrac{168}{2{,}500} = -0.0672$ in.; $R_2 = 1{,}200$ lb.

Example II

The beam of Example I carries 600 lb. 20 in. from the left support and 720 lb. 20 in. from the right support. Find the moment at the second support and find all the reactions when this support is 0.06 in. below the line of the others.

$$200\ M_2 + \frac{600 \times 20 \times 1{,}200}{40} + \frac{720 \times 20 \times 3{,}200}{60} - \frac{6 \times 0.06 \times 10^8 \times 100}{40 \times 60} = 0;$$

$$200\ M_2 + 360{,}000 + 768{,}000 - 1{,}500{,}0000 = -372{,}000;$$

$$M_2 = 1{,}860; \quad R_1 = 346.5 \text{ lb.}; \quad R_2 = 462.5 \text{ lb.}; \quad R_3 = 511 \text{ lb.}$$

Example III

Solve Example II by integration between limits.

$$M = R_1x - 600(x - 20) + R_2(x - 40) - 720(x - 80)$$

$$E\,I\,\theta = E\,I\,\theta_1 + \frac{R_1x^2}{2} - 300(x - 20)^2 + \frac{R_2(x - 40)^2}{2} - 360(x - 80)^2; \quad (2)$$

$$E\,I\,y = E\,I\,\theta_1 x + \frac{R_1x^3}{6} - 100(x - 20)^3 + \frac{R_2(x - 40)^3}{6} - 120(x - 80)^3. \quad (3)$$

If θ is the angle measured from the line which joins the end supports, the first span gives

$$-6 \times 10^6 = 4 \times 10^9\theta_1 + \frac{R_1\,64{,}000}{6} - 8 \times 10^5; \quad (4)$$

$$24 \times 10^6\theta_1 + 64\,R_1 + 31{,}200 = 0. \quad (5)$$

The entire beam gives

$$0 = 10^{10}\theta_1 + \frac{R_1 10^6}{6} - 512 \times 10^5 + \frac{R_2\,216{,}000}{6} - 96 \times 10^4; \quad (6)$$

$$6 \times 10^7\theta_1 + 1{,}000\,R_1 + 216\,R_2 - 312{,}960 = 0. \quad (7)$$

From the moment at the right end,

$$\begin{aligned} 10\,R_1 + 6\,R_2 - 6{,}240 &= 0; \\ 360\,R_1 + 216\,R_2 - 224{,}640 &= 0; \\ 6\,E\,I\,\theta_1 10^7 + 640\,R_1 - 88{,}320 &= 0; \\ 6\,E\,I\,\theta_1 10^6 + 64\,R_1 - 8{,}832 &= 0; \\ 6\,E\,I\,\theta_1 10^6 + 16\,R_1 + 7{,}800 &= 0; \\ 48\,R_1 = 16{,}632; \quad R_1 &= 346.5. \end{aligned}$$

Example IV

A 6-in. by 5-in. beam is supported at the left end and 40 in. from the left end and is fixed 100 in. from the left end. The second support is 0.03 in. below the line of the ends. The beam carries 600 lb. 20 in. from the left end and 720 lb. 20 in. from the right end. If $E = 1{,}600{,}000$, find the moment at the second support and at the fixed end. Find the reaction of each support.

From the two-moment equation for the second span,

$$60\,M_2 + 120\,M_3 + \frac{720 \times 40(3{,}600 - 1{,}600)}{60} = -\frac{6 \times 0.03 \times 10^8}{60};$$

$$M_2 + 2M_3 + 16{,}000 + 5{,}000 = 0.$$

For the two spans,

$$200\ M_2 + 60\ M_3 + \frac{600 \times 20 \times 1{,}200}{40} + \frac{720 \times 20 \times 3{,}200}{60} - \frac{18 \times 10^6 \times 100}{2{,}400} = 0;$$

$$200\ M_2 + 60\ M_3 + 378{,}000 = 0;$$

$$17\ M_2 = 25{,}200;\quad M_2 = 1{,}482.3;\quad M_3 = -11{,}241.2;\quad R_1 = 337.06;$$
$$R_2 = 290.88.$$

Problem

2. Solve Example IV by integration between limits. Except for the deflection term, one equation is identical with Eq. (4). Another is exactly the same as Eq. (6). A third equation is obtained from Eq. (2) with the condition that $\theta = 0$ when $x = l$.

Example V

A 3-in. by 4-in. beam, for which $E = 1{,}250{,}000$ lb. per sq. in., is 100 in. long. It is supported at the left end, 40 in. from the left end, 70 in. from the left end, and 10 in. from the right end. The support 40 in. from the left end is 0.05 in. below the line of the other three. Find the reaction of each support for a load of 12 lb. per in. over the entire length.

$$140\ M_2 + 30\ M_3 + 3(40^3 + 30^3) - \frac{6 \times 2 \times 10^5 \times 0.05 \times 70}{1{,}200} = 0$$
$$14\ M_2 + 3\ M_3 - 7{,}700 = 0.$$

Since the second support is 0.05 in. below the line of the third and fourth supports, the third support is $0.05 \times {}^{20}\!/_{50} = 0.2$ in. above the line joining the second and fourth. By using this deflection of 0.02 in. in the equation of three moments, the result is

$$30\ M_2 + 100\ M_3 - 12{,}000 + 3(30^3 + 20^3) + \frac{6 \times 2 \times 10^7 \times 0.02 \times 50}{600} = 0.$$
$$3\ M_2 + 10\ M_3 + 29{,}300 = 0.$$
$$131\ M_2 = 164{,}900;\quad M_2 = 1{,}258.8;\quad M_3 = -3{,}307.7;\quad R_1 = 271.47;$$
$$R_2 = 236.31;\quad R_3 = 587.21;\quad R_4 = 104.61 \text{ lb.}$$

137. Deflection, Moment about a Secondary Axis.—When the bending moment applied to a beam is not about a principal axis of the section, it was shown in Art. 84 that the deformation does not take place about an axis perpendicular to the plane of the moment. In Fig. 146, the component of 103.92 pounds is resisted by the minimum moment of inertia, which is 9 in.^{4} The component of 60 pounds is resisted by the maximum moment of inertia, which is 16 in.^{4} Relatively greater deflection occurs about the axis of minimum moment of inertia. If Fig. 146 represents the end of a cantilever, the deflection is easier about

an axis parallel to $A\,B$ than about an axis perpendicular to $A\,B$. The beam bends to the right of the vertical about the neutral axis $G\,F$.

Example

Find the deflection of the beam of the example of Art. 84 if $E = 1{,}500{,}000$ lb. per sq. in.

The component of 103.92 lb. causes a deflection of 0.55424 in. downward at 30° to the right of the vertical. The other component causes a deflection of 0.180 in. downward at 60° to the left of the vertical.

$$y = -0.47997 - 0.0900 = -0.56997 \text{ in.}$$
$$x = 0.27712 - 0.15588 = 0.12124 \text{ in.}$$

Problems

1. In the example, find the angle which the resultant deflection makes with the vertical. Compare with the neutral axis of the example of Art. 84.
2. A 3-in. by 10-in. cantilever is 10 ft. long. The upper edge of each 10-in. face is 6 in. to the right of the vertical plane through the lower edge. Find the components of the deflection at the end if the load on the end is 240 lb. and $E = 1{,}200{,}000$ lb. per sq. in.
   Ans. 3.072 in.; 0.3686 in.; $x = 2.2364$ in.; $y = -2.1381$ in.
3. In Problem 2, find the unit stress at each corner at the fixed end. Find the direction of the neutral axis. Compare with the angle which the resultant deflection makes with the vertical.
   Ans. 1,612.8 tension, 691.2 compression at top; 46° 18′ with vertical.
4. A 4-in. by 4-in. cantilever is 10 ft. long and carries a load of 85 lb. at the end. The cosine of the angle which two parallel faces make with the horizontal is $\frac{15}{17}$. Find the components of the deflection if $E = 1{,}000{,}000$.
   Ans. $y = -2.295$ in.; $x = 0$. The moment of inertia of any regular polygon is the same in all directions.
5. Find the fiber stress at each corner of the beam of Problem 4 at the fixed end. Solve by means of the components.
6. Solve Problem 5 by means of $\frac{I}{c}$. What is c for the highest and lowest corners? For the other corners?
   $$\text{Max. } c = \frac{8 \times 2}{17} + \frac{15 \times 2}{17};\ S_{\max} = 1{,}293.75 \text{ lb./in.}^2$$

138. Deflection by Moments in Different Planes.—When the forces acting on a beam are not all parallel to one plane which passes through the beam, it is necessary to resolve these forces into components parallel to two axes which are perpendicular to each other and to the length of the beam. If the beam is circular, square, or of any other section for which the moment of inertia is the same in every direction, these axes may be taken in

any convenient way. For all other sections the resolutions must be made parallel to one of the principal axes of inertia. The two components of the deflection at any point are calculated separately, and the resultant deflection found from their vector sum.

Example

A 3-in. solid shaft, weighing 24 lb. per ft., is 10 ft. long and is supported at the ends. A pulley weighing 160 lb. is 3 ft. from the left end and is subjected to a pull of 400 lb. 30° below the horizontal in a plane perpendicular to the length of the shaft. Find the deflection at the pulley, if E is 29,000,000 lb. per sq. in.

Resolved vertically, the total vertical load at the pulley is 360 lb. The horizontal pull if 346.4 lb. The deflections at 36 in. from one end are

From concentrated load of 360 lb.	0.0793 in.
From load of 2 lb. per in.	0.0381 in.
Total vertical deflection.	0.1174 in.

The horizontal deflection from load of 346.4 lb. is 0.0763 in.

Problems

1. A 10-in. 25.4-lb. standard I-beam, 15 ft. long, is used as a purlin plate. Its web makes an angle of 18° with the vertical. It carries a vertical load of 240 lb. per ft. and a force of 250 lb. per ft. parallel to the web. Find the unit stress at the corners at the middle and the components of the deflection parallel and perpendicular to the web. $E = 29{,}000{,}000$.
   Ans. $S = 8{,}344 \pm 6{,}615 = 14{,}959$ lb./in.^{2} and 1,729 lb./in.^{2}
   Deflections = 0.1538 in. parallel to web and 0.4222 in. parallel to flange.
2. A vertical post, 6 in. square and 10 ft. long, is fixed at the bottom. A horizontal force of 200 lb., south 20° west, is applied 1 ft. from the top, and a uniform force of 60 lb. per ft. east is applied for the entire length. Find the maximum stress at each corner at the bottom and at the middle. Find the south and east deflections at the top and at the middle if E = 1,500,000 lb. per sq. in.
3. A vertical post, 6 in. square and 10 ft. long, is fixed at the bottom with two faces in the meridian. A horizontal force of 200 lb., south 20° west is applied at the top and a uniform force of 60 lb. per ft., directed east, is applied to the west face. Find the unit stress at each corner at the bottom. Find the deflection at the top and 20 in. from the top if E = 1,500,000.
   Ans. East component at top = 0.5568. S = 1,398 lb./in.^{2} at southeast corner.

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CHAPTER XII

SHEAR IN BEAMS

139. Direction of Shear.—The total vertical shear in a beam is calculated by the methods of Art. 49, but this gives no information in regard to the distribution of the shearing stress in the section. In Art. 21 it was shown that shearing stresses occur in pairs, and that a small block subjected to shearing stress of given intensity along two parallel faces is subjected to a shearing stress of the same intensity along two other faces at right angles to these.

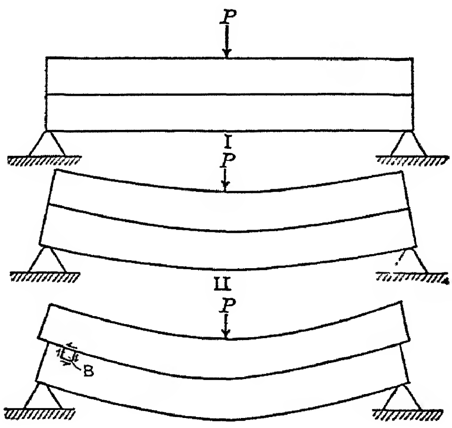


FIG. 210.—Horizontal shear in beams.

Figure 210, I, represents a beam made by placing one plank on top of another. Figure 210, II, is the same beam under load, provided that the planks are held from slipping with reference to each other by being glued or bolted together to form a single beam. If the planks are free to move, they take the form III, in which the upper plank is moved outward over the lower one at each end. A small block B in the upper portion of the lower plank may be treated as a free body. The plank above this block has been displaced toward the left. If the planks were glued together, the upper plank would have exerted a horizontal

shearing stress upon the upper surface of the block. To prevent rotation there must be a vertical shear upward at the left side. The actual shearing stresses upon this block from the surrounding material, if the upper plank were glued to the lower, would take the directions of the arrows.

The shear at the left of the block is vertically upward, which is the direction of the external shear. If a block were taken to the right of the load P, it would be found that the shear on its left side is vertically downward, which is the direction of the vertical shear in that part of the beam. One of the planks of Fig. 210 may be thicker than the other, but the *direction* of the shear will remain the same.

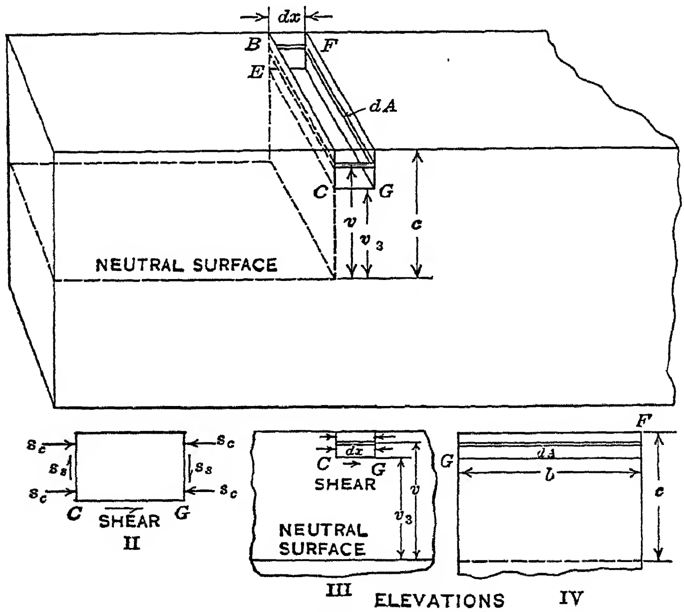


FIG. 211.—Horizontal shear in rectangular section.

140. Intensity of Shearing Stress.—Figure 211 represents a part of a beam subjected to vertical shear and to a bending moment. The shear is assumed to be positive from left to right and the moment is assumed to produce compression in the fibers above the neutral surface. A small block is shown extending across the beam between vertical planes dx apart and reaching from the top of the beam to a horizontal plane at a distance v_3 from the neutral surface. Two elevations of this block and the adjoining parts of the beam are shown in Fig. 211, III and IV,

and an enlarged elevation of the block in Fig. 211, II. The block is in equilibrium under the action of the compressive stress on the ends (the rectangles whose diagonals are $C\,B$ and $G\,F$), the vertical shearing stress on the same surfaces, the horizontal shear from the material below (on the rectangle $G\,E$), and the vertical compression or tension across the base.

In Fig. 211, a filament of cross section dA and length dx extends through the block parallel to the neutral surface. The unit compressive stress on the left end of this filament is $\frac{M_1 v}{I_1}$, in which M_1 is the bending moment at the section, and I_1 is the moment of inertia with respect to the neutral axis of the entire cross section of the beam. The compression of the left end of the filament is $\frac{M_1 v}{I_1}\,dA$. The total compression on the left end of the block is the integral from v_3 to c of the compression on the left end of the filament.

$$\text{Total compression on left end} = \frac{M_1}{I_1}\int_{v_3}^{c} v\,dA. \tag{1}$$

$$\text{Total compression on right end} = \frac{M_2}{I_2}\int_{v_3}^{c} v\,dA. \tag{2}$$

The resultant horizontal push on the block in the direction of the length of the beam is the difference of these integrals (1) and (2). If the section of the beam is uniform, $I_1 = I_2$, and v_3 and c, are the same for both expressions. The resultant horizontal push on the block is

$$\frac{M_2 - M_1}{I}\int_{v_3}^{c} v\,dA. \tag{3}$$

This resultant horizontal force must be balanced by the horizontal shear at the bottom of the block. If the breadth CE at the bottom of the block is b, the total area in horizontal shear is $b\,dx$, and the total shear is $s_s b\,dx$. When the resultant horizontal compression is equated with the horizontal shear at the bottom of the block,

$$s_s b\,dx = \frac{M_2 - M_1}{I}\int_{v_3}^{c} v\,dA; \tag{4}$$

$$s_s = \frac{M_2 - M_1}{I\,b\,dx}\int_{v_3}^{c} v\,dA. \tag{5}$$

Since $M_2 - M_1$ is equal to dM,

$$\frac{M_2 - M_1}{dx} = \frac{dM}{dx} = V, \tag{6}$$

in which V is the total vertical shear.

$$s_s = \frac{V}{Ib}\int_{v_3}^{c} v\,dA, \tag{7}$$

in which s_s equals the unit horizontal shear at a distance v_3 from the neutral axis and also equals the unit vertical shear at the same

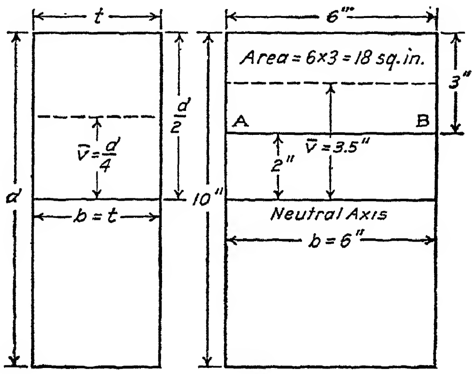


FIG. 212.—Rectangular section in shear.

place. The term $\int_{v_3}^{c} v\,dA$ is the moment of the area of the end of the block with respect to the neutral axis.

$$\bar{v} = \frac{\int_{v_3}^{c} v\,dA}{A}; \quad \int_{v_3}^{c} v\,dA = \bar{v}\,A. \tag{8}$$

When the area and location of the center of gravity of the portion of the plane section above the line $C\,E$ are known, the integral may be replaced by the equivalent expression of (8). Equation (7) then becomes

$$s_s = \frac{V}{Ib}\,\bar{v}\,A', \qquad \text{Formula XXVIII}$$

in which V is the total vertical shear at the section, I is the moment of inertia of the *entire section* with respect to the neutral axis, b is the breadth of the section at the plane at which s_s is

calculated, A' is the area of the *portion* of the section above (or below) this plane, and $\bar{v}$ is the distance of the center of gravity of A' from the neutral axis.

Figure 212, I, shows a rectangular section of thickness t and height d. It is desired to find the unit shearing stress at the neutral surface when the total vertical shear is V. The moment of inertia $= \frac{t\,d^3}{12}$; $b = t$ at any plane; $A' = t\,\frac{d}{2}$; and $\bar{v} = \frac{d}{4}$.

$$s_s = \frac{V}{t} \times \frac{12}{t\,d^3} \times \frac{d}{4} \times \frac{t\,d}{2} = \frac{3\,V}{2\,t\,d}.$$

Since the average vertical shear is $\frac{V}{\text{total area}} = \frac{V}{t\,d}$, the unit horizontal (or vertical) shearing stress at the neutral surface of a rectangular section is three halves as great as the average unit vertical shearing stress.

Example

Find the unit shearing stress in a 6-in. by 10-in. rectangular section at a plane 3 in. from the top, when the total vertical shear is 6,000 lb. (Fig. 212, II).

$$\frac{V}{I\,b} = \frac{6{,}000}{500 \times 6} = 2; \quad \bar{v}\,A' = 3.5 \times 18 = 63;$$
$$s_s = 2 \times 63 = 126 \text{ lb. per sq. in.}$$

In this solution, the unit shearing stress in the plane $A\,B$ of Fig. 212, II, has been found by means of the area above the plane. The area between $A\,B$ and the bottom of the section might have been used. For that area $A' = 7 \times 6$; $b = 3.5 - 2 = 1.5$; $\bar{v}\,A' = 1.5 \times 42 = 63$ as before.

Problems

1. Find the unit shearing stress in the foregoing example at 3 in. above the neutral axis. Check by means of the area below the plane of the shearing stress. *Ans.* $s_s = 2 \times 48 = 96$ lb./in.^{2}
2. Find the unit shearing stress at the neutral surface for the foregoing example by Formula XXVIII. Check by means of the average vertical shearing stress.
3. A 6-in. by 8-in. beam, 8 ft. long, is supported at the ends and carries a load of 7,680 lb. 5 ft. from one end. Find the maximum unit shearing stress at the section at which total vertical shearing stress is greatest. *Ans.* 150 lb./in.^{2} at neutral surface.
4. Find the unit stress for the beam of Problem 3 at 2 in. from the top and at 3 in. from the top at each end.

*5. A 7-in. by 14-in. beam of longleaf yellow pine, placed on supports 13 ft. 6 in. apart, was subjected to equal loads at points 4 ft. 6 in. from the supports. When the total load was 57,500 lb., the beam failed by shear at the neutral axis at one end. Find the ultimate shearing strength of this timber parallel to the grain. Compare the result with the figures given by the U. S. Department of Agriculture (see handbook). *Ans.* 440 lb./in.^{2}

*6. A 7-in. by 16-in. beam of Douglas fir, supported at points 13 ft. 6 in. apart and loaded at the third points with equal loads, failed by shear when the total load was 45,000 lb. Find the ultimate shearing strength of this timber parallel to the grain. *Ans.* 301 lb./in.^{2}

7. Timber having an allowable shearing stress of 90 lb. per sq. in. and an allowable bending stress of 1,350 lb. per sq. in. is used as a 6-in. by 8-in. beam to carry a load at the middle. What is the maximum allowable load on any length?
Ans. Max. $V = 60 \times 48 = 2{,}880$ lb. Maximum load at middle = 5,760 lb.

8. Below what length does shear govern in Problem 7? On account of shear the load at the middle cannot exceed 5,760 lb.
Ans. $Z = 64$ in.^{3}; $M = 64 \times 1{,}350 = 86{,}400$; $2{,}880 \times \frac{l}{2} = 86{,}400$; $l = 60$ in.
For a length greater than 60 in., the bending calculation gives a load less than 5,760 lb.; bending governs. For a length smaller than 60 in. the bending calculation gives a load greater than 5,760. Since 5,760 lb. is the maximum in shear, shear governs.

9. Solve Problem 8 for a uniformly distributed load.

141. Non-rectangular Beams.—Formula XXVIII gives the unit shearing stress in the horizontal plane $G\ E$ of Fig. 211. If the section of the beam is not rectangular, the unit shearing stress may not be uniform over the horizontal surface. Figure 213, I, is a circular section in which $A\ D$ is the trace of a horizontal plane. The short lines crossing $A\ D$ are traces of planes in which the shear is transmitted from one side of $A\ D$ to the other. At the middle of $A\ D$ the shear is transmitted from a filament above the plane to a filament directly below it, and the line B is vertical. At A and D, the shear is transmitted from a filament on the outer surface above the plane to a slightly larger filament, also on the outer surface, below the plane. The short lines at A and D are tangent to the section. At the diameter $E\ F$, the shear is transmitted from a filament above the plane to an equal filament

* Problems 5 and 6 are from tests made by Prof. A. N. Talbot, described in *Bulletin* No. 41 of the Engineering Experiment Station of the University of Illinois.

directly below it, and it is customary to assume that the distribution is uniform.

Figure 213, II, is part of an I-beam section. At the plane where the web joins the flange, there must be a great difference in the intensity of the shearing stress. At $K L$, at some little distance down the web, the shearing stress becomes practically uniform over the section.

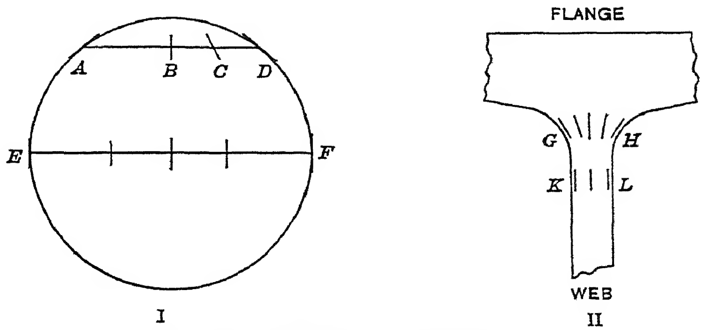


FIG. 213.—Shear in curved sections.

Formula XXVIII may be used for beams having faces which are not parallel. The results are approximately correct.

Example

The section of a beam (Fig. 214, I) is an isosceles triangle of base 9 in. and altitude 12 in. Find the unit shearing stress at the neutral surface under a total shear of 6,480 lb.

$$s_s = \frac{6{,}480}{432 \times 6} \times \frac{8}{3} \times 24 = 160 \text{ lb./in.}^2$$

Problem

1. In Fig. 214, find the unit shearing stress at the plane $E F$, which is midway between the vertex and the base. *Ans.* $s_s = 180$ lb./in.²

The unit shearing stress is greater at the middle $E F$ than at the neutral axis $C D$. The total shearing stress across $E F$ is less than that at $C D$, but the ratio of the breadth at $E F$ to the breadth at $C D$ is still smaller.

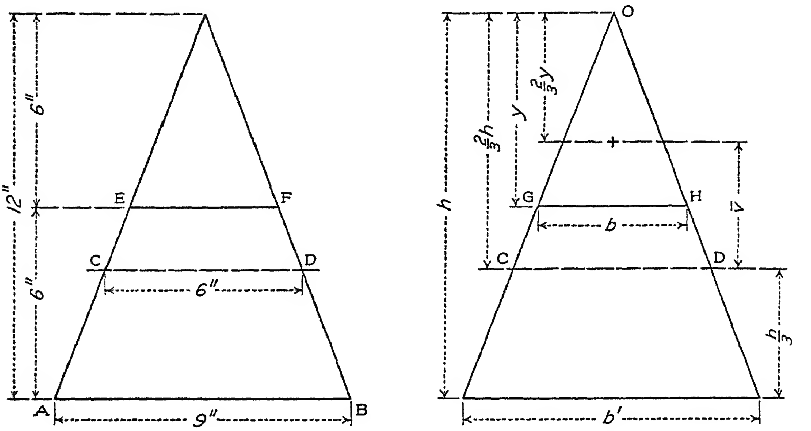
Figure 214, II, may be used to calculate the unit shearing stress at any plane $G H$ at a distance y from the vertex of the beam. If b' is the base of the triangle, and b is the length of $G H$,

$$b = \frac{b'y}{h}; A' = \frac{b'y^2}{2h}; \bar{v} = \frac{2h}{3} - \frac{2y}{3} = \frac{2(h-y)}{3}; s_s = \frac{V}{3I}(hy - y^2).$$

The position of maximum unit shearing stress is given by

$$\frac{d}{dy}(h\,y - y^2) = 0; \quad h - 2\,y = 0; \quad y = \frac{h}{2}.$$

The maximum unit shearing stress in a beam of triangular section with base horizontal is at the middle of the height.



I II

FIG. 214.—Shearing stress in a triangular section.

Problems

2. Find the unit shearing stress at the middle of the height of a triangular beam and find the ratio of this stress to the average unit stress.

Ans. $s_s = \frac{3\,V}{b'h} = \frac{3\,V}{2\,A}$; ratio $= \frac{3}{2}$.

3. Find the unit shearing stress at the neutral surface of a triangular beam and find the ratio of this stress to average unit stress. *Ans.* Ratio $= \frac{4}{3}$.

4. Check example and Problem 1 by means of the ratios of Problems 2 and 3.

5. For a beam of circular section, what is the ratio of the unit shearing stress at the neutral surface to the average vertical shearing stress?

Ans. Ratio $= \frac{4}{3}$.

142. Shearing Stress in I-beams.—It is customary to calculate the unit shearing stress in the web of an I-beam by dividing the total vertical shear by the cross section of the web, which is regarded as extending the entire depth of the beam. If t is the thickness of the web, and d is the depth of the beam, this empirical formula is

$$\text{Unit shearing stress} = \frac{\text{total vertical shear}}{t\,d}.$$

In a 12-inch 31.5-pound I-beam (Fig. 215), the thickness of the web is 0.35 inch; the area $t\,d$ is 4.2 square inches, and the average unit shearing stress, as computed by this method, is 0.238 V. To find the unit shearing stress at the neutral surface the upper half of the section is divided into a vertical rectangle, a horizontal rectangle, and two triangles, and the moment of each area is computed.

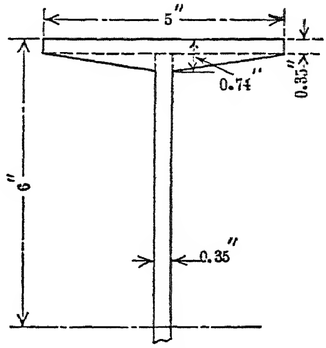


FIG. 215.—I-beam section.

	A'	$\bar{v}$	$A'\,\bar{v}$
Horizontal rectangle	1.750	5.825	10.194
Two triangles	0.907	5.520	5.006
Vertical rectangle	1.977	2.825	5.585
Total	.....	.....	20.785

$$\frac{V}{I\,b}\bar{v}\,A' = \frac{20.785\ V}{215.8 \times 0.35} = 0.275\ V.$$

To find the unit shearing stress in a plane 5 inches above the neutral surface, which is a little below the flange, the moment of the vertical rectangle, 5 inches high and 0.35 inch thick, is subtracted from the moment of the entire upper half of the section.

$$\bar{v}\,A' = 20.785 - 5 \times 0.35 \times 2.5 = 20.785 - 4.375 = 16.410.$$

$$s_s = \frac{16.410\ V}{215.8 \times 0.35} = 0.217\ V.$$

The average of 0.275 V and 0.217 V is 0.246 V, which differs very little from 0.238 V. It is evident that the method of calculating average unit shear in an I-beam section gives a result which is practically correct.

Problem

Find the unit shearing stress at the neutral axis and at 1.5 in. from the top for a 24 × 12 100-lb. wide-flange section, neglecting the fillets. Compare with $\frac{V}{t\,d}$.

Ans. $s_s = 0.0984\ V$ and $0.0800\ V$; mean $= 0.0892\ V$; $\frac{V}{24 \times 0.468} = 0.0890\ V$.

143. Failure of Beams.—The nature of the failure in a beam depends principally upon the relative ultimate strength of the material in the different directions and the value of the different maximum stresses. In a beam which is short relative to its depth, the unit tensile and compressive stresses at the dangerous section are small compared with the unit shearing stress at the neutral surface at the ends. Owing to the fact that timber has a small shearing strength parallel to the grain, such a beam, if made of timber, will usually fail by shear. Figure 216 shows four wooden beams each about 40 inches long. The upper beam is a

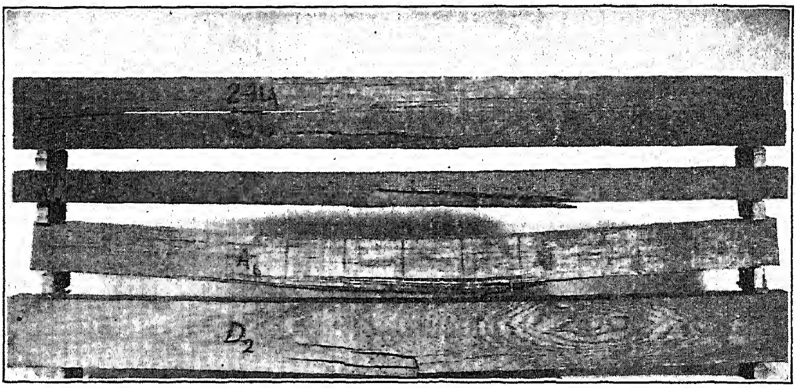


FIG. 216.—Failure of timber beams.

yellow-pine beam glued to a white-pine beam. The total depth was 3.80 inches and breadth 1.57 inches. The beam was supported at points 36 inches apart and loaded at the third points; this beam failed by longitudinal shear at one end when the total load was 1,950 pounds. The failure followed the glued surface but began in the white pine.

Beams of brittle material, such as cast iron, hard steel, stone, or plain concrete, fail by tension. Soft-steel beams fail by buckling the compression flange or by buckling the web.

144. Deflection Caused by Shear.—When a beam is bent, part of the deflection is caused by shear, unless the moment is constant. (For constant moment, $V = 0$.) If the unit shearing stress across a section were constant, the shearing deformation in a length dx would be $\frac{s_s}{E_s}\,dx$, and the total deflection in a length l would be

$$y_s = \frac{1}{E_s}\int_0^l s_s dx. \qquad (1)$$

If s_s is constant throughout the length l, this becomes

$$y_s = \frac{s_s l}{E_s}. \tag{2}$$

In an I-beam section the unit shearing stress is *assumed* to be constant and the equations above apply to give an *approximate* result.

Example I

Find the deflection of a 10-in. 25-lb. I-beam, supported at points 12 in. apart, and loaded with 49,600 lb. at the middle of the span. $E_s = 12{,}000{,}000$, $E = 29{,}000{,}000$.

The vertical shear is 24,800 lb. Since the web area is 3.1 sq. in., the unit shearing stress is 8,000 lb. per sq. in. If the middle is regarded as fixed, the shear of either end upward is

$$y_s = \frac{8{,}000 \times 6}{12{,}000{,}000} = 0.004 \text{ in.}$$

The deflection which is due to bending is

$$y = \frac{49{,}600 \times 12^3}{48 \times 29{,}000{,}000 \times 122.1} = 0.0005 \text{ in.}$$

In this extreme case, the deflection which is due to shear is greater than that which is due to bending. If the beam were made twice as long, the bending deflection would be eight times as great, while the shear deflection would be only doubled. For beams of any considerable length relative to their cross section, the deflection which is due to shear may be neglected.

The shearing stress in a beam is not uniformly distributed. It is possible, however, to calculate the true shear deflection for beams when the distribution of shearing stress is known. In Art. 201 it will be shown, by a method of work and energy, that the deflection of a beam of rectangular section may be calculated by multiplying the average unit shearing stress by the factor 1.2.

Example II

A steel cantilever, 2 in. square and 40 in. long, has a load of 240 lb. on the free end. If E_s is 12,000,000 lb. per sq. in., find the shear deflection.

$$y_s = \frac{1.2 \times 60 \times 40}{12{,}000{,}000} = 0.00024 \text{ in.}$$

Problems

1. A 2-in. by 3-in. steel beam rests on supports 12 in. apart and carries a load of 12,000 lb. midway between the supports. If $E_s = 12{,}000{,}000$

and $E = 30{,}000{,}000$ lb. per sq. in., what is the shear deflection and what is the bending deflection? *Ans.* $y_s = 0.0006$ in.; $y = 0.0032$ in.

2. Solve Problem 1 if the length of the beam is 24 in. and the load is 6,000 lb.

3. Solve Problem 1 if the length of the beam is 60 in. and the load is 2,400 lb.

4. The beam of Problem 1 carries a distributed load of 1,600 lb. per in. Find the deflection which is due to shear and the deflection which is due to bending. *Ans.* $y_s = 0.00048$ in.; $y = 0.0032$ in.

*

CHAPTER XIII

SPECIAL BEAMS

145. Beams of Constant Strength.—In a beam of "constant strength" the unit stress in the outer fibers is the same at all sections. Since $S = \frac{M\,c}{I} = \frac{M}{Z}$, the stress is constant when the section modulus varies as the bending moment. In a cantilever with a load on the end, for instance, the moment is directly proportional to the distance from the end. If the depth is constant and the width increases uniformly from the free end to the fixed end, the section modulus varies directly as the moment, and the unit stress in the outer fibers is constant. If it were not necessary to make some allowance for shear and compression at the free end, this beam would be only one-half as heavy as a uniform beam of equal strength. Even with the additional material to meet the requirements of shear and compression, a great saving in weight is secured by the use of "beams of constant strength."

146. Cantilever with Load on the Free End.—With the origin of coördinates at the free end of the cantilever, the moment at a distance x from the end is $P\,x$. (It is not necessary to consider the sign of the moment, since the unit stress depends upon the magnitude only.) If S is the allowable bending stress and Z is the section modulus,

$$P\,x = S\,Z.$$

Since the section modulus for a rectangular section is $\frac{b\,d^2}{6}$,

$$P\,x = \frac{S\,b\,d^2}{6}.$$

Problems

1. A cantilever of rectangular section, with a load P on the free end, has a constant depth of 5 in. If the allowable stress in the outer fibers at any section is 1,200 lb. per sq. in., what is the equation for the breadth in terms of the load on the end and the distance x from the end?

Ans. $b = \frac{P\,x}{5{,}000}$.

2. A cantilever of constant strength, loaded at the free end, has a constant depth of 4 in. The maximum breadth is 8 in. Find the breadth at the middle and at one-fourth the length from each end. *Solve without writing.*
3. What is the allowable load on the beam of Problem 2 if the allowable stress is 1,000 lb. per sq. in.?
4. A cantilever of rectangular section, with the load on the free end, has a constant breadth b. What is the expression for the depth at any section at a distance x from the load?

$$Ans.\ d^2 = \frac{6\,P\,x}{S\,b}.$$

5. A rectangular cantilever of constant strength, 5 ft. long, with a load of 600 lb. on the free end, has a constant breadth of 4 in. Derive an expression for the depth at any section for an allowable unit stress of 1,000 lb. per sq. in., and calculate the depth for intervals of 10 in. *Ans.* $d^2 = 0.9\,x$.

x..............	10	20	30	40	50	60
d^2..............	9	18	27	36	45	54
d, inches........	3	4.24	5.20	6.00	6.71	7.35

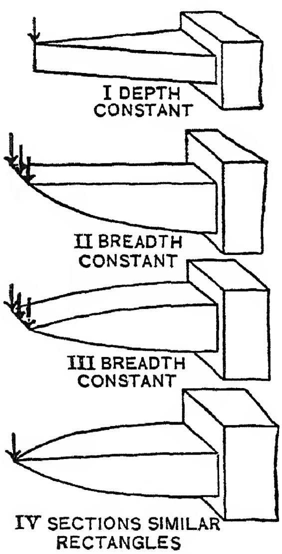


FIG. 217.—Cantilevers of constant strength.

6. A cantilever of constant strength and square section, 5 ft. long, carries a load of 720 lb. on the free end. Derive an expression for the depth of any section for a unit stress of 1,200 lb. per sq. in. Arrange the work as in Problem 5 and calculate the depth at 12-in. intervals. Use table of cube roots.

Ans. $d^3 = 3.6\,x$.

7. A cantilever of constant strength, loaded at the free end, has all sections similar rectangles. The breadth is 4 in. and the depth is 8 in. at 80 in. from the load. Derive an expression for the depth in terms of the distance from the load and calculate the breadth at each 10-in. interval.

Ans. $d^3 = 6.4\,x$; $b = 3.42$ in. at 50 in. from load.

8. A cantilever of constant strength and circular section carries a load of P lb. on the free end. Derive the expression for the diameter at any section in terms of the allowable stress and the distance from the load. $Ans.\ d^3 = \dfrac{32\,P\,x}{\pi\,S}.$

Figure 217 shows some cantilevers of constant strength, which are loaded at the ends and have rectangular sections. Figure 217, I, is a beam of constant depth. The breadth varies as

x—the equation of a straight line. Figure 217, II, represents a beam of constant breadth. The depth varies as the square root of x—the equation of a parabola. One surface may be plane as in Fig. 217, II, or both may be curved as in Fig. 217, III. In either case, the equation gives the total depth. Figure 217, IV, represents a cantilever in which all sections are similar rectangles. The equation is that of the cubical parabola.

147. Shearing and Bearing Stresses at the End.—In Fig. 218, the load P is represented at the extreme ends of the beams. Allowance must be made at the ends for the bearing and shearing stresses. For instance, in Problem 5 of Art. 146, suppose the allowable unit shearing stress to be 150 pounds per square inch. The *average* unit shearing stress in a rectangular section will be 100 pounds per square inch and the minimum area of cross-

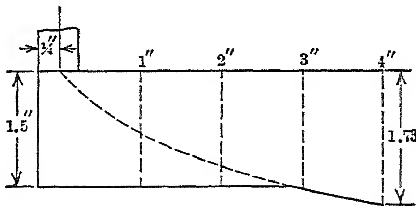


FIG. 218.

section will be 6 square inches. The depth at the end should not be less than 1.5 inches.

Suppose also that the allowable bearing stress is 300 pounds per square inch, and that the *center* of the load must be 5 feet from the wall; the bearing area must be at least 2 square inches. If the load extends the entire width of the beam, the bearing area must be 4 inches by ½ inch. The actual beam must extend at least ¼ inch beyond the center of the load. Figure 218 shows the details for these conditions. The dotted lines are the limits for the beam figured for bending only. The solid lines show the *minimum* dimensions figured for all stresses. The actual beam should be somewhat larger at the end than shown, as a great increase in safety can be secured here with practically no increase in cost and weight. Artistic appearance and convenience of construction may cause further modifications *outside* the *minimum dimensions*.

Problems

1. Design a cantilever of constant strength and constant depth of 4 in. to carry a load of 600 lb. 4 ft. from the fixed end. The allowable bending

stress is 1,200 lb. per sq. in.; the allowable bearing stress is 240 lb. per sq. in.; and the allowable shearing stress parallel to the grain is 180 lb. per sq. in.

Ans. Maximum width, 9 in.; minimum width, 1.25 in.; minimum bearing surface extends 1 in. on each side of the line of application of the load.

2. A cantilever of constant strength and constant breadth of 4 in. is designed to carry 900 lb. 60 in. from the fixed end. The allowable bending stress is 1,000 lb. per sq. in., the allowable bearing stress is 225 lb. per sq. in., and the allowable horizontal shearing stress is 150 lb. per sq. in. Design the beam.

Ans. Maximum depth, 9 in.; minimum depth, 2.25 in.; beam extends 60.5 in. from the fixed end. The depth at each 10-in. interval is?

148. Cantilever with Uniformly Distributed Load.—The only difference between a cantilever with uniformly distributed loading and a cantilever with a concentrated load on the end is in the expression for the external moment.

Problems

1. A cantilever of constant strength, which carries a uniformly distributed load of w per unit length, has a constant breadth b. Derive the expression for the depth at any distance x from the free end.

$$Ans.\ d^2 = \frac{3\,w\,x^2}{S\,b};\ d = \sqrt{\frac{3\,w}{S\,b}}x.$$

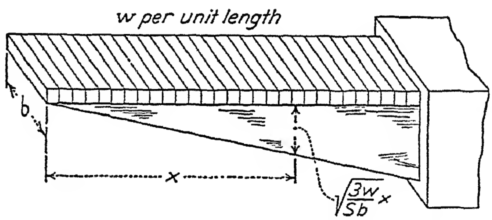


FIG. 219.—Cantilever of constant breadth.

2. A cantilever of constant strength and constant breadth of 4 in. carries a uniformly distributed load of 240 lb. per ft. Find the depth at 4 ft. from the free end if the allowable stress is 960 lb. per sq. in. *Ans.* 6 in.

3. A uniformly loaded cantilever of constant strength has a constant depth d. If all sections are rectangles, find the expression for the breadth (Fig. 220).

$$Ans.\ b = \frac{3\,w\,x^2}{S\,d^2}.$$

4. A cantilever of constant strength, uniformly loaded, is 5 ft. long and 4 in. deep. The maximum breadth is 8 in. Find the breadth at each foot interval. If the allowable bending stress is 1,080 lb. per sq. in., what is the total load?

$$b = \frac{3\,w}{S\,d^2}x^2 = K\,x^2;\quad 8 = 3{,}600\,K;\quad K = \frac{1}{450}.$$

x, feet	1	2	3	4	5
b, inches	0.32	1.28	..	..	8

Ans. $W = 768$ lb.; $w = 12.8$ lb./in.

5. If the allowable horizontal shearing stress of the beam of Problem 4 is 120 lb. per sq. in., for what distance from the free end must the beam be designed for shear, and what is the form of this portion?

Total vertical shear per inch of width $= 80 \times 4$.

$$320\,b = \frac{320\,x^2}{450}; \quad \frac{320\,x^2}{450} = 12.8\,x; \quad x = 18 \text{ in.}$$

The first 18 in. is a triangle which is 0.48 in. wide at 1 ft. from the end.

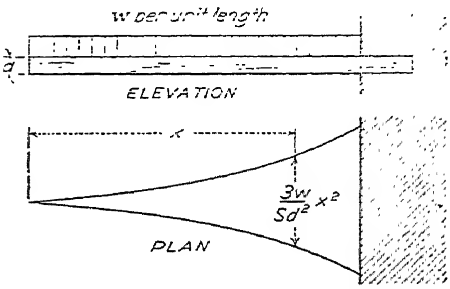


FIG. 220.—Cantilever of constant depth.

NOTE: *No provision is made for bearing in this design.*

6. A cantilever of constant strength and constant depth of 4 in. is 6 ft. long. It carries 120 lb. per ft. and 300 lb. 1 ft. from the free end. If the bending stress is 1,200 lb. per sq. in., find the breadth at each foot interval.

x, feet	1	2	3	4	5	6
M, inch-pounds	720	6480				
Z, inches^3	0.6	5.4				
b, inches	0.225	......	4.275	..	10.125	

7. The beam of Problem 6 is 6 in. wide throughout. Find the depth at each foot interval.

Ans. x, feet	1	2	3	4	5	6
d, inches	0.775	2.324	..	4.313	..	6.050

8. Derive an expression for the depth of a cantilever of square section to carry a uniformly distributed load.

Ans. $d^3 = \dfrac{3\,w\,x^2}{S}$.

9. A uniformly loaded beam of constant strength and square section carries a load of 9 lb. per in. The allowable stress is 1,000 lb. per sq. in. Find the dimensions at 64 in. from the free end. *Ans.* $d = 4.8$ in.

10. Find the dimensions of the beam of Problem 9 at 8 in., 15.625 in., 27 in., and 42.875 in. from the free end.

11. A cantilever which carries 9 lb. per in. has all sections similar rectangles for which the depth is twice the breadth. A load of 180 lb. is placed

10 in. from the free end. Find the dimensions for 10-in. intervals up to 5 ft. if S = 1,000 lb. per sq. in.

Ans. x, inches......	10	20	30	40	50	60
d, inches......	1.754	3.509	..	..	6.050	

12. How does the volume of the beam of Problem 1 compare with that of a uniform beam of equal strength? *Ans.* One-half as great.

13. How does the volume of the beam of Problem 3 compare with that of a uniform beam of equal strength if no correction is made for shear? *Ans.* One-third as great.

14. How does the volume of the beam of Problem 5 compare with that of a uniform beam of equal strength. *Ans.* 33.78 per cent.

149. Beam of Constant Strength, Simply-supported.—When a beam is supported at the ends and carries a single load at the middle, the problem is exactly the same as that of a cantilever of one-half the length which is pushed up by the end reaction. When the load is not at the middle, the portion from the load to each end is equivalent to a cantilever. Ample allowance must be made for shear and bending at the supports.

A uniformly loaded beam which is simply-supported is equivalent to cantilevers which are pushed up by the end reactions and bent down by the distributed loads. Since the moment equation is composed of two terms, the calculations are not so simple as for a cantilever.

Problems

1. A cast-steel beam is made for a span of 8 ft. to carry a distributed load of 2,400 lb. per ft. with a maximum unit stress of 12,000 lb. per sq. in. Find the section modulus at each 12 in.

Ans. x, feet...........	1	2	3	4
Z, in.^{3}...........	8.4	14.4	18	19.2

2. A timber beam of constant strength and constant breadth of 8 in. is used for a span of 10 ft. to carry a load of 240 lb. per ft. with a maximum stress of 1,200 lb. per sq. in. Find the depth at each foot.

Ans. x, feet..........	1	2	3	4	5
d^2, in.^{2}..........	8.1	14.4	18.9	21.6	22.5
d, inches........	2.848	3.795	4.347	4.648	4.743

3. Derive the expression for the depth of a uniformly loaded, simply-supported beam of constant breadth.

Ans. $d = \sqrt{\frac{3\,w}{S\,b}(l\,x - x^2)}$.

Example I

Find the ratio of the volume of the beam of Problem 3 to the volume of a uniform beam of equal strength. Simplify the integration by taking the origin of coördinates at the middle of the span.

From the general moment equation,

$$M = \frac{w\,l^2}{8} - \frac{w\,x^2}{2} = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right) = \frac{S\,b\,d^2}{6}, \tag{1}$$

in which x is measured from the middle of the span.

$$d = \sqrt{\frac{3\,w}{S\,b}\left(\frac{l^2}{4} - x^2\right)}; \tag{2}$$

$$V = b\int_0^{\frac{l}{2}} d\,dx = \int_0^{\frac{l}{2}} \sqrt{\frac{3\,w\,b}{S}\left(\frac{l^2}{4} - x^2\right)}\,dx, \tag{3}$$

in which V is the volume of one-half the beam.

Let $x = \frac{l}{2}\cos\theta$, then $dx = -\frac{l}{2}\sin\theta\,d\theta$ and

$$\sqrt{\frac{l^2}{4} - x^2} = \frac{l}{2}\sin\theta.$$

$$\int\left(\frac{l^2}{4} - x^2\right)^{\frac{1}{2}}dx = -\frac{l^2}{4}\int_{\frac{\pi}{2}}^{0}\sin^2\theta\,d\theta = -\frac{l^2}{8}\int_{\frac{\pi}{2}}^{0}(1 - \cos 2\,\theta)\,d\theta \tag{4}$$

$$= -\frac{l^2}{8}\left[\theta - \frac{\cos 2\,\theta}{2}\right]_{\frac{\pi}{2}}^{0} = \frac{\pi\,l^2}{16}.$$

Volume of one-half beam $= \sqrt{\frac{3\,w\,b}{S}} \times \frac{\pi\,l^2}{16}$.

From Equation (1), maximum depth $= \sqrt{\frac{3\,w}{S\,b}} \times \frac{l}{2}$, and volume of one-half beam $= \sqrt{\frac{3\,w\,b}{S}} \times \frac{l^2}{4}$.

$$\text{Ratio} = \frac{\pi}{4}.$$

Example II

A 20-in. 95-lb. I-beam is used for a span of 30 ft. to carry 180,000 lb. uniformly distributed. The beam is reinforced by 12-in. by 1-in. plates welded to the top and bottom flanges. If the maximum bending stress is 15,000 lb. per sq. in., find the length of each pair of plates. For the first pair of plates,

$$90{,}000\,x - 250\,x^2 = 15{,}000 \times 160;$$
$$x^2 - 360\,x + 9{,}600 = 0;$$
$$x = 29 \text{ in.}$$

Each of the first pair of plates is $360 - 58 = 302$ in. in length.

The additional moment of inertia is $\dfrac{12(22^3 - 20^3)}{12}$;

$$I = 2{,}648 + 1{,}600 = 4{,}248 \text{ in.}^4;$$
$$Z = \frac{4{,}248}{11} = 386.2 \text{ in.}^3;$$
$$x^2 - 360\,x + 23{,}172 = 0;\ x = 84 \text{ in.}$$

Each of the second pair of plates is 16 ft. long. Will another pair of plates be required?

Shapes are not rolled as beams of constant strength, but combinations of shapes and plates, as in Example II, are riveted or welded together in such way that the section modulus varies approximately as the bending moment. In machinery and vehicles, where weight is important, beams of constant strength are much used. In the frames of stationary machines, these are frequently made of cast iron. In other places, cast steel or forged steel is employed. Cast-steel members, of approximately constant strength, are used in the construction of railway cars and trucks, and steel forgings in automobiles.

A tree is a vertical beam of constant strength. A bamboo rod or a wheat straw is a hollow beam of constant strength, which has a large section modulus relative to its weight.

150. Deflection of Beam of Constant Strength.—Since the moment of inertia is not constant, the fundamental equation for successive integration is written

$$E\,\frac{d^2y}{dx^2} = \frac{M}{I}; \quad \text{(Formula XVII, Art. 87)}$$

and the equations for area moments are written

$$E\,y = \int \frac{M}{I}(x - x_a)\,dx \quad \text{(Formula XXIV, Art. 105)}$$

and

$$E\,y = \int \frac{M}{I}\,x\,dx. \quad \text{(Formula XXV, Art. 105)}$$

The moment of inertia is expressed in terms of x before integration.

In the following derivations, the maximum moment of inertia is written I_m. Constant depth and depth at the maximum section are expressed by D; and constant breadth and breadth at the maximum section are expressed by B.

When the depth is constant, the moment of inertia and the breadth vary directly as the section modulus, which, in turn, varies directly as the moment. For all beams of constant strength and constant depth, the calculations are the same as for a beam of constant moment, *provided the breadth of every surface parallel to the neutral surface varies directly as the moment.*

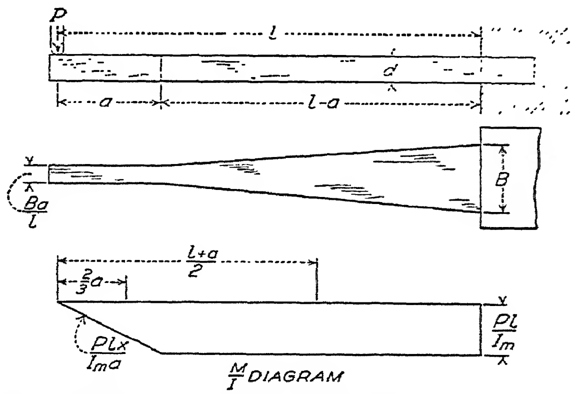


FIG. 221.—Cantilever of constant depth with provision for shear.

151. Cantilever of Constant Depth.—For a cantilever of constant strength and constant depth, loaded at the free end,

$$I = \frac{I_m x}{l}; \quad M = -P\,x; \quad \frac{M}{I} = -\frac{P\,l}{I_m};$$

$$E\,y = -\frac{P\,l}{I_m}\int_0^l x\,dx = -\frac{P\,l^3}{2\,I_m}. \tag{1}$$

The deflection at the end is $1\frac{1}{2}$ times as great as the deflection of a beam of constant section for which $I = I_m$. To find the deflection at a distance a from the end by area moments,

$$E\,y = -\frac{P\,l}{I_m}\int_a^l (x-a)dx = -\frac{P\,l}{2\,I_m}\Big[(x-a)^2\Big]_a^l = -\frac{P\,l}{2\,I_m}(l-a)^2. \tag{2}$$

Example

A cantilever of constant depth d, loaded at the free end, has a constant section of width $\frac{B\,a}{l}$ for a distance a from the load. The remainder of the

beam is designed for constant strength. Find the expression for the deflection at the free end by graphic integration of area moments.

The $\frac{M}{I}$ diagram is shown in Fig. 221. The diagram for the uniform portion of length a is a triangle. The remainder of the diagram, of length $l - a$, is a rectangle. The deflection at the end is given by

$$E\,y = -\frac{P\,l}{I_m} \times \frac{a}{2} \times \frac{2\,a}{3} - \frac{P\,l}{I_m}(l - a)\frac{l + a}{2} = -\frac{P\,l}{I_m}\left(\frac{l^2}{2} - \frac{a^2}{6}\right).$$

Problems

1. A cantilever 6 ft. long has triangular sections which are 6 in. deep throughout. The first 18 in. has a constant width of 2 in. at the top. The remainder is designed as a beam of constant strength for a load 6 ft. from the fixed end. The maximum width is 8 in. When the load is 200 lb., find the maximum stress. Find the deflection at the free end if E = 1,000,000.

   Ans. S = 1,200 lb./in.^{2}; $y_{\max}$ = 0.7614 in.

2. If the beam of Problem 1 has elliptical sections with all vertical axes 6 in. and the maximum horizontal axis 8 in., with horizontal axes of 2 in. for the first 18 in., while the remainder is designed for constant strength, find the maximum stress, the deflection at the end, and the deflection 2 ft. from the free end. *Ans.* $y_{\max}$ = 0.4309 in.; S = 509.3 lb./in.^{2}

3. The beam of Problem 2 has a circular section at 54 in. from the free end. Calculate the stress at this section.

For a *uniformly loaded cantilever of constant depth,*

$$I = \frac{I_m x^2}{l^2}; \quad M = -\frac{w\,x^2}{2}; \quad \frac{M}{I} = -\frac{w\,l^2}{2\,I_m}.$$

$$E\,y_{\max} = \frac{w\,l^2}{2\,I_m}\int_0^l x\,dx = -\frac{w\,l^4}{4\,I_m}, \tag{3}$$

which is twice maximum deflection of cantilever of uniform section, similarly loaded.

To find the deflection at any distance a from the free end,

$$E\,y = -\frac{w\,l^2}{2\,I_m}\int_a^l (x - a)\,dx = -\frac{w\,l^2(l - a)^2}{4\,I_m}. \tag{4}$$

Problems

4. A cantilever of constant strength, uniformly loaded, has a deflection at a distance x from the end which is equal to the maximum deflection of a uniform beam having a constant moment of inertia I_m. Find x.

   Ans. $x = l\left(1 - \frac{1}{\sqrt{2}}\right) = 0.2929\,l.$

5. Find the maximum deflection and the deflection 18 in. from the free end for the beam of Problem 4 (Art. 148) if E = 1,200,000 lb. per sq. in.

   Ans. 0.81 in.; 0.3969 in.

6. Find the deflection at the free end of the beam of Problem 5 (Art. 148) if E = 1,200,000 lb. per sq. in. For the triangle, 18 in. long, $I = \frac{0.64\,x}{3}$; $\frac{M}{I} = -30\,x.$

$$E\,y_{\max} = -30\int_0^{18} x^2\,dx - 540\int_{18}^{60} x\,dx = -58{,}320 - 884{,}520 = -942{,}840;$$

$$y_{\max} = -0.7857 \text{ in.}$$

7. Solve Problem 5 by means of the moment of the area of the $\frac{M}{I}$ rectangle.

8. Solve Problem 6 by means of the area of the $\frac{M}{I}$ rectangle for 18 in. to 60 in. and the triangle from 0 in. to 18 in.

152. Simply-supported Beam of Constant Depth.—For a *simply-supported beam of constant strength and constant depth, loaded at the middle,* the maximum moment at the middle is $\frac{P\,l}{4}$, and $\frac{M}{I} = \frac{P\,l}{4\,I_m}$.

The deflection of the end upward from the tangent at the middle by area moments from the $\frac{M}{I}$ rectangle is

$$\frac{P\,l}{4\,I_m} \times \frac{l}{2} \times \frac{l}{4} = \frac{P\,l^3}{32\,I_m} = E\,y_{\max}. \tag{1}$$

The slope at the left support is given by

$$0 = E\,\theta_1 + \frac{P\,l^2}{8\,I_m}; \quad E\,\theta_1 = -\frac{P\,l^2}{8\,I_m}.$$

$$E\,y = -\frac{P\,l^2\,x}{8\,I_m} + \frac{P\,l\,x}{4\,I_m} \times \frac{x}{2} = -\frac{P\,l\,x}{8\,I_m}(l - x). \tag{2}$$

For a *simply-supported beam of constant strength and constant depth, uniformly loaded,* $\frac{M}{I} = \frac{w\,l^2}{8\,I_m}$.

At the middle,

$$0 = E\,\theta_1 + \frac{w\,l^3}{16\,I_m}; \quad E\,\theta_1 = -\frac{w\,l^3}{16\,I_m}.$$

$$E\,y = -\frac{w\,l^3x}{16\,I_m} + \frac{w\,l^2x}{8\,I_m} \times \frac{x}{2} = -\frac{w\,l^2x}{16\,I_m}(l - x). \tag{3}$$

At the middle, where $x = \frac{l}{2}$,

$$E\,y = -\frac{w\,l^4}{64\,I_m}, \tag{4}$$

which is six-fifths as great as the maximum deflection of a beam of constant section.

153. Beam of Constant Strength, Breadth Constant.—When the breadth is constant, the moment of inertia varies as the cubes of the vertical dimensions.

$$\frac{I_m}{I} = \frac{D^3}{d^3}; \qquad \frac{1}{I} = \frac{D^3}{I_m d^3}.$$

For constant strength,

$$\frac{M}{Z} = \frac{M_m}{Z_m}; \quad \frac{M_m}{M} = \frac{D^2}{d^2};$$
$$\frac{1}{I} = \frac{M_m^{3/2}}{I_m M^{3/2}}; \quad \frac{M}{I} = \frac{M_m^{3/2}}{I_m M^{1/2}}. \tag{1}$$

Equation (1) applies to any beam which has sections of vertical similarity and has a principal axis horizontal so that a vertical load will bend it vertically downward. Sections have vertical similarity when the heights of vertical elements in one section have the same ratio as the heights of corresponding elements in any other section. A beam for which all sections are isosceles triangles of constant width at the top has vertical similarity. If d_a is the altitude of section A, d_b the altitude of section B, and if h_a and h_b are the heights of elements at the same distance from the vertical plane of symmetry of the beam, in sections A and B, respectively,

$$\frac{h_a}{h_b} = \frac{d_a}{d_b}.\text{*}$$

For a cantilever with a load on the free end, $M = -P\,x$. Since I is positive $\frac{M}{I}$ is negative.

$$\frac{M}{I} = -\frac{P^{3/2}\,l^{3/2}}{I_m P^{1/2} x^{1/2}}; \quad E\,y_{\max} = -\int \frac{M}{I}\,x\,dx = -\frac{P\,l^{3/2}}{I_m}\int_0^l x^{1/2}\,dx;$$
$$E\,y_{\max} = -\frac{2\,P\,l^{3/2}}{3\,I_m}\Big[x^{3/2}\Big]_0^l = -\frac{2\,P\,l^3}{3\,I_m}, \tag{2}$$

which is twice the deflection of a uniform cantilever for which the moment of inertia is I_m.

* The terms "vertical similarity" and "horizontal similarity" were suggested by Prof. P. W. Ott.

The deflection at any point is given by

$$E\,y = \int \frac{M}{I}(x - a)\,dx = -\frac{P\,l^{3/2}}{I_m}\int_a^l (x^{1/2} - a\,x^{-1/2})dx;$$

$$E\,y = -\frac{P\,l^{3/2}}{I_m}\left[\frac{2\,x^{3/2}}{3} - 2\,a\,x^{1/2}\right]_a^l = -\frac{P\,l^{3/2}}{I_m}\left(\frac{2\,l^{3/2}}{3} - 2\,a\,l^{1/2} + \frac{4\,a^{3/2}}{3}\right);$$

$$E\,y = -\frac{P}{3\,I_m}(2\,l^3 - 6\,a\,l^2 + 4\,a^{3/2}\,l^{3/2}) =$$

$$-\frac{2\,P}{3\,I_m}(l^3 - 3\,a\,l^2 + 2\,a^{3/2}l^{3/2}). \quad (3)$$

Problems

1. A cantilever of constant strength and constant width, 80 in. long, has sections in the form of isosceles triangles which are 4 in. wide at the top. The depth is 6 in. at the fixed end. Find the deflection at the free end end and 20 in. from the free end caused by a load of 90 lb. on the end. The modulus of elasticity is 1,000,000.

Ans. $y_{max} = 1.28$ in.; $y = 0.64$ in.

2. The cantilever of Problem 1 is reinforced at the end for shear. The end is a triangle 1.5 in. high. The section at 20 in. from the end is 3 in. high. The homologous sides of these triangles are connected by planes to form the frustum of a pyramid. Find the deflection at the free end.

Substituting the limits 20 and 80 in Eq. (2) to get the deflection at the load caused by the moment in 60 in. of beam designed for constant strength, $-\frac{180 \times 80^{3/2}}{72 \times 10^6}(80^{3/2} - 20^{3/2}) = -1.1200$ in.

To get the deflection caused by the frustum, imagine the surfaces extended an additional 20 in. to form a pyramid.

$$I = \frac{3\,x^3}{64{,}000};\ M = -90(x - 20);$$

$$\int \frac{M}{I}\,(x - 20)\,dx = -1{,}920{,}000\int_{20}^{40}\left(\frac{1}{x} - \frac{40}{x^2} + \frac{400}{x^3}\right)dx$$

$$= -1{,}920{,}000\left(\log_e x + \frac{40}{x} - \frac{200}{x^2}\right) = -1{,}920{,}000(\log_e 2 - 0.625);$$

$$y_{max} = -1.1200 - 0.13085 = -1.25085 \text{ in.}$$

For a *cantilever uniformly loaded,* $M = -\dfrac{w\,x^2}{2}$;

$$\frac{M_m^{3/2}}{I_m\,M^{1/2}} = -\frac{w\,l^3}{2\,I_m\,x};\quad \int \frac{M}{I}\,x\,dx = -\frac{w\,l^3}{2\,I_m}\int_0^l dx = -\frac{w\,l^4}{2\,I_m}.$$

$$E\,y_{max} = -\frac{w\,l^4}{2\,I_m} = -\frac{W\,l^3}{2\,I_m}, \quad (4)$$

which is four times as great as the deflection of a uniform cantilever for which the moment of inertia is I_m. The deflection at a distance a from the free end is given by

$$E\,y = -\int \frac{w\,l^3}{2\,I_m x}(x - a)dx = -\frac{w\,l^3}{2\,I_m}\int_a^l \left(1 - \frac{a}{x}\right)dx;$$

$$E\,y = -\frac{w\,l^3}{2\,I_m}\Big[x - a\log_e x\Big]_a^l = -\frac{w\,l^3}{2\,I_m}\left(l - a - a\log_e \frac{l}{a}\right);$$

$$E\,y = -\frac{W\,l^2}{2\,I_m}\left(l - x\left(1 + \log \frac{l}{x}\right)\right). \tag{5}$$

Problems

3. A cantilever of constant strength for a uniformly distributed load is 80 in. long, has all sections isosceles triangles 4 in. wide at the top, and a maximum section 6 in. high at the fixed end. If E is 1,000,000 lb. per sq. in., what is the deflection at the end under a total load of 120 lb.?

Ans. $y_{\max} = 1.28$ in.

4. Find the deflection of the beam of Problem 3 at 20 in. from the free end.

Ans. $y = 0.5164$ in.

5. In Problem 3, find the deflection 10 in. from the fixed end.

A *uniformly loaded, simply-supported beam of constant breadth and constant strength* has a maximum moment of $\frac{w\,l^2}{8}$. If the origin of moments is taken at the middle,

$$M = \frac{w\,l^2}{8} - \frac{w\,x^2}{2}; \quad \frac{M}{I} = \frac{w\,l^3}{16\,I_m\left(\frac{l^2}{4} - x^2\right)^{1/2}}.$$

The deflection of a point at a distance a from the middle upward from the tangent at the middle is

$$E\,y = \int_0^a \frac{M}{I}(a - x)dx = \frac{w\,l^3}{16\,I_m}\int_0^a \frac{a - x}{\left(\frac{l^2}{4} - x^2\right)^{1/2}}dx;$$

$$E\,y = \frac{w\,l^3}{16\,I_m}\left[a\sin^{-1}\frac{2\,x}{l} + \left(\frac{l^2}{4} - x^2\right)^{1/2}\right]_0^a$$

$$E\,y = \frac{w\,l^3}{16\,I_m}\left(a\sin^{-1}\frac{2\,a}{l} + \left(\frac{l^2}{4} - a^2\right)^{1/2} - \frac{l}{2}\right). \tag{6}$$

At the end $a = \frac{l}{2}$,

$$E\,y_{\max} = \frac{w\,l^3}{16\,I_m}\left(\frac{l}{2}\sin^{-1}1 - \frac{l}{2}\right) = \frac{w\,l^4}{16\,I_m}\left(\frac{\pi}{4} - \frac{1}{2}\right); \tag{7}$$

$$y_{\max} = \frac{0.01784\,w\,l^4}{E\,I_m},$$

which is positive upward from the tangent at the middle. To find the deflection of a point at a distance a from the middle downward from the supports, add the results from Equation (6) to $-\dfrac{0.01784\, w\, l^4}{E\, I_m}$.

Problems

6. A simply-supported, uniformly loaded rectangular beam of constant strength is 80 in. long between supports, 4 in. wide throughout, and 6 in. deep at the middle. If $E = 1{,}000{,}000$, find the deflection at the middle caused by a total load of 2,160 lb.

$$Ans.\ \frac{w\, l^3}{16\, E\, I} = 0.0120;\ 40\left(\frac{\pi}{2} - 1\right) = 22.832;\ y_{\max} = 0.2740 \text{ in.}$$

7. In Problem 6, find the deflection at 30 in. from the middle upward from the tangent at the middle by Eq. (6). Then calculate the deflection of this point downward from the level of the supports.

$$\begin{aligned} 30 \times 0.84806 \text{ radian} &= 25.4418 \\ \sqrt{700} &= 26.4575 \\ &\ -40. \\ \hline &\ 11.8993 \times 0.012 = 0.1428 \text{ in.} \\ y &= -0.2740 + 0.1428 = -0.1312 \text{ in.} \end{aligned}$$

8. The beam of Problem 6 was designed for a maximum unit shearing stress of 202.5 lb. per sq. in. at the neutral surface at the supports as shown by the half beam of Fig. 222. For the last 8 in., the vertical faces are

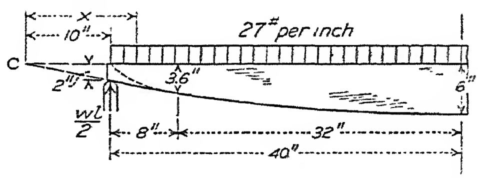


FIG. 222.—End of simply-supported beam with provision for shear.

trapezoids which are 2 in. high at the end and 3.6 in. at the point of tangency with the curve. Find the deflection at the middle.

To find that part of the deflection of the end upward from the tangent at the middle which is caused by the moment in the 32 in. of constant strength beam, the bracketed term of Eq. (6) is integrated from zero to 32, while a remains 40.

$$40 \sin^{-1} {}^{64}\!/\!_{80} + 24 - 40 = 40 \times 0.92735 - 16 = 21.0940;$$
$$y = 0.0120 \times 21.094 = 0.2531 \text{ in.}$$

To find the deflection of the 8-in. length upward from the tangent at its right end, the origin of coördinates is taken at C (Fig. 222), which is the intersection of the upper and lower plane surfaces of this portion. This choice of origin gives a monomial term for I which is the denominator of the $\dfrac{M}{I}$ expression and thereby greatly simplifies the integration.

$$d = 0.2\,x;\quad I = \frac{4 \times 0.008\,x^3}{12} = \frac{0.008\,x^3}{3};\quad M = 1{,}080(x - 10) - 13.5(x - 10)^2;$$

$$\int \frac{M}{I}(x - 10)dx = \int \frac{8{,}100}{16\,x^3}(-x^3 + 110\,x^2 - 1{,}900x + 9{,}000)dx;$$

$$E\,y = \frac{81{,}000}{16}\left[-x + 110\log_e x + \frac{1{,}900}{x} - \frac{4{,}500}{x^2}\right]_{10}^{18};$$

$$E\,y = \frac{81{,}000}{16}(-8 + 64.6569 + 105.5555 - 190 - 13.8888 + 45);$$

$$E\,y = \frac{3.3236 \times 81{,}000}{16} = 16{,}826;\quad y = 0.016826 \text{ in.}$$

Total deflection $= 0.2531 + 0.0168 = 0.2699$ in.

154. Beam of Constant Strength, Sections Similar.—For a beam with similar sections, I varies as the fourth power of the dimensions, and Z varies as the third power.

$$\frac{M_m}{M} = \frac{D^3}{d^3};\quad \frac{I_m}{I} = \frac{D^4}{d^4} = \frac{M_m^{4/3}}{M^{4/3}}$$

$$\frac{M}{I} = \frac{M_m^{4/3}}{I_m M^{1/3}}. \tag{1}$$

For *a cantilever with load on the free end*, $M = -P\,x$;

$$M_m = -P\,l;\quad \frac{M}{I} = -\frac{P\,l^{4/3}}{I_m x^{1/3}};\quad E\,y = -\int \frac{P\,l^{4/3}}{I_m x^{1/3}}(x - a)dx;$$

$$E\,y = -\frac{P\,l^{4/3}}{I_m}\int (x^{2/3} - a\,x^{-1/3})dx = -\frac{P\,l^{4/3}}{I_m}\left[\frac{3\,x^{5/3}}{5} - \frac{3ax^{2/3}}{2}\right]_a^l;$$

$$E\,y = -\frac{P\,l^{4/3}}{I_m}\left(\frac{3\,l^{5/3}}{5} - \frac{3\,a^{5/3}}{5} - \frac{3\,a\,l^{2/3}}{2} + \frac{3\,a^{5/3}}{2}\right);$$

$$E\,y = -\frac{P\,l^{4/3}}{I_m}\left(\frac{3\,l^{5/3}}{5} - \frac{3\,a\,l^{2/3}}{2} + \frac{9\,a^{5/3}}{10}\right). \tag{2}$$

When $a = l$,

$$E\,y_{\max} = -\frac{3\,P\,l^3}{5\,I_m}, \tag{3}$$

which is 1.8 times as great as the deflection of a beam of uniform section.

For *a uniformly loaded cantilever*, $M_m = -\dfrac{w\,l^2}{2}$;

$$\frac{M}{I} = -\frac{w\,l^{8/3}}{2\,I_m x^{2/3}};\quad E\,y = -\frac{w\,l^{8/3}}{2\,I_m}\int (x^{1/3} - a\,x^{-2/3})dx;$$

$$E\,y = -\frac{w\,l^{8/3}}{2\,I_m}\left[\frac{3\,x^{4/3}}{4} - 3\,a\,x^{1/3}\right]_a^l =$$

$$-\frac{w\,l^{8/3}}{2\,I_m}\left(\frac{3\,l^{4/3}}{4} - 3\,a\,l^{1/3} + \frac{9\,a^{4/3}}{4}\right); \quad (4)$$

$$y_{\max} = -\frac{3\,w\,l^4}{8\,E\,I_m}, \quad (5)$$

which is three times as great as the deflection of a beam of uniform section.

155. Beams of Two or More Materials.—Beams are frequently made of two or more materials which have different moduli of elasticity. The most common types are combinations of timber and steel or of concrete and steel.

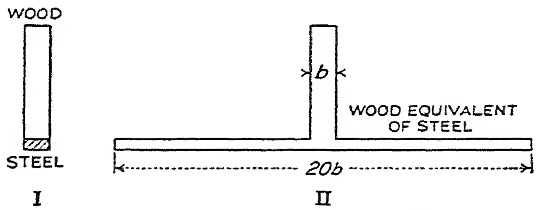


FIG. 223.—Beam of two materials.

Figure 223 shows a steel plate which is bolted to the bottom of a wooden beam. The position of the neutral surface depends upon the ratio of the moduli of elasticity of the two materials. If the modulus of the steel is 30,000,000 pounds per square inch and that of the wood is 1,500,000 pounds per square inch, the unit stress in the steel at a given unit deformation is twenty times as great as the unit stress in the wood.

The location of the neutral axis for Fig. 223 may be computed on the assumption that the density of the steel is twenty times as great as the density of the wood; or, for purpose of calculation, the steel may be replaced by a wooden strip which is twenty times as wide and has the same thickness. Figure 223, II, illustrates this substitution.

Example

A 4-in. by 6-in. wooden beam has a steel plate 1 in. wide and ½ in. thick fastened to the lower surface. Find the neutral axis and the maximum fiber stress in each material if the modulus of elasticity of the steel is twenty times as great as that of the wood, and the bending moment is 30,000 in.-lb.

The steel may be replaced by a wooden strip 20 in. wide and ½ in. thick. To get the distance of the center of gravity from the bottom of the wood,

$$\bar{y} = \frac{24 \times 3 - 10 \times \frac{1}{4}}{34} = 2.04 \text{ in.}$$

To get the moment of inertia of the equivalent wooden section about the common surface.

$$\frac{4 \times 6^3}{3} = 288,$$

$$\frac{20 \times (\frac{1}{2})^3}{3} = 0.83,$$

$$I = 288.83.$$

$$I_c = 288.83 - 34 \times 2.04^2 = 147.34 \text{ in.}^4$$

To get the unit stress in the top fibers of the wood,

$$S = \frac{30{,}000 \times 3.96}{147.34} = 806 \text{ lb. per sq. in.}$$

In the bottom steel fibers,

$$S = \frac{30{,}000 \times 2.54 \times 20}{147.34} = 10{,}344 \text{ lb. per sq. in.}$$

The result for steel is multiplied by 20 because the moment of inertia used was calculated on the assumption that the steel was replaced by wood.

Problems

(Use E for steel, twenty times E for timber, in these problems.)

1. A 4-in. by 4-in. timber beam has a 4-in. by ½-in. steel plate on the lower surface and a 2-in. by 1-in. plate on the upper surface. Find the neutral axis of the combination. What is the maximum fiber stress in the steel when that in the wood is 600 lb. per sq. in.?
   Ans. Neutral axis, 2.10 in. above bottom of timber; fiber stress in steel, 16,571 lb./in.^{2}
2. A 6-in. by 6-in. timber beam, 10 ft. long, has a 6-in. by ½-in. steel plate on the top and bottom surfaces. Find the unit stress in the steel when a load of 9,000 lb. is put on the middle. *Ans.* 13,720 lb./in.^{2}

Figure 224 is called a *flitched beam.* It is built up of alternating wooden beams and steel plates, all fastened together by a few bolts. When the vertical depth of the wooden beams and steel plates is the same, the bolts transmit no shear below the elastic limit of the steel or wood. With vertical loads, parallel to the plane of the plate, the steel is not efficiently used in this type of beam.

Flitched beams were once used to some extent, but very little at present. Steel I-beams are usually preferable. Wooden beams are frequently bolted to the web of an I-beam. This is

generally done for convenience in attaching woodwork rather than for reinforcing the beam.

When steel is fastened to the top and bottom of a wooden beam, it is used efficiently. The combination is equivalent to an I-beam section, the timber acting as the web and the steel as the flange.

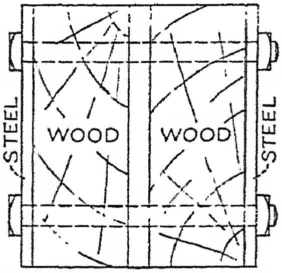


Fig. 224.—Flitched beam.

Formerly, combinations of steel and timber had considerable application in the construction of vehicles. Thin plates of steel in compression were kept from buckling by bolts which fastened them to the thicker and, therefore, more rigid timber. This form of construction requires no expensive equipment. With the present development of portable welding apparatus these combinations are little used.

156. Reinforced-concrete Beams.—Reinforced concrete represents another form of combination beam. A reinforced-concrete beam has steel rods embedded in the concrete near the surface in the tension side. Sometimes both tension and compression sides are reinforced. These rods may be ordinary round or square steel bars. Usually they are corrugated or otherwise deformed or made of cable or twisted square bar. Such deformed or twisted bars are better fitted to transmit the stress from the concrete, since they do not slip when the grip of the concrete is weakened.

Figure 225 represents a section and a portion of the length of a reinforced-concrete beam, which is 8 inches wide and 11 inches deep, Figure 262 is a photograph of a beam of this size after failure. The reinforcement consists of three rods with centers 1 inch from the bottom of the concrete.

In the development of the theory of concrete beams, it is customary to consider that the steel takes all the tension. This assumption is correct for loads commonly used in practice. For small loads, giving tensile stresses of less than 300 pounds in the concrete (depending upon the quality), considerable of the tensile stress is carried by the aggregate. For larger loads, fine cracks form in the tension side, and experiments show that the steel takes practically all of the tension.

The *percentage of reinforcement* in a beam is calculated by dividing the area of the steel by the area of the beam section above the

center of the steel. In Fig. 225 the beam is regarded as an 8-inch by 10-inch section; the inch of concrete below the center of the rods is considered as simply protecting the steel. With three $\frac{5}{8}$-inch rods, each of which has a cross section of 0.307 square inch, the reinforcement in the beam of Fig. 225 is $0.921 \div 80 = 0.0115 = 1.15$ per cent. While it is customary to speak of the percentage of reinforcement, when used in formulas it is expressed as a ratio.

Elaborate formulas have been proposed for the calculation of reinforced-concrete beams. Some of these formulas assume that the compression curve of concrete is a parabola, *which it is not.*

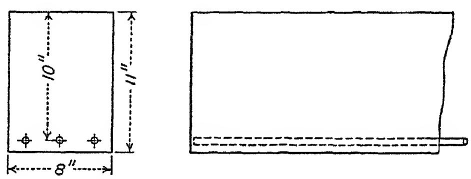


FIG. 225.—Reinforced beam.

The form and the constants of the compression curve vary greatly with the materials, the proportions, the care in mixing, the age, and the stresses to which it has been subjected. The modulus of elasticity is lowered greatly by slight overloads. For these reasons there is little use for great refinement of calculation unless the computer is provided with carefully determined compression curves of the actual concrete under consideration, and it is now customary to work on the assumption that the compression curve is a straight line.

A Joint Committee from The American Society of Civil Engineers, The American Society for Testing Materials, The American Railway Engineering Association, and the Association of American Portland Cement Manufacturers has prepared a report on Concrete and Reinforced Concrete and has recommended certain formulas and constants. In the articles that follow, the important formulas are given in the form recommended by the Joint Committee, and with the same symbols except that s is used for unit stress instead of f.*

157. Location of the Neutral Axis.—The line OF of Fig. 226 represents the compressive stress. The depth from the extreme compressive fibers to the center of the reinforcement is d and the

* *Proc. A. S. T. M.*, vol. 13, 1913.

distance from the neutral surface to the extreme fibers is kd, in which k is a fraction less than unity. The distance from the neutral surface to the center of the reinforcement is $(1 - k)d$. The ratio of the modulus of elasticity of the steel to that of the concrete is represented by n;

$$n = \frac{E_s}{E_c}. \tag{1}$$

If the modulus of elasticity of the steel is 30,000,000 and that of the concrete in compression is 2,000,000 pounds per square inch

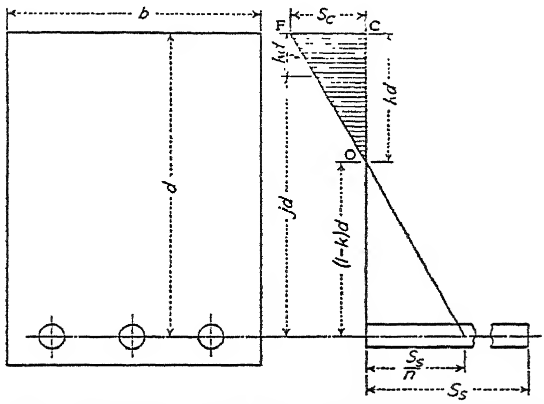


FIG. 226.—Stress in rectangular reinforced-concrete beam.

the value of n is 15. This is the value which is recommended for general use. However, when tests show the ultimate strength of the concrete to be between 2,200 pounds per square inch and 2,900 pounds per square inch, the Joint Committee recommends that n should be 12; and for concrete stronger than 2,900 pounds per square inch, the committee recommends that n should be 10.

The unit deformation at the center of the reinforcement is to the unit deformation in the outer fibers of the concrete as $1 - k$ is to k. Since the modulus of the steel is n times the modulus of the concrete, the unit stress at the center of the reinforcement is

$$s_s = \frac{n(1 - k)}{k}S_c, \tag{2}$$

in which s_s is the unit tensile stress at the center of the steel reinforcement, and S_c is the maximum unit compressive stress in the concrete. The tensile stress at the center of the reinforce-

ment is the average tensile stress, and it is assumed that the resultant tensile stress is at the center of the section. (The error is negligible.) The area of the concrete in compression in a rectangular section is $b\,k\,d$, and the average unit stress over this area is $\frac{S_c}{2}$.

$$\text{Total compressive stress} = \frac{S_c\,k\,b\,d}{2}. \qquad (3)$$

$$\text{Total tensile stress in steel} = \frac{A\,n\,S_c(1-k)}{k}, \qquad (4)$$

in which A is the area of the steel. The ratio of the area of the steel to the area of the concrete is represented by p;

$$p = \frac{A}{b\,d}; \quad A = p\,b\,d, \qquad (5)$$

As the concrete below the neutral surface is not regarded as taking any of the tensile stress, the total tension in the steel equals the total compression in the concrete. Equating (3) and (4) and substituting for A,

$$\frac{S_c\,k\,b\,d}{2} = \frac{S_c\,p\,b\,d\,n(1-k)}{k}, \qquad (6)$$

$$k^2 = 2\,p\,n(1-k), \qquad (7)$$

$$k^2 + 2\,p\,n\,k - 2\,p\,n = 0, \qquad (8)$$

$$k = \sqrt{2\,p\,n + (p\,n)^2} - p\,n. \qquad (9)$$

Problems

1. If the modulus of the steel be taken as fifteen times that of the concrete and the area of the steel is 1 per cent of the total area $b\,d$, find the distance of the neutral axis from the extreme compression fibers. *Ans.* $k = 0.418$.

2. Solve Problem 1 for a reinforcement of 1.2 per cent and for 1.6 per cent. *Ans.* 0.446, 0.493.

158. Relative Stresses in Concrete and Steel.—When the location of the neutral axis has been determined by means of Equation (9) of the preceding article or by experiment, the relative values of the average unit compressive stress in the concrete and the average unit tensile stress in the steel may be computed from the relation that the total tension in the steel is equal to the total compression in the concrete. In Problem 1 of the preceding article, for instance, since the area of concrete in compression is $0.418\,b\,d$ and the area of the steel in tension is

0.01 $b\,d$, the average unit stress in the concrete is $^{10}/_{418}$ as great as the average unit stress in the steel. If the average unit stress in this steel is 12,000 pounds per square inch, the average unit compressive stress in the concrete is 287 pounds per square inch, and the maximum stress in the extreme fibers is 574 pounds per square inch.

The unit compressive stress in the extreme fibers may also be computed from the distances from the neutral axis and the ratio of the two moduli of elasticity. From Equation (2) of the preceding article,

$$S_c = \frac{k\,s_s}{(1 - k)n}. \tag{1}$$

TABLE XXII.—ALLOWABLE UNIT COMPRESSIVE STRESSES IN EXTREME FIBERS OF CONCRETE BEAMS, IN POUNDS PER SQUARE INCH

Aggregate	1:2:4	1:3:6
Gravel or hard limestone or sandstone......	650	425
Soft limestone or sandstone...............	500	325
Cinder.................................	200	125

Problems

1. In Problem 2 of the preceding article, calculate the unit stress in the extreme fibers when the average unit tensile stress in the steel is 12,000 lb. per sq. in. *Ans.* 645 and 778 lb./in.^{2}
2. Solve Problem 1 if the allowable unit stress in the steel is 16,000 lb. per sq. in.

With a unit stress in the steel of 12,000 pounds per square inch, the unit compressive stress in the concrete with 1.6 per cent reinforcement is above the allowed value for 1:2:4 concrete for the best material ordinarily used. If the allowable unit stress in the steel is 16,000 pounds per square inch (which is the maximum recommended by the Joint Committee) even 1 per cent of reinforcement gives a unit stress in the concrete of over 700 pounds per square inch. In order to use the steel efficiently in a beam of rectangular section, it is necessary to have a richer mix than 1:2:4 or to keep the reinforcement below 1 per cent.

Problem

3. If $n = 15$ and $p = 0.008$, what will be the unit stress in the steel when the unit stress in the outer fibers of the concrete, calculated on the

assumption that the compression curve is a straight line, is 650 lb. per sq. in.? *Ans.* 15,600 lb./in.^{2}

159. The Resisting Moment.—The resultant compressive stress is at the center of gravity of the triangle $C\,F\,O$ of Fig. 226. The resultant tensile stress is regarded as being at the center of the reinforcement; therefore the arm of the resisting moment is $\left(1 - \frac{k}{3}\right)d$. The term $\left(1 - \frac{k}{3}\right)$ is represented by the single letter j.

$$\text{Resisting moment arm} = \left(1 - \frac{k}{3}\right)d = j\,d. \qquad (1)$$

Problems

1. What is the resisting moment arm in Problem 1 of Art. 158? *Ans.* $j\,d = 0.86\,d$.

The resisting moment is either total stress multiplied by the moment arm,

$$M = \frac{S_c j\,k\,b\,d^2}{2} = s_s A\,j\,d; \qquad (2)$$

$$S_c = \frac{2\,M}{j\,k\,b\,d^2}; \qquad (3)$$

$$s_s = \frac{M}{A\,j\,d}. \qquad (4)$$

(In the following problems $n = 15$, $S_c = 650$ lb. per sq. in.)

2. A reinforced-concrete beam for a span of 15 ft. is 10 in. wide and 12 in. deep to center of reinforcement. The reinforcement consists of three deformed bars, each having a cross section of 0.39 sq. in. The beam weighs 125 lb. per linear foot. What is the maximum safe load on the middle, based on the compressive strength of the concrete? What is the unit tensile stress in the steel at this load.
Ans. $M = 167{,}000$ in.-lb.; maximum safe load 2,770 lb.
3. In Problem 2 find the unit stress in the steel by dividing the moment by the resisting arm, to get the total tension, and then dividing by the area of the steel. *Ans.* 13,800 lb./in.^{2}
4. Design a reinforced-concrete beam for a span of 20 ft. to carry a load of 800 lb. per ft. including its own weight, using 1 per cent reinforcement.

An approximate value of the resisting moment may be computed from the expression:

$$M = 0.8\,d \times A\,s_s.$$

The moment arm is always a little greater than $0.8\,d$ and the total tensile stress in the reinforcement is $A\,s_s$. Of course, if the percentage of reinforcement is too great, the compressive stress in the concrete will be too

160. Steel Ratio.—It was shown in Art. 158 that, when the percentage of reinforcement is too great, the concrete stress will exceed its allowable value before the steel is fully stressed. The ratio of the steel area to total area may be found for any allowable unit stresses. From the equality of the total tensile and compressive stress,

$$\frac{S_c\, k\, b\, d}{2} = s_s A, \tag{1}$$

from which

$$k = \frac{2\, s_s A}{S_c b\, d} = \frac{2\, s_s p}{S_c}. \tag{2}$$

From Equation (8) of Art. 157,

$$k^2 + 2\, p\, n\, k = 2\, p\, n. \tag{3}$$

Eliminating k between Equations (2) and (3):

$$\frac{4 s_s^2 p^2}{S_c} + \frac{4 s_s p^2 n}{S_c} = 2\, p\, n; \tag{4}$$

$$p = \frac{n}{2\frac{s_s}{S_c}\left(\frac{s_s}{S_c} + n\right)} = \frac{1}{2\frac{s_s}{S_c}\left(\frac{s_s}{n\, S_c} + 1\right)}. \tag{5}$$

Problem

Find the steel ratio if the allowable unit compressive stress in the concrete is 600 lb. per sq. in., the allowable tensile stress in the steel is 15,000 lb. per sq. in., and the ratio of the modulus of elasticity of the steel to that of the concrete is 15. *Ans.* $p = 0.0075$.

See Report of the Joint Committee on Concrete and Reinforced Concrete, *Proceedings of The American Society for Testing Materials*, 1913, page 278.

CHAPTER XIV

BENDING COMBINED WITH TENSION OR COMPRESSION

161. Transverse and Longitudinal Loading.—A beam may be subjected to an *axial* (*in the direction of its length*) tension or compression and transverse forces which produce bending moments. Since the bending moments cause axial stresses, these stresses may be added to the direct tensile or compressive stresses which are caused by the axial loads, in the same way as the stresses which are caused by bending around the two principal axes of a section were added in Art. 84.

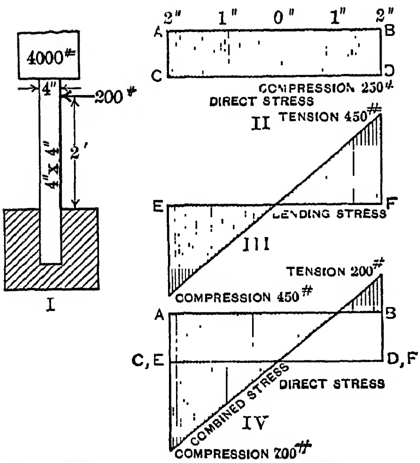


Fig. 227.—Post with compression and bending.

In Fig. 227, a 4-inch by 4-inch vertical post is fixed at the bottom and carries a load of 4,000 pounds on the top. The direct compressive stress is 250 pounds per square inch in all parts of the post between the load and the support. A horizontal force, perpendicular to one face, is applied 2 feet above the plane at which the post is fixed. The transverse force causes a tensile stress of 450 pounds per square inch at the fixed plane on the right side and an equal compressive stress on the left side. The combined stress is 700 pounds per square inch compression on the left side and 200 pounds per square inch tension on the right side. Figure 227, IV, shows the distribution of stress with compression represented by a negative ordinate and tension by a positive ordinate. Figure 227, II, shows the compression alone which is caused by the direct load of 4,000 pounds. Figure 227, III, shows the stress which is caused by bending. This stress is 450 pounds compression at the left, 450 pounds tension at the right, and zero at the middle. These stresses are combined in Fig.

227, IV. The line EF, which is the zero line for the bending stress, is placed on the line CD, which represents the compressive stress in Fig. 227, II. The combined unit compressive stress on the left side is $-250 - 450 = -700$ pounds per square inch. On the right side the combined stress is $-250 + 450 = 200$ pounds per square inch tension. The unit stress is zero at $\frac{8}{9}$ inch from the right face of the post.

The unit stress is

$$\frac{P}{A} + \frac{Mv}{I}, \quad S = \frac{P}{A} \pm \frac{M}{Z} \text{ for outer fibers} \qquad \text{Formula XXIX}$$

in which P is the total axial load and M is the bending moment from any source whatever. Since v is positive on one side of the neutral axis and negative on the other, the second term of Formula XXIX may be positive or negative, depending upon the position of the fiber in the section.

Problems

1. A timber post, 6 in. square, is fixed at the bottom with two parallel faces in the meridian (north and south vertical plane). The post is 4 ft. high from the fixed section. It carries a load of 10,800 lb. on the top and resists a horizontal force of 360 lb. west 6 in. from the top. Find the unit stress in the east and west outer fibers at the bottom. *Ans.* 720 lb./in.^{2} compression and 120 lb./in.^{2} tension.
2. In Problem 1, how far from the bottom is the stress zero in the extreme eastern fibers? *Ans.* 12 in.
3. In Problem 1, how far from the east surface at the bottom is the stress zero? Solve by Formula XXIX and check by interpolating the answers of Problem 1.
4. A rectangular pier is 6 ft. wide, 3 ft. thick, and 50 ft. high above the ground. If the material weighs 120 lb. per cu. ft. and is unable to resist tension, what wind pressure uniformly distributed over one entire face will overthrow it? Use all units in feet. *Ans.* 7.2 lb./ft.^{2}
5. Solve Problem 4 if the material has a tensile strength of 10 lb. per sq. in.
6. A 6-in. by 8-in. post, 61 in. long from the fixed end, leans parallel to the 8-in. faces until the top of each 6-in. face is 11 in. to the right of the vertical plane through the bottom. Find the stress in the outer fibers at the bottom when a load of 3,660 lb. is placed on the middle of the top. *Ans.* 75 + 629 lb./in.^{2}

162. Eccentric Loading.—Figure 228 represents a rigid bar G supported by three equal rubber bands (or springs) which are symmetrically placed and suspended from a rigid horizontal support. Each of the bands is stretched the same amount and the bar hangs in a horizontal position. Figure 228, II, shows the

same bar with a load P at the middle. The rubber bands are equally stretched and the bar remains in a horizontal position. If the load P be moved to the right, as in Fig. 228, III, the middle band receives the same elongation as in the preceding case, while the left band is elongated less and the right band more. If the load be moved still farther to the right, a place is reached where

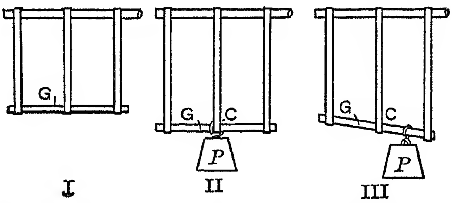


FIG. 228.—Eccentric loading of rubber bands.

the left end is elevated above the position which it occupied before the load was applied, so that no load whatever comes on the left band. If, instead of the rubber bands, helical springs were used, the spring on the side away from the load would come into compression when the load is shifted a considerable distance from the middle.

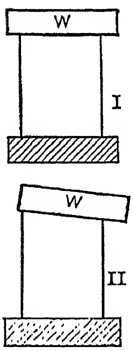


FIG. 229.—Compression with direct and with eccentric oading.

Instead of the rubber bands of Fig. 228, a continuous sheet of rubber or metal might be used. If such a sheet is fastened to a rigid body at the top, and to another rigid body at the bottom, and if a load is then applied to the lower rigid body considerably to one side of the center of the sheet, the sheet is elongated on that side and shortened or buckled on the other. A similar result obtains when a compressive load is applied to a body. Figure 229, I, shows a block of soft rubber with the load central, and Fig. 229, II, shows the effect of moving this load a little to one side.

Figure 228, III, shows that the effect of the eccentric load is a translation downward, of the same magnitude as that caused by the central load in II, together with a rotation about the bottom of the middle band as an axis. Equilibrium occurs when the moment of the load P about the lower end of the middle band is equal to the moment of the excess of tension in the right band plus the moment of the deficiency of tension in the left band. Suppose that the bands are 4 inches apart, and that a load of

1 pound at the middle stretches all three bands 0.4 inch. One pound will stretch a single band 1.2 inches. Now move the load of 1 pound 2 inches to the right of the middle. The moment with respect to the middle is 2 inch-pounds, which may be balanced by a load of 0.5 pound 4 inches from the axis. The tension in the right band is 0.25 pound more and the tension in the left band is 0.25 pound less than that in the middle band. The total tension is $\frac{1}{12}$ pound in the left band, $\frac{1}{3}$ pound in the middle band, and $\frac{7}{12}$ pound in the right band. These may be checked by moments around the left band, the right band, or about any other axis whatever. If the load is moved more than 3 inches from the middle, the tension in the left band becomes less than it was before this 1-pound load was applied.

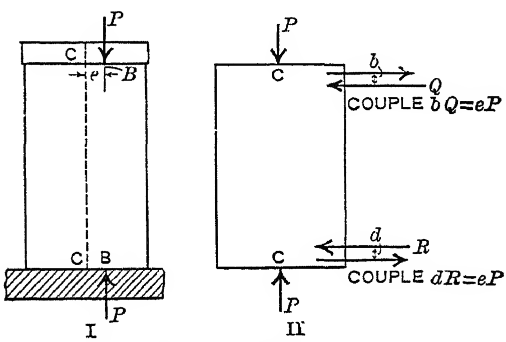


FIG. 230.—Block with eccentric loading.

Figure 230, I, shows a block which is subjected to a load P at a distance e from its axis. A force along any line may be replaced by an equal force along any parallel line and a couple the moment of which is equal to the product of the force by the distance between the lines.*

In Fig. 230, the force P at the top, at a distance e from the axis, may be replaced by a force P along the axis and a clockwise couple of moment $e \times P$. The reaction at the bottom may likewise be regarded as equivalent to a reaction P along the axis and a counterclockwise couple of moment $e \times P$. These equal couples are shown in Fig. 230, II. An eccentric load may be regarded as equivalent to a load along the axis combined with a bending moment, which is the product of the load multiplied by the eccentricity.

* See any textbook of mechanics.

Figure 231 shows examples of large eccentricity in which the existence of direct stress combined with bending stress is almost self-evident. The portion above G in Fig. 231, I, may be treated as a free body. A vertical resolution shows that the vertical reaction across the section at G is equal to the load P. The moment of the load P about an axis perpendicular to the plane of the paper through the center of the section at G is $e \times P$.

Figure 231, II, shows the deflection which is caused by compression. The deflection of the bar increases the eccentricity.

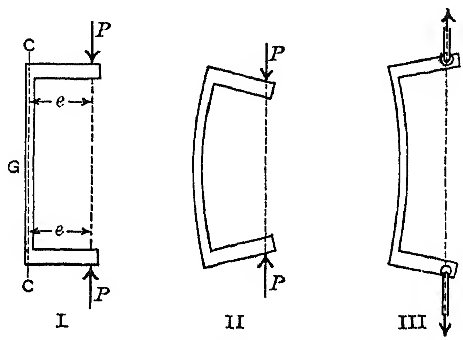


FIG. 231.—Large eccentricity.

Figure 231, III, shows the effect of tension. The deflection of the bar reduces the eccentricity.

Since an eccentric load is equivalent to a central load and a couple of moment eP, the unit stress at a distance v from the center of gravity of any section is

$$s = \frac{P}{A} \pm \frac{Mv}{I}. \qquad \text{Formula XXIX}$$

At the outer fibers,

$$S = \frac{P}{A} \pm \frac{Mc}{I} = \frac{P}{A} \pm \frac{M}{Z}.$$

In these formulas, $M = e \times P$ is the moment of the load about the axis through the center of gravity of the section, v is the distance from the center of gravity of the section to any given fiber, and c is the distance from the center of gravity of the section to the extreme outer fiber.

Formula XXIX for eccentric loading assumes that the section at which the load is applied remains plane. This condition is approximately fulfilled when the load is applied through a rela-

tively rigid plate or capstone to a body which is comparatively elastic. When the load is concentrated, the maximum stresses are usually greater than those given by the formula. If the block under stress is of some length, the sections near the middle are practically plane and the formula applies with greater accuracy.

The derivation of Formula XXIX assumes that E is constant. This assumption limits the formula to stresses below the elastic limit. Since the stress on one side is greater than on the other,

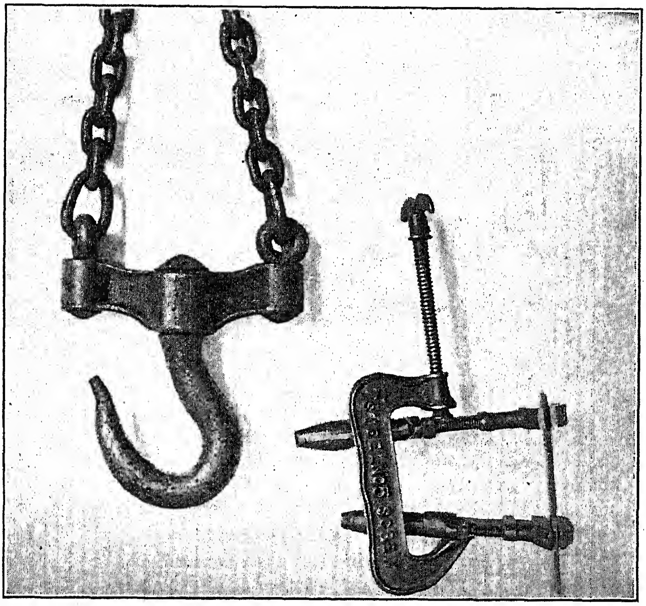


FIG. 232.—Eccentric loading.

it will reach the elastic limit before the other and cause a shifting of the neutral axis for the bending stresses. The error caused by this displacement of the neutral axis is greater than in a beam subjected to bending alone.

Subject to a correction for curvature (see Chapter XIX), Formula XXIX applies approximately to hooks.

Some examples of compression combined with bending are shown in Fig. 232. The forces P are applied to the wrenches by means of the screw clamp. The wrenches as cantilevers transmit the direct compression and the bending moments to the bar. The experiment may be performed with two wrenches and a metal or wooden bar with the forces applied to the wrenches

by the hands instead of the clamp. The bar will bend as in Fig. 231, II, when the forces are toward each other and as in Fig. 231, III, when the forces are away from each other.

The clamp of Fig. 232 is subjected to *tension* and bending. The eccentricity is the distance from the center of the screw to the center of gravity of any section. In a hook, the line of the load joins the shank with the point which is immediately below it when loaded. This is, of course, the point in the concave portion which is farthest from the shank. The eccentricity is the distance of the center of gravity from this load line.

Problems

1. A short timber block, 6 in. square, is subjected to a load of 18,000 lb. The line of the resultant force is ¾ in. from the axis in a plane midway between the parallel faces. Find the maximum and minimum compressive stress. *Ans.* 875 lb./in.^{2}; 125 lb./in.^{2}

2. Solve Problem 1 if the force is 1 in. from the axis. *Ans.* 1,000 lb./in.^{2}; 0.

3. Solve Problem 1 for an eccentricity of 2 in. How far is the point of zero stress from the nearest face? *Ans.* 1.5 in.

4. Solve Problem 1 if the eccentricity is 1 in. from the axis on the diagonal of the square section.
Ans. 1,207 lb./in.^{2} compression; 207 lb./in.^{2} compression.

5. A solid cylinder, 2 in. in diameter, is subjected to a pull of 15,708 lb. along a line 0.25 in. from the axis. Find the maximum and minimum unit stress. *Ans.* 10,000 lb./in.^{2}; 0.

6. In Problem 5, how far from the surface is the tensile stress 4,000 lb. per sq. in.? Solve by the equation and by interpolation. *Ans.* 0.8 in.

7. Solve Problems 5 and 6 for an eccentricity of 0.3 in.

8. Solve Problem 5 if the cylinder is hollow with inside diameter 1 in.
Ans. 12,000 lb./in.^{2} tension; 1,333 lb./in.^{2} tension

9. A short piece of 10-in. 15.3-lb. standard channel carries a load of 10,000 lb. The resultant load lies in the plane of symmetry 1 in. from the back of the web. Find the maximum and minimum stress.
Ans. 5,237 lb./in.^{2} compression at tip of flanges; 1,235 lb./in.^{2} compression at back of web.

10. A solid wall has the resultant load 2 ft. from the front edge. The load is 12 tons per running foot. Assuming that the load is so distributed that the top remains plane, find the unit stress in tons per square foot at the front edge if the breadth of the wall is 4 ft., 6 ft., 8 ft., 10 ft.
Ans. 3 tons/ft.^{2}, 4 tons/ft.^{2}, 3.75 tons/ft.^{2}, 3.36 tons/ft.^{2} compression.

11. A solid post in the form of an isosceles triangle, 4 in. wide at the base and 6 in. high, carries a load of 4,800 lb. on the line of symmetry 3 in. from base. Find the maximum and minimum stress.
Ans. 1,200 lb./in.^{2} at vertex; 0 at base.

12. In a hook of circular section the distance from the center of gravity of the section to the line of the load is 3 in. The load is 1,600 lb. and

the diameter of the section is 2 in. Using Formula XXIX, find the *approximate* value of the maximum tensile and compressive stress.

Ans. 6,621 lb./in.^{2} tension; 5,602 lb./in.^{2} compression.

163. Maximum Eccentricity without Reversing Stress.—A brick pier laid in lime mortar has no tensile strength, and the tensile strength of masonry laid in cement mortar is uncertain. For this reason, the load on a masonry pier or wall should be so placed that the stress over the entire section shall be compressive.

Example

A solid masonry wall has a width b. How far may the resultant load be placed from the middle without having tensile stress on the opposite face?

For a length l of the wall, the section modulus is

$$Z = \frac{l \times b^2}{6}; \quad A = l \times b;$$

$$\frac{P}{A} - \frac{Pe}{Z} = 0; \quad \frac{P}{l \times b} = \frac{6Pe}{l \times b^2};$$

$$e = \frac{b}{6}.$$

The resultant load must not be more than one-sixth the breadth from the middle.

Architects and engineers have the rule that *the resultant load on a rectangular pier or wall shall not fall outside the middle third,* unless the material has tensile strength. When the load has an eccentricity of one-sixth the width, the stress in the outer fibers on the side opposite the eccentricity is zero, while the stress on the outer fibers on the same side is twice the average stress. It is desirable, therefore, to have eccentricity as small as possible, and to have rigid capstones and foundations which spread the load evenly over the section.

Problems

1. What eccentricity of the load on a short solid cylinder of radius a will make the stress zero on one side?

Ans. $e = \frac{a}{4}$.

2. A hollow cylinder of inside radius b and outside radius a is so loaded as to give zero stress on one side. What is the eccentricity?

Ans. $e = \frac{a^2 + b^2}{4a}$.

3. A rectangular pier is 18 in. wide from east to west and 60 in. long. It carries a slightly eccentric, uniformly distributed load of 54,000 lb. which makes the stress on the west face 40 lb. per sq. in. Find the eccentricity and the stress on the east face. *Ans.* $e = 0.6$ in.

4. A hollow rectangular pier is 24 in. wide from east to west and 60 in. long. The walls are 8 in. thick. It carries a load of 100,000 lb., which is applied at the top by means of a rigid capstone. The compressive stress in the east face is 40 lb. per sq. in. Find the eccentricity of the load.

Ans. $e = 2.91$ in.

The answer to Problem 1 shows that the stress is reversed in a short solid cylinder in the surface opposite the eccentricity when the resultant load is more than one-eighth the diameter from the axis. For a hollow cylinder, the eccentricity may be much greater than one-fourth the radius without reversing stress. For a very thin hollow cylinder, the possible eccentricity without reversal of stress approaches one-half the radius.

With uncertain eccentricity of the loading, it is often advisable to build brick piers hollow. The maximum compressive stress, of course, is greater, but the possibility of dangerous tensile stresses is greatly reduced. The center of a pier when the load is eccentric acts as a fulcrum about which the section may turn.

Problem

5. Find the maximum and minimum stresses in the pier of Problem 4 if the eccentricity is 2.91 in. and the pier is solid.

For any section, the maximum eccentricity without reversing stress may be computed by Formula XXIX. Since the fibers under zero stress are on the side of the center of gravity opposite the load, the negative sign is used in the formula.

$$S = 0 = \frac{P}{A} - \frac{M\,c}{I}; \tag{1}$$

$$\frac{P}{A} = \frac{M\,c}{I} = \frac{P\,e\,c}{A\,r^2}, \tag{2}$$

in which r is the radius of gyration of the section with respect to the axis through the center of gravity.

$$e = \frac{r^2}{c}. \tag{3}$$

Problems

(*Solve* 6, 7, *and* 8 *without writing.*)

6. What is the radius of gyration of a circle with respect to its diameter? What is the distance to the outer fibers? Find maximum eccentricity without reversing stress.
7. What is the square of the radius of gyration of a rectangle of sides b and d with respect to the line of symmetry parallel to d? What is the distance to the outer fiber? What is the maximum eccentricity without reversing stress?

8. What is the square of the radius of gyration of an isosceles triangle of height h and base b with respect to a line through the center of gravity parallel to the base? What is the maximum eccentricity without reversing stress at the vertex? What is the maximum eccentricity without reversing stress at the base?

Ans. $\frac{h}{12}$ toward the base; $\frac{h}{6}$ toward the vertex.

9. What is the maximum eccentricity without reversing stress for a square section when the resultant load falls on a diagonal?

164. Resultant Load Not on Principal Axis.—In all the problems of the preceding articles, the resultant load fell on one principal axis, and rotation took place about the other principal axis. When the section is a circle, a square, an equilateral triangle, or any other regular polygon, the moment of inertia is the same for every axis which lies in the plane of the section and passes through its center of gravity, and every such axis is a principal axis. For other sections, when the load does not fall on a principal axis, the axis of rotation is not the line $O\,E$ of Fig. 233 but is some other line, such as $O\,G$, which lies between $O\,E$ and the axis of minimum moment of inertia.

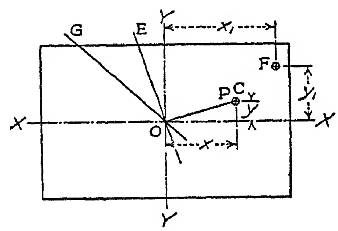


FIG. 233.—Resultant load off principal axis.

To find the bending stress caused by an eccentric load, the eccentricity is resolved into two components parallel to the two principal axes of inertia, and the stress at any point is calculated separately for each component. In Fig. 233, the load perpendicular to the plane of the paper is applied at the point C. The eccentricity is $O\,C$. The components of the eccentricity are x and y. The moment of the load P about the X axis is $P\,y$. At any point F, whose coördinates are x_1 and y_1 the bending stress caused by the moment $P\,y$ is

$$S = \frac{M\,v}{I} = \frac{P\,y\,y_1}{I_x}.$$

The moment about the Y axis is $P\,x$, and the stress at F caused by this moment is $\frac{P\,x\,x_1}{I_y}$.

The combined direct stress and bendings tress at any point (x_1, y_1) when the load is applied at (x, y) is

$$s = \frac{P}{A} + \frac{P\,x\,x_1}{I_y} + \frac{P\,y\,y_1}{I_x} \tag{1}$$

If x and x_1 have the same sign, the second term of the second member of Equation (1) is positive; and if y and y_1 have the same sign, the third term is positive.

Example I

A rectangular block is 12 in. long, measured from east to west, and 10 in. wide, from south to north. It is subjected to a load of 3,600 lb., which is applied 2 in. from the east edge and 2 in. from the north edge. Find the unit stress at each corner.

If the east line through the center is taken as the X axis and the north line is taken as the Y axis, $I_x = 1{,}000$ and $I_y = 1{,}440$. The moment about the X axis is 10,800 in.-lb. This moment causes a bending stress of 54 lb. per sq. in. at the north and south edges. The moment about the Y axis is 14,400 in.-lb. This moment causes a bending stress of 60 lb. per sq. in. at the east and west edges. The direct compression is 30 lb. per sq. in. The total stresses at the corners are 144 lb. compression at the northeast corner, 24 lb. compression at the northwest corner, 84 lb. tension at the southwest corner, 36 lb. compression at the southeast corner.

Problems

1. Solve the example above if the load is placed 3 in. from the east edge and 3 in. from the north edge.
   Ans. 111 lb. compression at the northeast corner.
2. In Problem 1, what is the average of the stress at all four corners?
3. A 10-in. by 6-in. post stands vertical with its 10-in. faces running north and south. A load of 2,400 lb. is placed on the top 2 in. from the east face and 2 in. from the north face. A horizontal force of 75 lb. toward the west is applied to the east face 24 in. above the bottom, and a horizontal force of 80 lb. toward the south is applied to the north face 20 in. above the bottom. Find the unit stress at each corner at the bottom.
   Ans. At the northeast corner, compression $= 40 + 72 + 40 - 16 - 30 = 106$ lb. in.^{2}
4. Find the average of the stress at all four corners.
5. From the results for the example above, find the location of the points on the south and west edges at which the stress is zero.
   Ans. 8.4 in. from the southwest corner on the south edge, and $7\frac{7}{9}$ in. on the west edge.

The maximum eccentricity of the load without reversing stress at any given point (x_1, y_1) may be found by equating the second member of Equation (1) to zero.

$$\frac{P}{A} + \frac{P\,x\,x_1}{I_y} + \frac{P\,y\,y_1}{I_x} = 0; \tag{2}$$

$$\frac{P}{A}\left(1 + \frac{x\,x_1}{r_y^2} + \frac{y\,y_1}{r_x^2}\right) = 0; \tag{3}$$

$$1 + \frac{x\,x_1}{r_y^2} + \frac{y\,y_1}{r_x^2} = 0, \tag{4}$$

in which r_x and r_y are the radii of gyration with respect to the X axis and the Y axis, respectively.

Example II

A short block of rectangular section is b wide and d thick. Find the eccentricity of the load without reversing stress.

To have zero stress at E, at the middle of the right side of Fig. 234,

$$x_1 = \frac{b}{2}; \quad y_1 = 0; \quad r_y^2 = \frac{b^2}{12}.$$

Equation (4) becomes

$$1 + \frac{\frac{b\,x}{2}}{\frac{b^2}{12}} = 0, \quad x = -\frac{b}{6}. \tag{5}$$

For zero stress at F, at the middle of a side of length b,

$$y = -\frac{d}{2}.$$

To find the equation of a straight line upon which a load will give zero stress at the corner C of Fig. 234,

$$x_1 = \frac{b}{2}, \quad y_1 = \frac{d}{2}.$$

$$1 + \frac{\frac{b\,x}{2}}{\frac{b^2}{12}} + \frac{\frac{d\,y}{2}}{\frac{d^2}{12}} = 0;$$

$$-\frac{x}{\frac{b}{6}} - \frac{y}{\frac{d}{6}} = 1. \tag{6}$$

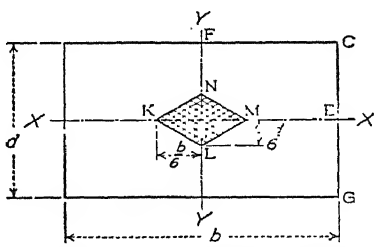


FIG. 234.—Area in which load does not reverse stress.

Equation (6) may represent a straight line with x intercept $-\frac{b}{6}$ and y intercept $-\frac{d}{2}$ ($K\,L$ of Fig. 234). For the lower right corner G of Fig. 234, $x_1 = \frac{b}{2}$,

$$y_1 = -\frac{d}{2},$$

and

$$-\frac{x}{\frac{b}{6}} + \frac{y}{\frac{d}{6}} = 1. \tag{7}$$

The line $K\,N$ of Fig. 234 is a part of the line represented by Eq. (7). A load at any point on this line gives zero stress at G; but if the load were below K, the line of zero stress would fall inside C. The parallelogram $L\,K\,N\,M$ encloses an area anywhere in which a load may be applied without causing reversed stress at the corners or at other points of the rectangular section. This area is sometimes called the *kernel* of the section.

Example III

Find the kernel of an equilateral triangle (Fig. 235).

Since this is a regular polygon, the moment of inertia is the same for any axis through the center of gravity. In Fig. 235, the X axis is taken parallel to the base.

$$I = \frac{b\,h^3}{36}, \quad h = \frac{b\sqrt{3}}{2}, \quad r^2 = \frac{h^2}{18} = \frac{b^2}{24}.$$

To find the equation of the line upon which a load will give zero stress at the vertex C, where $x_1 = 0$, and $y_1 = \frac{2\,h}{3}$.

$$\frac{2\,h\,y}{3} \times \frac{18}{h^2} + 1 = 0; \quad y = -\frac{h}{12}. \tag{8}$$

The load is on the line $F\,G$, which is parallel to the base at a distance of one-twelfth the height below the center of gravity. It is evident that the

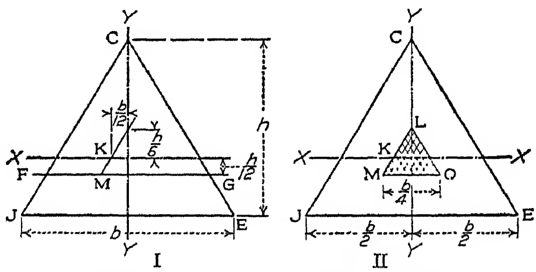


FIG. 235.—Kernel for triangular section.

lines for the other two vertices will combine with this one to give an equilateral triangle. However, it is worth while to derive the equation of the line for the vertex E by means of the coördinates of the figure.

$$x_1 = \frac{b}{2}, \quad y_1 = -\frac{h}{3}.$$

$$\frac{\frac{b\,x}{2}}{\frac{b^2}{24}} - \frac{\frac{h\,y}{3}}{\frac{h^2}{18}} + 1 = 0; \quad -\frac{x}{\frac{b}{12}} + \frac{y}{\frac{h}{6}} = 1. \tag{9}$$

Equation (9) represents a straight line which intercepts the X axis at $-\frac{b}{12}$ and the Y axis at $\frac{h}{6}$. Since its slope is $\frac{h}{\frac{b}{2}}$, this line is parallel to the side JC of the section. The kernel is an equilateral triangle (Fig. 235, II). Its height is $\frac{h}{12} + \frac{h}{6} = \frac{h}{4}$; and its base is $\frac{b}{4}$.

CHAPTER XV

COLUMNS

165. Definition.—In the discussion of eccentric loading in the preceding chapter, no account was taken of the deflection of the body, and of the effect of this deflection upon the eccentricity and the bending stress. Eccentric tension produces a deflection which reduces the eccentricity, as is shown in Fig. 231, III. Eccentric compression, on the other hand, produces a deflection which increases the eccentricity. A yardstick may be placed with one end on the floor and pushed down by the hand at the other end until the middle is bent several inches from the straight line. The original eccentricity, of possibly 0.01 inch, is increased several hundred times, and the bending stress may be sufficient to cause rupture at the middle. If the stick is placed with one end on a platform scale, as shown in Fig. 236, it is found that the load which causes a deflection of 2 inches is little, if any, greater than the load which causes a deflection of 1 inch. While the resisting moment has been doubled, the external moment arm has likewise been nearly doubled. The applied force, therefore, changes very little.

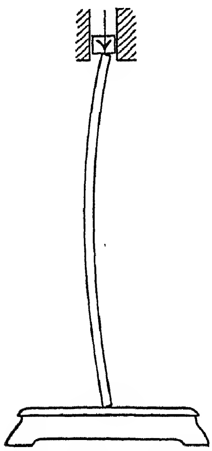


FIG. 236.—A long column.

A compression member whose length is several times as great as its smallest transverse dimension is called a *column* or *strut*. Long vertical compression members of buildings and the posts of bridges are usually called columns. The compression members of roof trusses and the vertical compression members of airplanes are called struts. The top chord of a bridge usually acts as a column. The connecting rod of an engine is a column during the forward stroke.

When a column is vertical, the only bending moment is that which is due to the eccentricity of the load and to the deflection. When a column is horizontal or inclined, its own weight applied

as in a beam becomes an appreciable factor. The rafters which support a roof act as columns and inclined beams.

Example

A cold-rolled steel rod, 1.250 in. in diameter and 60 in. long, was loaded in compression with cylindrical ends, which were eccentric 0.0202 in. When a load of 6,413 lb. was applied, the rod deflected 0.0451 in. at the middle; and when a load of 8,116 lb. was applied, the rod deflected 0.1136 in. at the middle. Find the maximum stress in the rod at the middle.

The maximum stress was at the concave side of the beam. The total eccentricity for the first load was 0.0202 + 0.0451 = 0.0653,

$$S = \frac{6{,}413}{1.2272} + \frac{6{,}413 \times 0.0653}{0.19175} = 5{,}226 + 2{,}184 = 7{,}410 \text{ lb. per sq. in.}$$

For the second load, S = 6,614 + 5,663 = 12,277 pounds per square inch. From this example, it is evident that the maximum unit stress rises much faster than the load, on account of the rapid increase of eccentricity at the middle of the length of the rod.

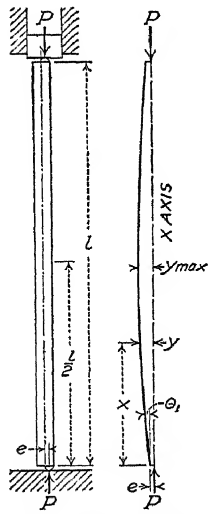


FIG. 237.—Eccentrically loaded column.

166. Column Theory.—Figure 237 shows a vertical column with ends free to turn about a horizontal axis perpendicular to the plane of the paper. The left figure represents the column as it appears, with no apparent deflection. The right figure represents the central axis of the column with deflections greatly magnified. In order that the beam formulas may apply, the X axis is vertical (parallel to the length of the column) and the Y axis is horizontal and positive toward the left. In this figure, the X axis is on the line of the applied forces, and the origin is at the bottom of the column. The eccentricity, which is the distance of the centers of gravity at the ends from the line of the load, is regarded as positive. In a section at a distance x from the origin, the moment arm is y and the moment is $P\,y$. This moment turns the lower end counterclockwise and, therefore, is negative.

From Equation (7) of Art. 78,

$$M = E\,I\frac{d\theta}{dl}; \quad E\,I\frac{d\theta}{dl} = -P\,y; \tag{1}$$

$$E\,I\frac{d\theta}{dl}\,dy = -\,P\,y\,dy. \tag{2}$$

Since $\dfrac{dy}{dl} = \sin\theta = \theta$ for small angles,*

$$E\,I\,\theta\,d\theta = -P\,y\,dy; \tag{3}$$

$$\frac{E\,I\,\theta^2}{2} = -\frac{P\,y^2}{2} + C_1. \tag{4}$$

Since $\theta = 0$ when $x = \dfrac{l}{2}$ and $y = y_{\max}$, $C_1 = \dfrac{P\,y^2_{\max}}{2}$;

$$\theta^2 = \frac{P}{E\,I}(y^2_{\max} - y^2). \tag{5}$$

If $\dfrac{dy}{dx}$ is substituted for θ,* Equation (5) becomes

$$\frac{dy}{\sqrt{y^2_{\max} - y^2}} = \sqrt{\frac{P}{E\,I}}\,dx. \tag{6}$$

$$\sin^{-1}\frac{y}{y_{\max}} = \sqrt{\frac{P}{E\,I}}\,x + C_2. \tag{7}$$

When $x = \dfrac{l}{2}$; $y = y_{\max}$;

$$\sin^{-1} 1 = \frac{\pi}{2} = \sqrt{\frac{P}{E\,I}}\,\frac{l}{2} + C_2; \tag{8}$$

$$\sin^{-1}\frac{y}{y_{\max}} = \frac{\pi}{2} - \sqrt{\frac{P}{E\,I}}\left(\frac{l}{2} - x\right). \tag{9}$$

$$y = y_{\max}\sin\left[\frac{\pi}{2} - \sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right)\right] = y_{\max}\cos\sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right). \tag{10}$$

When $x = 0$, $y = e$ and

$$e = y_{\max}\cos\sqrt{\frac{P}{E\,I}}\,\frac{l}{2}; \tag{11}$$

$$y_{\max} = e\sec\sqrt{\frac{P}{E\,I}}\,\frac{l}{2} = e\sec\sqrt{\frac{P\,l^2}{4\,E\,I}}. \qquad \text{Formula XXX}\dagger$$

$$y = e\sec\sqrt{\frac{P}{E\,I}}\,\frac{l}{2}\cos\sqrt{\frac{P}{E\,I}}\left(\frac{l}{2} - x\right). \tag{12}$$

* These approximations are equivalent to those used in deriving Formula XVII of Art. 87 from Equation (7) of Art. 78.

† $\sqrt{\dfrac{P\,l^2}{4\,E\,I}}$ is an angle in radians, the secant (or cosine) of which is a numerical quantity involved in the solution of these column problems. *It does not refer to any angle on the figure.*

Example

A 2-in. by 2-in. timber strut, 5 ft. long, is tested as a column with round ends. What is the deflection at the middle under a load of 3,200 lb., if $E = 1{,}500{,}000$ lb. per sq. in. and the eccentricity is 0.100 inch?

$$\frac{Pl^2}{4EI} = \frac{3{,}200 \times 60 \times 60 \times 3}{4 \times 4 \times 1{,}500{,}000} = 1.44,$$

$$\sqrt{\frac{Pl^2}{4EI}} = 1.2.$$

$y_{\max} = 0.100 \sec 1.2$ radians $= 0.100 \sec 68° \, 45' = 0.276$ in. Deflection $= y_{\max} - e = 0.276 - 0.100 = 0.176$ in.

Problems

1. In the example above what is the bending moment at the middle, and what is the maximum unit stress?
   Ans. $M = 883.2$ in.-lb.; $S = 800 + 662 = 1{,}462$ lb./in.^{2}
2. In the example above, if the load is increased to 3,872 lb., what is the deflection? *Ans.* $y_{\max} = 0.403$ in. $S = 968 + 1{,}170 = 2{,}138$ lb./in.^{2}
3. A timber beam 6 in. square and 10 ft. long is used as a strut with round ends and eccentricity of 0.5 in. Find the deflection at the middle under a load of 18,000 lb. if $E = 1{,}200{,}000$ lb. per sq. in. Find the maximum and minimum compressive stress at the middle if the eccentricity is parallel to the sides.
   Ans. $D = 0.5(\sec 40° \, 31' - 1) = 0.1576$ in.; max. $S = 829$, min. $S = 171$ lb./in.^{2}
4. Solve Problem 3 for the stress if the eccentricity is parallel to a diagonal of the section. *Ans.* $S = 965$ lb./in.^{2} and 35 lb./in.^{2}
5. Solve Problem 3 for a load of 23,040 lb. *Ans.* $D = 0.2176$ in.; $S = ?$
6. A 15-in. 33.9-lb. standard channel, 80 in. long, is used as a column to carry a load of 99,000 lb., which is applied through cylindrical bearings in a plane midway between the surfaces of the web. Find the deflection at the middle and the maximum and minimum stress if $E = 29{,}000{,}000$ lb. per sq. in.
   Ans. $D = 0.2713$ in. $S = 18{,}215$ lb./in.^{2} compression and 16,646 lb./in.^{2} tension, using $Z = 3.2$.
7. Solve Problem 6 if the load is 1 in. from the back of the web.
8. A 2-in. round steel rod, 10 ft. long, is used as a column with ends free to turn. Find the deflection at the middle and the maximum fiber stress on the concave side when the load is 8,000 lb. and the eccentricity is 0.1 in., if E is 30,000,000 lb. per sq. in.
   Ans. $y_{\max} = 0.1 \sec 63° \, 21' = 0.2230$ in.; max. $S_c = 4{,}814$ lb./in.^{2}
9. A column with ends free to turn is made of a 2-in. round steel rod for which E is 30,000,000 lb. per sq. in. The length is 5 ft. Find the deflection at the middle and the maximum unit stress for loads of 20,000 lb., 30,000 lb., 50,000 lb., 60,000 lb. and 70,000 lb. for eccentricities of 0.01 in. and 0.1 in.

Ans. Load............	20,000	30,000	50,000	60,000	70,000
Unit stress:					
For e = 0.01....	6,763	10,340	19,295	32,483	Infinite
For e = 0.1.....	10,337	17,500	49,800	154,000	Infinite

167. Euler's Formula.—The answers to Problem 8 of the preceding article show that the stress becomes infinite for a load of 70,000 with an eccentricity of 0.01 inch or with an eccentricity ten times as great. Since the secant of 90 degrees is infinite, Formula XXX shows that any column will deflect without limit and finally fail if the total load and the dimensions are such that

$$\sqrt{\frac{P\,l^2}{4\,E\,I}} = \frac{\pi}{2}. \tag{1}$$

From Equation (1),

$$P = \frac{\pi^2 E\,I}{l^2}, \qquad \text{Formula XXXI}$$

which is the common form of Euler's formula. The unit load is

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}, \qquad \text{Formula XXXII}$$

in which r is the radius of gyration of the section with respect to the axis about which it bends.

The ratio $\frac{l}{r}$ is called the *slenderness ratio* of the column. In Formula XXXII, the slenderness ratio and the *unit* load $\frac{P}{A}$ are the two variables. The formula is of the form

$$y = \frac{a}{x^2},$$

in which a replaces $\pi^2 E$, $y = \frac{P}{A}$, and $x = \frac{l}{r}$. The unit load is a constant divided by the square of the slenderness ratio.

Euler's formula contains the moment of inertia but does not include the distance to the outer fibers. When the eccentricity is negligible, and the slenderness ratio is relatively large, the ultimate load does not depend upon the form of the column, except in so far as the form changes the moment of inertia.

Figure 238 is Euler's curve for a modulus of elasticity of 29,400,000. As a mathematical curve, it is infinite in both directions. As an engineering curve, it must not be used above the point at which the unit load is the elastic limit of the material. It will be shown later that it is best not to use Euler's formula for

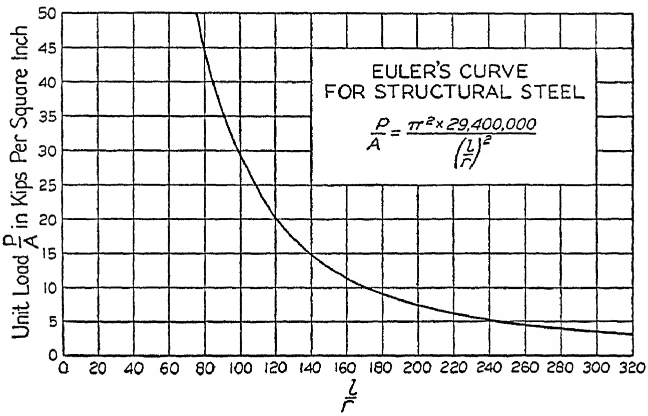


FIG. 238.—Euler's curve for a round-end column.

values of the unit load above one-third the elastic limit of the material.

Example

Find the total load with a factor of safety of 2 for a round steel rod 2 in. in diameter if $E = 29{,}400{,}000$ lb. per sq. in. and the elastic limit is 30,000 lb. per sq. in., for lengths of 40 in., 50 in., 60 in., and 80 in.

As these lengths are multiples of 10 in., begin with this length and find the others by multiplying by the square of the ratio of 10 to the required length.

Since the moment of inertia of a circle with respect to a diameter is $\frac{\pi a^4}{4}$ and the area is πa^2, the square of the radius of gyration is $\frac{a^2}{4}$, in which a is the radius. The radius of gyration of a 2-in. circle is ½ in. For $l = 10$ in.

$$\frac{P}{A} = \frac{9.87 \times 29{,}400{,}000}{400} = 725{,}445, \text{ pounds per square inch,}$$

which is many times the ultimate strength of the material

Length, inches	$\frac{l}{r}$	Unit load, pounds per square inch	Total safe load, pounds
40	80	45,340	
50	100	29,017	45,580
60	120	?	31,652
80	160	11,335	17,805

For the 40-in. length $\frac{P}{A}$, as calculated by the formula, is 45,340 lb. per sq. in., which is above the elastic limit of the material. For the 50-in. length the unit load is nearly the elastic limit. The result may be used with the factor of safety of 2 if it is certain that the eccentricity is very small.

Problems

1. In the example find the unit load for values of the slenderness ratio from 100 to 300 at intervals of 20. Plot.
2. When the strut used in the example of Art. 165 was adjusted to small eccentricity (less than 0.0003 in.), the deflection at the middle caused by a load of 9,400 lb. was 0.0061 in. When the load was 9,800 lb., the deflection was 0.0104 in. When the load reached 10,020 lb., the deflection was 0.1064 in. When the machine was run still further, the deflection increased with no increase of load. Calculate $E\,I$ for this cold-rolled steel from Euler's critical load. Calculate E.

   Ans. $E\,I = 3{,}654{,}700$; $E = 30{,}480{,}000$ lb./in.^{2}
3. A spruce strut, tested at the Bureau of Standards, was 1.75 in. square and 6 ft. 3.75 in. long. The ultimate load was 3,020 lb. Find E by Euler's formula. What was the value of $\frac{l}{r}$?

   Ans. $E = 2{,}248{,}000$; $\frac{l}{r} = 150$.
4. Compression readings for the strut of Problem 3 were taken from a gage length of 30 in. When the load changed from 305 lb. to 2,593 lb., the compression in the gage length was 0.0098. What was the unit stress and what was the unit deformation? Calculate E from these readings. When the load changed from 305 lb. to 2,745 lb., the compression was 0.0107 in. Calculate E. Compare with the results of Problem 3.
5. In what respect does the calculation of E in Problem 3 differ from the calculation in Problem 4? For which of the methods is it necessary to have the zero load correct?
6. A yardstick with ends rounded was placed vertical, with the lower end on a platform scale and the upper end loaded. When the load was 5 lb., the deflection at the middle was 0.03 in. When the load was 6 lb., the deflection was 0.20 in. When the load was 6.40 lb., the deflection was 1.00 in. The load decreased to 6.28 lb. with a deflection of 2.50 in. Calculate $E\,I$ from the last two readings. *Ans.* 851; 825.
7. The yardstick of Problem 6, supported as a beam at points 34 in. apart, was deflected $^{31}\!/_{32}$ in. at the middle by a load of 1 lb. at the middle. Find $E\,I$ and compare with Problem 6.
8. The yardstick of Problem 6 was 1.08 in. wide and 0.18 in. thick. Find E and I. Find the maximum stress as a beam in Problem 7. Find the maximum stress at the middle for the 6.28-lb. load of Problem 6, assuming that the eccentricity was negligible.

9. Find the maximum load per square inch area for a 5-in. by 4-in. by ½-in. angle section 10 ft. long if $E = 29{,}000{,}000$. Get the minimum radius of gyration from a similar section.

Ans. $\frac{P}{A} = 13{,}666$ lb./in.²

10. By Euler's formula calculate the unit load in terms of the slenderness ratio for structural steel, for values of $\frac{l}{r}$ from 80 to 300 at intervals of 20. Use $E = 29{,}400{,}000$. Compare with Fig. 242.

168. Classification of Columns.—Columns may be divided, according to the nature of the ends, into the following classes:

I. Both ends free to turn about parallel, horizontal axes, but not free to move laterally (Figs. 236, 237, and 239, I).

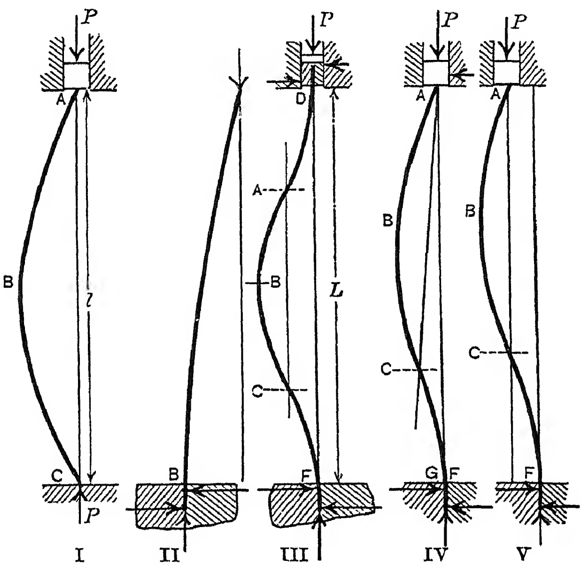


FIG. 239.—Types of ideal columns.

II. One end fixed and one end free to turn and free to move laterally (Fig. 239, II).

III. Both ends fixed so that the tangents at the ends do not change (Fig. 239, III).

IV. One end fixed and the other free to turn about one or more horizontal axes, but not free to move laterally (Fig. 239, IV).

Only Class I has been considered in the preceding articles.

If L is the entire length of the free portion of a column, and l is the length of the cosine (sine curve if the origin is shifted) curve of Fig. 237, $L = l$ for Class I.

For Class II, the entire length of the column, from the fixed end to the top, corresponds to one-half of the cosine curve; hence $l = 2L$ in Formulas, XXIX, XXX, and XXXI.

For Class III, the slope is zero at the ends and at the middle. The middle half $A\ B\ C$ corresponds to the cosine curve of Class I. This portion is represented by l in the column formulas. If L is the entire length $D\,F$, then $L = 2\,l$. A column with both ends *rigidly fixed* will carry as great a load as a column of half its length with ends free to turn. The points A and C of Fig. 239, III, are points of inflection (or contraflexure) at which the moment and the curvature change signs. The portion $A\ D$ is equal to one-half of the cosine curve $A\ B\ C$. If the portion $A\ D$ is rotated 180 degrees about an axis through A perpendicular to the plane of the paper, the point D will fall on B and the curves $A\ B$ and $A\ D$ will coincide. The moment is zero at A and C.

The column of Class IV is fixed at one end and free to turn at the other, but not free to move laterally. The point of inflection is at C, of Fig. 239, IV. Since the column is free to turn, there is no moment at the top A. As the load is applied, a very small eccentricity at the top, inequality of the material, or slight crookedness causes bending (just as these causes produce bending in Classes I, II, and III). Since the moment at the point of inflection C is zero and there are no transverse forces between A and C, the resultant reaction at A is directed from A toward C. The force A, therefore, must have a horizontal component. The resultant of the horizontal forces at the fixed end is equal to and opposite the horizontal component at A. The portion $A\ B\ C$ of Fig. 239, IV, forms a cosine curve with the X axis parallel to $C\ A$. The lower portion $C\,F$ forms part of the cosine curve as far as the plane of the body which holds it. Below that plane it is straight. If this portion continued to curve until it became parallel to $A\ C$, it would form a complete half of the cosine curve and its length would be equal to $A\ B$ or $B\ C$. Since the portion is vertical at the fixed end, its length is less than one-half of $A\ B$, and less than one-third of the entire length of the column. The solution of the differential equation shows that $A\ C$ is nearly $0.7\,L$. For practical purposes, $l = 0.7\,L$ and $l^2 = 0.5\,L^2$, nearly.

It is sometimes stated that l is equal to two-thirds L in a column which is fixed at one end and free to turn at the other. This can be true only under the impractical conditions of Fig. 239,

V. In this figure, the top of the column is displaced laterally toward the left. If this displacement is such that the point B is as far from the line $A\ C$ as the top A is from the vertical line through the fixed end F, then the line $A\ C$ from the end to the point of contraflexure becomes vertical. In this position, $A\ C$ is two-thirds of the total length L; there is no horizontal component of the force at the top; and the vertical force is greater than in Fig. 239, IV. The position is unstable. Under a slight vibration the column will deflect to the right of the vertical line through F at the lower end, and the ultimate load will be greatly reduced.

Problems

1. A thin yardstick is clamped vertically in a vise at 4 in. from the lower end. When a load of 2 lb. is placed on the top, the stick deflects with gradually *increasing* speed and, unless supported or the load removed, finally breaks. Find $E\ I$. *Ans.* $E\ I = 830$.
2. A yardstick, with ends rounded, was supported and loaded as in Fig. 239, I, and was deflected a large amount by a load of 6.1 lb. on the top. Find $E\ I$ by Euler's formula.
3. The yardstick of Problem 2 was clamped 4 in. from one end and the load was applied as in Fig. 165, IV. A deflection of 1.5 in. was caused by a load of 15.42 lb. Find $E\ I$ by Euler's formula.
4. The load in Problem 3 was displaced 1 in. south of the vertical line through the bottom. The vertical component of the load when the maximum deflection was 2 in. south was 17.12 lb. Find $E\ I$ from this experiment.

169. Experimental Check of Theory.—Euler's formula and the secant equation, Formula XXIX of Art. 166, can be tested best on columns which are free to turn and not free to move laterally at the ends (Class I of Fig. 239). A slender column of Class II, rigidly clamped at the bottom, may be used for demonstration with an actual load on the top. For accurate results the center of gravity of this load must coincide with the center of the upper end of the column. For stronger struts, the inconvenience of applying actual loads of several tons and the difficulty of fixing the lower end make this form impracticable. Difficulty of fixing the ends and uncertainty as to the eccentricity rule out Classes III and IV.

Figure 240 shows the apparatus used by the writer for testing struts of Class I.* Half cylinders, hardened and ground, are

* *Bull.* No. 25, Ohio State University Engineering Experiment Station, pp. 11 and 12.

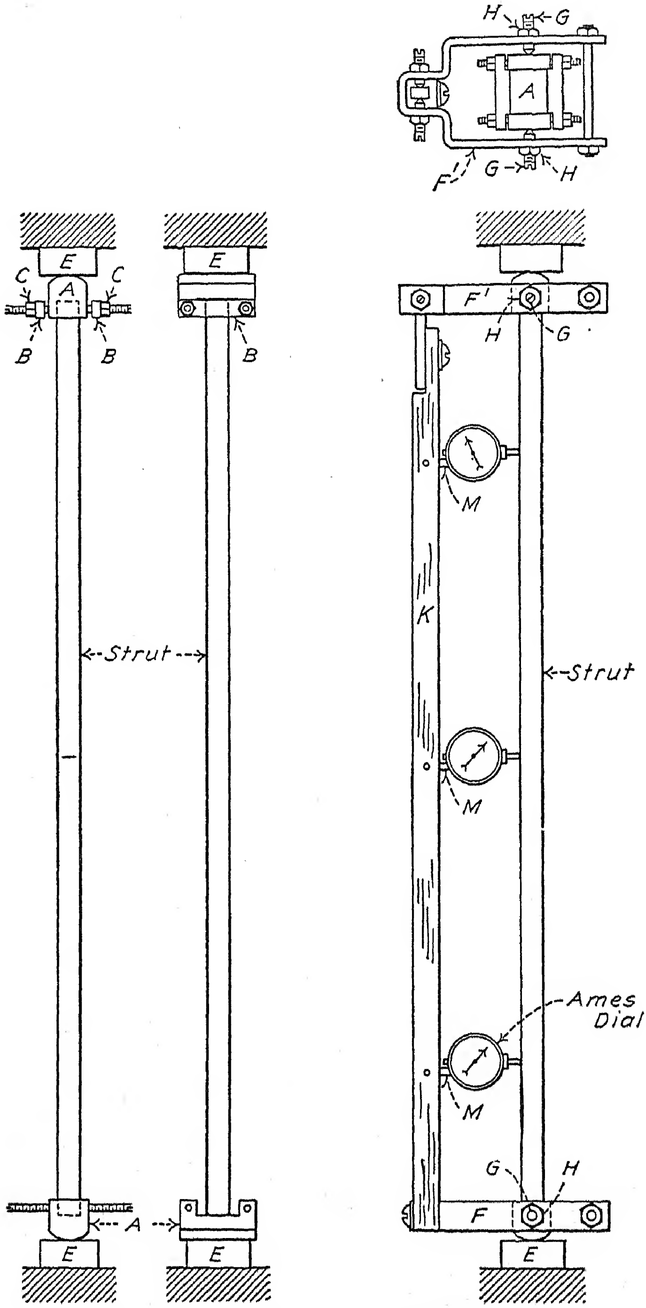


FIG. 240.—Deflection apparatus for columns.

attached to the ends of the column. In the machine these cylinders roll on parallel plates which were case hardened and ground. Clamp bars B (which are not drawn on the lower cylinder) hold the column in place and make it possible to adjust the eccentricity at each end. To measure the deflection and adjust *the eccentricity*, a wooden bar carrying three Ames dials is attached to the heads by means of steel yokes F and F'. These yokes are connected to the centers of the cylinders through cone bearings which permit free rotation around the axis. The wooden bar is rigidly fastened to the lower yoke and connected to the upper yoke through cone bearings.

The column is first adjusted approximately to less than 0.01 inch by measurement. It is then loaded and readings are taken on the upper and lower dials. The eccentricity is shifted under a small load until the deflection readings are the same at both dials. The eccentricity is then shifted equal amounts as read by the dials until a considerable load gives a very small deflection.

The uppermost curve of Fig. 241 shows the deflection of a 1-inch round bar of cold-rolled steel. The deflection at the middle was 0.0002 inch at 9,500 pounds, 0.0011 inch at 10,500 pounds, and 0.0019 at 10,800 pounds. At 11,200 pounds the deflection was 0.0168 inch, and at 11,250 pounds it reached 0.1789 inch, far beyond the limits of the drawing. The column was then pushed toward the left by a slight horizontal force applied at the middle with one finger. The beam balanced at 11,250 pounds and the deflection of the middle dial was -0.163 inch.

Slight crookedness may cause two of the dials to read in opposite directions at the start and it is impossible at first to predict which way the strut will finally bend.

With zero eccentricity known, it is then possible to move the column to get any desired eccentricity. Since there must be some initial load to hold the column in place in the testing machine, allowance must be made for the initial deflection. The lower curve of Fig. 241 was obtained by nearly equal positive and negative eccentricities, which made it possible to eliminate the readings at the initial load.*

Figure 241 also gives deflection curves for eccentricity of 0.0020 inch and 0.0100 inch. These were calculated by Formula

* *Bull.* No. 25, Ohio State University Engineering Experiment Station, p. 18.

XXX from the measurements of the test column and the modulus of elasticity obtained by compression test as a square-end column. The points marked by solid circles represent the experimental deflections. The close agreement of these and other tests fully verifies the equations.

A committee of the American Society of Civil Engineers under the chairmanship of Dean F. E. Turneaure has recently

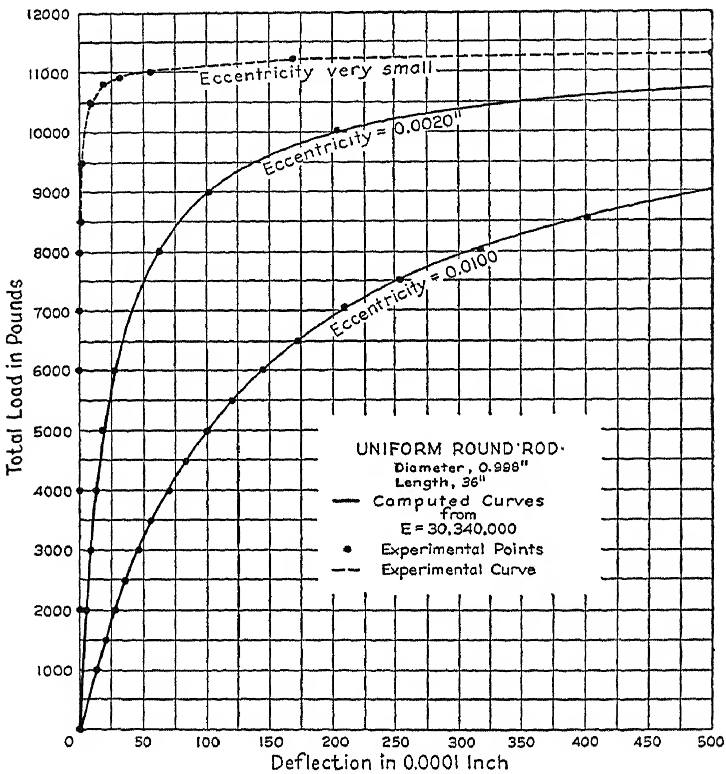


FIG. 241.—Load-deflection curves for three eccentricities.

finished a very complete study of columns.* The experiments, which were made by Prof. M. O. Withey, at the University of Wisconsin, included single shapes of structural steel and various forms of built-up columns. The proportional limit and yield point in both tension and compression and the ultimate strength in tension were measured on specimens taken from the material of each column. Since it had been discovered that there is a great difference in material taken from different parts of a rolled sec-

* *Trans. A.S.C.E.*, No. 98, 1933, p. 1376.

tion, sufficient test pieces were taken to investigate these variations. For instance, from a 10-inch 20-pound channel, four test pieces from the toe and root gave 28.9 kips per square inch as the proportional limit and 38 as the yield point; four pieces from the root and web gave 26.6 and 35.8, respectively; two pieces from the root gave 25.3 and 34.6; and two pieces from the web gave 28.0 and 36.8 kips per square inch. The thinner web and toe, which had been subjected to more work in rolling than the root, gave higher values for the proportional limit and yield point. The completeness of the data concerning the properties of the material greatly enhances the value of this investigation.

These columns were tested on roller end bearings designed to support loads of more than 300 tons, and to rotate with relatively small friction. Each consists of a nest of rollers which bear on a curved surface having its axis of curvature at the center of the bearing plate (see *Transactions American Society of Civil Engineers*, vol. 95, 1931, page 1131, Fig. 9).

The test columns were adjusted to small eccentricity and then moved to give an eccentricity such that $\frac{e\,c}{r^2} = 0.5$, or other convenient values of the eccentric ratio.

170. Application of the Secant Formula.—Formula XXX of Art. 166 and Formula XXIX of Art. 162 together give the unit stress in a round-end column under a known load. However, when it is necessary to design or select a column to carry a given load, these formulas are not convenient, since neither the total load nor the unit stress is explicitly given. A problem of this kind must be solved by the method of trial and error.

When a number of columns are to be designed, it is a great saving of time to represent these formulas by means of a table or curve. From Formula XXX,

$$\text{Maximum moment} = e\,P \sec \sqrt{\frac{P\,l^2}{4\,E\,I}}; \tag{1}$$

$$\text{Maximum unit stress} = S_u = \frac{P}{A} + \frac{e\,P\,c}{I} \sec \sqrt{\frac{P\,l^2}{4\,E\,I}}; \tag{2}$$

$$S_u = \frac{P}{A}\left(1 + \frac{e\,c}{r^2} \sec \sqrt{\frac{P}{A\,E}}\,\frac{l}{2\,r}\right), \tag{3}$$

in which r is the radius of gyration of the column, and $\frac{l}{r}$ is the *slenderness ratio.*

To determine the relation of $\frac{P}{A}$ to $\frac{l}{r}$ when the unit stress at the concave surface is the ultimate strength of the material, Equation (3) may be written,

$$\frac{e\,c}{r^2} \sec \sqrt{\frac{P}{A\,E}}\,\frac{l}{2\,r} = \frac{S_u}{\frac{P}{A}} - 1. \tag{4}$$

It is difficult to solve for $\frac{P}{A}$ in terms of the slenderness ratio but it is easy to solve for slenderness ratio in terms of the unit load.

The expression $\frac{e\,c}{r^2}$ occurs in the equations above. In this expression e is the eccentricity, which is the distance of the line of the resultant load from the center of gravity of all sections before the column is bent. After bending begins, e is the distance of the resultant load from the center of gravity of the end sections for Classes I, III, and IV. The distance of the extreme outer fiber at any section from the axis through the center of gravity of that section about which it rotates is represented by c, and the radius of gyration with respect to this same axis is r. If a column is equally free to bend in all directions, it will bend about the principal axis of minimum radius of gyration. For this reason the structural-steel handbooks give the minimum radius of gyration for angle sections.

Columns with relatively large slenderness ratio fail by bending. The uppermost curve of Fig. 241 illustrates the behavior of a slender column when the eccentricity is extremely small and the ends are free to turn with the minimum friction. The deflection is very small until the load approaches Euler's critical value. When the critical load is reached, the deflection continues to increase with no increase of load. If there had been a live load on the column, which would follow up at its full value, the column would have continued to bend indefinitely until completely ruined.

With some friction at the compression heads, the load may exceed the critical value with very little deflection. However, when the slight deflection produces sufficient moment, the heads turn suddenly, a large deflection occurs, and the load (on a testing machine) drops to Euler's critical value or lower.

With a small eccentricity of 0.002 inch the load approaches the critical value. However, the sum of the direct stress and the bending stress which is caused by the larger deflection reaches the yield point of the material at a somewhat smaller load. With the eccentricity of 0.01 inch (which is still quite small) the total stress at the concave side at the middle would cause failure at considerably lower loads.

The slenderness ratio of the 1-inch rod of Fig. 241 is 144. Euler's formula gives the ultimate load correctly if the eccentricity is very small. An 18-inch length of the same rod would have a slenderness ratio of 72. The ultimate load would be 44,100 pounds, which is 56,000 pounds per square inch. Since the material is cold-rolled steel with high yield point and ultimate strength, Euler's formula gives the true ultimate load for this length if the eccentricity is negligible. Relatively shorter struts fail by compression before Euler's load is reached.

The yield point is taken as the ultimate unit load for structural steel. While very short columns may be stressed beyond this point without failure, a permanent distortion is not desirable. The average yield point for structural steel is about 36,000 pounds per square inch. This value was used in the preceding editions of the "Strength of Materials" in the calculation of tables and curves from the secant formula. The American Society of Civil Engineers Column Committee has adopted a lower value which is called the *useful limit point* (abbreviated U.L.P.). The useful limit point has been defined as that point on the stress-strain diagram of a centrally loaded column at which the slope of the tangent is one-half of the slope of the straight portion. The committee adopted the value of 32,000 pounds per square inch, which is designated as the yield point in secant equations. In the tables which follow, 32,000 pounds will be used as the yield point; 29,400,000 as the modulus of elasticity; and $0.25 = \frac{ec}{r^2}$ to make the calculations comparable with the work of the committee.

Example

Using the constants of the A.S.C.E. committee, calculate $\frac{l}{r}$ for structural steel for unit load of 10,000 lb. per sq. in.

$$\frac{32{,}000}{10{,}000} - 1 = 2.2 = 0.25 \sec \sqrt{\frac{P}{A\,E}}\frac{l}{2\,r};$$

$$\sec \sqrt{\frac{10{,}000}{29{,}400{,}000}}\frac{l}{2\,r} = 8.8;$$

$$\log \sec \sqrt{\frac{1}{2{,}940}}\frac{l}{2\,r} = 0.94448;$$

$$\sqrt{\frac{1}{2{,}940}}\frac{l}{2\,r} = 83^\circ\, 29' = 1.45705 \text{ radians};$$

$$\frac{l}{r} = \sqrt{2{,}940} \times 2 \times 1.45705 = 158.0.$$

Table XXIII gives the values of $\frac{l}{r}$ for a series of values of $\frac{P}{A}$.

TABLE XXIII.—ULTIMATE UNIT LOAD ON A STEEL COLUMN WITH ROUND ENDS

$E = 29{,}400{,}000$; $S_u = 32{,}000$; $\frac{e\,c}{r^2} = 0.25$.

$\frac{P}{A}$	$\frac{S_u}{\frac{P}{A}} - 1$	$\sec\sqrt{\frac{P}{A\,E}}\frac{l}{2\,r}$	Log of $\sec\sqrt{\frac{P}{A\,E}}\frac{l}{2\,r}$	$\sqrt{\frac{P}{A\,E}}\frac{l}{2\,r}$ Degrees	$\sqrt{\frac{P}{A\,E}}\frac{l}{2\,r}$ Radians	$\frac{l}{r}$
2,000	15	60	1.77815	89°03′	1.5542	376.9
4,000	7	28	1.44716	87°57′	1.5350	263.2
6,000	$1\frac{2}{3}$	$5\frac{2}{3}$	1.23888	86°42′	1.5132	211.8
8,000	3	12	1.07918	85°13′	1.4873	180.3
10,000	2.2	8.8	0.94448	83°29′	1.4571	158.0
12,000	$\frac{5}{3}$	$2\frac{2}{3}$	0.82391	81°22′	1.4201	140.6
14,000	$\frac{9}{7}$	$3\frac{6}{7}$	0.71120	78°47′	1.3750	126.0
16,000	1	4	0.60206	75°31′	1.3180	113.0
18,000	$\frac{7}{9}$	$2\frac{8}{9}$	0.49292	71°15′	1.2435	100.5
20,000	0.6	2.4	0.38021	65°23′	1.1412	87.5
21,000	$\frac{11}{21}$	$\frac{44}{21}$	0.32123	61°30′	1.0734	80.3
22,000	$\frac{5}{11}$	$\frac{20}{11}$	0.25964	56°38′	0.9884	72.3
23,000	$\frac{9}{23}$	$\frac{36}{23}$	0.19457	50°17′	0.8776	62.8
24,000	$\frac{1}{3}$	$\frac{4}{3}$	0.12494	41°25′	0.7229	50.6
25,000	0.28	1.12	0.04922	26°46′	0.4672	32.0

Problems

1. Calculate Euler's critical load for $\frac{l}{r} = 158$.

Ans. $\frac{P}{A} = 11{,}620$ lb./in.^{2}

2. Calculate Euler's critical load for a slenderness ratio of 376.9.

3. Find $\frac{e\,c}{r^2}$ for a 6-in. by 8-in. rectangular section if the resultant load is midway between the 6-in. sides at 2.8 in. from the nearest 8-in. side.

Ans. $\frac{e\,c}{r^2} = 0.2.$

4. Find $\frac{e\,c}{r^2}$ for a circular section 3 in. in diameter when the eccentricity is ⅙ in.

Figure 242 is a graph of Table XXIII with $\frac{P}{A}$ as ordinates and $\frac{l}{r}$ as abscissas. The figure includes Euler's curve for $E = 29{,}400{,}000$ pounds per square inch.

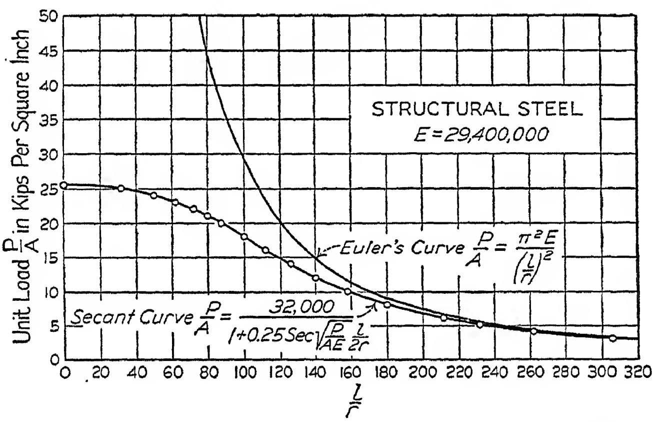


FIG. 242.—Ultimate unit load computed by two formulas.

The curve of ultimate strength from the secant formula approaches Euler's curve as the limit. With smaller eccentricity, the ultimate strength curve would be higher and approach Euler's curve sooner. If the eccentricity were zero, the ultimate strength curve would be a horizontal straight line $\frac{P}{A} = 32{,}000$. (On account of the fact that the useful limit point is below the real yield point, and the fact that short compression pieces may be loaded beyond the yield point without total collapse, the loads of complete failure would sometimes fall above the line $\frac{P}{A} = 32{,}000$.)

171. End Conditions in Actual Columns.—The classification of columns in the preceding article represents ideal conditions, which are only approximated in practice. The columns in actual use are

Round-end columns, which end with spherical or cylindrical surfaces. They sometimes end with knife edges, which may be regarded as cylinders of small radii. The round surfaces roll on plane surfaces with practically no friction. Round-end columns are not used in structures and are rarely used in machines. Since they meet very closely the conditions of Class I with ends free to turn, they are frequently used in tests to check the accuracy of theory.

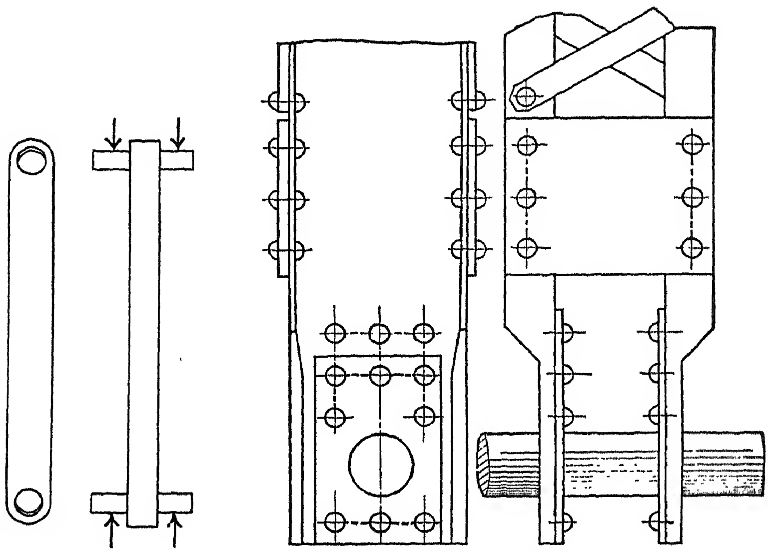


Fig. 243.—Pin-end columns.

A *pin-end* or *hinged-end column* ends with cylindrical surfaces which turn in *cylindrical bearings* (Fig. 243, I). Figure 243, II, shows one end of a pin-connected column made of two channels latticed together. This form of connection is commonly used in bridges. A column which ends with a ball and socket is practically the same as a hinged-end column, except that it is free to turn in any plane instead of in the single plane normal to the axis of the hinge.

If the pin of a hinged-end column rolled on a smooth plane surface, there would be little friction, and the conditions would be those of the ideal round-end column. Usually the pin turns in a closely fitting seat or bearing, which may introduce considerable friction. If the pin is small, the moment arm of the friction is small and there is little resistance to rotation at the end of the

column. If the pin is large, there is considerable resisting moment, and the column behaves at first approximately as a column with fixed ends. Figure 244 shows diagrammatically three stages of the deflection of a pin-end column.

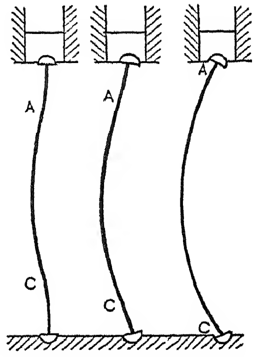


FIG. 244.—Deflection of hinged-end column.

For a series of tests at the Watertown Arsenal in 1909, built I-columns were made of one 10-inch by $\frac{3}{8}$-inch plate and four 4-inch by 3-inch by $\frac{3}{8}$-inch angles. The least radius of gyration was 1.65 inches. The pin-end columns of this series were tested with 3-inch pins which rested in $3\frac{1}{64}$-inch seats. The tests were made on a horizontal compression machine with the axis of each pin vertical and parallel to the 10-inch plate. The results of one test are given in Table XXIV.

Tests were made with slenderness ratios at intervals of 25 from 25 to 175, inclusive, for pin ends as in Table XXIV. The last two columns of Table XXV give the results of all the tests for pin ends. The broken curve of Fig. 245 is plotted from the averages in the last column of Table XXV. All pin-end columns of slenderness ratio of 25 and 50 failed by triple flexure with buckling of the flanges. All pin-end columns with slenderness ratio from 100 to 175, inclusive, failed by sudden springing laterally. For the longer columns the deflection was sufficient to give a moment arm which overcame the friction of the pins. In Table XXIV, the maximum deflection which was recorded was 0.24 inch at a load of 370,980 pounds. The load was released to the initial value of 13,740 pounds. When the load was again applied, sudden springing took place at about 20 pounds more than the previous load. The small deflection of 0.02 inch at 206,100 pounds which was reduced to zero at 274,800 pounds (indicating slight crookedness) would seem to show that the initial eccentricity was very small. If 0.24 inch is taken as the moment arm and 1.5 inches is the moment arm of the friction of the pin, the coefficient of friction was $0.24 \div 1.5 = 0.16$.

Square-end or flat-end columns end with plane surfaces in contact with plane surfaces. The ends must be accurately fitted to avoid eccentricity. If a beam which rests on a square-end column bends under a load, as shown in Fig. 246, II, the load

TABLE XXIV.—TEST OF BUILT I-COLUMN AT WATERTOWN ARSENAL

Area, 13.74 square inches; length, center to center of pins, 24 feet ½ inch; radius of gyration about axis of pin, 1.65 inches; $\frac{l}{r}$, 175; gage length, 100 inches.

Load		Compression in gage length, inches	Deflection, inches	
Total	Per square inch		Horizontal	Vertical
13,740	1,000	0	0	0
68,700	5,000	0.0131	0.01	0
137,400	10,000	0.0300	0.01	0
206,100	15,000	0.0468	0.02	0.02
274,800	20,000	0.0642	0	0.02
288,540	21,000	0.0680	0.02	0.02
302,280	22,000	0.0721	0.02	0.03
316,020	23,000	0.0760	0.03	0.04
329,760	24,000	0.0801	0.04	0.04
343,500	25,000	0.0840	0.06	0.04
13,740	1,000	0.0026 set	0.01	0.02
357,240	26,000	0.0874	0.11	0.04
370,980	27,000	0.0940	0.24	0.04
13,740	1,000	0.0052 set	0.04	0.03
371,000	27,010	Ultimate load		

Failed by suddenly springing laterally, after which the resistance was 71,000 pounds.

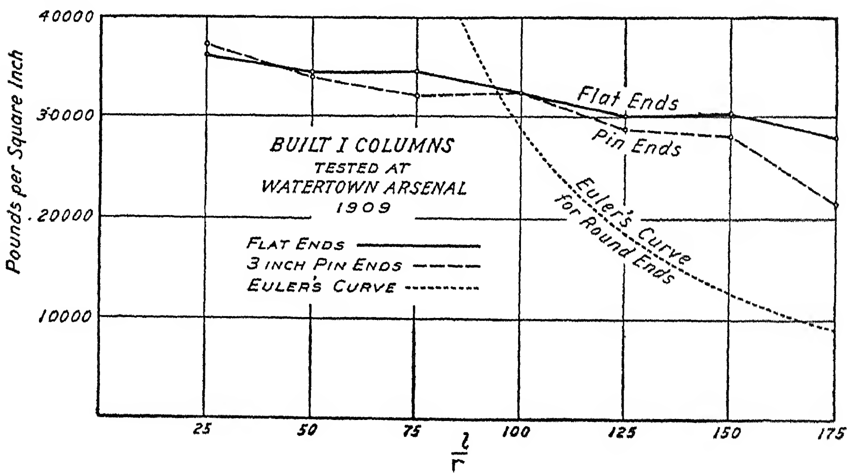


FIG. 245.

on the column becomes eccentric. Footings which support columns often settle unevenly and cause large eccentricity.

TABLE XXV.—COMPARATIVE TESTS OF PIN-END AND SQUARE-END COLUMNS

Slenderness ratio, $\frac{l}{r}$	Ultimate load, pounds per square inch			
	Square ends		Pin ends	
	Separate columns	Average	Separate columns	Average
25	37,450 36,000 35,580	36,343	37,670 37,870 36,720	37,420
50	34,460 34,560 34,750	34,590	33,800 33,640 34,080	33,840
75	34,690 34,740 34,420	34,617	32,270 32,000 32,110	32,160
100	31,670 32,800 32,800	32,423	31,940 31,950 33,070	32,320
125	29,930 31,300 28,880	30,037	30,000 28,850 28,740	29,197
150	30,300 30,080 30,520	30,300	27,400 28,310 29,190	28,320
175	24,730 26,650 26,720	25,033	13,130 27,010 23,200	21,110

All square-end columns failed by triple flexure. All pin-end columns of slenderness ratio 25 and 50 failed by triple flexure with buckling of the flanges. All pin-end columns with slenderness ratio from 100 to 175, inclusive, failed by sudden springing laterally.

Pin-end columns are square ended in the direction of the axis of the pin. The column of Table XXIV was horizontal in the testing machine with the pins vertical. The maximum vertical

deflection was only 0.04 inch, while the maximum horizontal deflection recorded was six times as great.

Fixed-end columns may be riveted, bolted, or welded to fixed footings, or to other members of a bridge or building. In a machine, a fixed-end column may be fastened in these ways to the frame or may be cast or forged continuous with it. Since the connection cannot be absolutely rigid, and since the member to which the column is "fixed" must suffer some distortion, the tangent at the end of the column does not remain entirely stationary and the conditions of Class III are never completely satisfied. If the column is very flexible in comparison with the body to which it is fixed, the ideal case may be approximated and one-half the total length of the column may be used for l in the formulas. In most practical columns, this assumption would introduce a dangerous error.

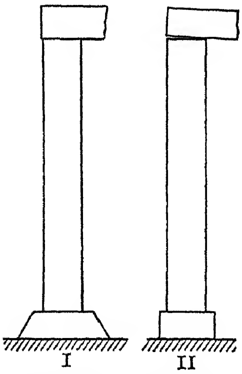


FIG. 246.—Square-end columns.

A column with a pin connection at one end and a square or fixed connection at the other is called a *pin-and-square column.* This column approximates the conditions of Class IV of Fig. 239. The yardstick of Problem 3 (Art. 168) shows the agreement of experiment with theory. In this experiment, the column was rigidly clamped to a 2-inch by 4-inch post and was relatively flexible. (The slenderness ratio was more than 800.) A column of ordinary slenderness fastened to a structure of comparable dimensions would not meet so closely the conditions of the theory, and the experimental and calculated results would not agree so well.

The second and third columns of Table XXV give the ultimate load on square-end columns of the same cross section as that of Table XXIV. The curves of Fig. 245 afford a comparison of hinged-end and square-end columns. Except for the slenderness ratio of 175, the results are much alike. One pin-end column of this last ratio had an ultimate strength of about half as much as the others, which greatly lowers the average for that length. Figure 245 gives Euler's curve for a round-end column with $E = 29{,}000{,}000$. For slenderness ratios of 100 and over, the strength of a pin-end column and of a square-end column are considerably above Euler's value. For smaller slenderness ratios

the ultimate strength is close to the yield point of the steel. It must be remembered, however, that these test columns were carefully made and were tested with small eccentricity. Moreover, the compression heads of this testing machine are very rigid, so that the square ends of the columns are not free to turn.

TABLE XXVI.—PENCOYD TESTS OF WROUGHT-IRON STRUTS

$\frac{l}{r}$	Average results for angles and tees. $\frac{P}{A}$ ultimate unit load, pounds per square inch			
	Round ends	Hinged ends	Flat ends	Fixed ends
20	44,000	46,000	49,000	45,000
40	36,500	40,500	41,000	38,000
60	30,500	36,000	36,500	34,000
80	25,000	31,500	33,500	32,000
100	20,500	28,000	30,250	30,000
120	16,500	24,250	26,500	28,000
140	12,800	20,250	23,250	25,500
160	9,500	16,350	20,500	23,000
180	7,500	12,750	18,000	20,000
200	6,000	10,750	15,250	17,500
220	5,000	8,750	13,000	15,000
240	4,300	7,500	11,500	13,000
260	3,800	6,500	10,250	11,000
280	3,200	5,750	8,750	10,000
300	2,800	5,000	7,350	9,000
320	2,500	4,500	5,750	8,000
340	2,100	4,000	4,650	7,000
360	1,900	3,500	3,900	6,500
380	1,700	3,000	3,350	5,800
400	1,500	2,500	2,950	5,200
420	1,300	2,250	2,500	4,800
440	......	2,100	2,200	4,300
460	......	1,900	2,000	3,800
480	......	1,700	1,900	

Valuable tests of columns were made by James Christie, in 1883, at the Pencoyd Iron Works. For some of these tests, the so-called *hinged-end columns* were fitted with hemispherical balls

turning in sockets. These balls were fitted as accurately as possible by measurement. The final adjustment was made in the testing machine. A small load was applied and the deflection measured. The hemispheres were shifted and the measurement repeated until a considerable load caused no appreciable deflection. The column was then loaded to failure. Since these ball-and-socket joints were lubricated, the friction was smaller than that of a hinged connection of a truss, but considerably more than that of the half cylinders which rolled on plates or in a nest of roller bearings. The ball-and-socket joint gives the strut opportunity to deflect in any direction, which is sometimes an advantage but more often a disadvantage for experimental studies. These tests were made on wrought-iron struts.*

With these hemispherical joints, "when the point of greatest strength was reached, the behavior of the specimen was peculiar. Under ordinary circumstances the bar, while bending under the strain, rotated from the start on its hinged ends. When correctly centered, no such rotation occurred at the beginning of the deflection, but the bar bent like a flat-ended strut, till the point of failure was reached, when it rotated on its ends suddenly, as sometimes to spring from the machine. These results could not be secured when the balls or pins rolled on plane surfaces, and were difficult to get when the pins were small."

The effect of the size of the pin was shown in these experiments. Two angles of the same length were cut from the same bar. One of these tested with a 2-inch ball and socket failed at 36,500 pounds per square inch; the other tested with a 1-inch ball and socket failed at 24,010 pounds per square inch.

These and other tests show how the friction at the ends of a hinged-end column partly fixes the ends and greatly increases the strength. It is a question, however, how much of this is lost on a railroad bridge on account of the vibration of moving trains.

Table XXVI gives the results of one series of tests which were made by the Pencoyd Company on rolled angles and tees of wrought iron. The hinged ends were ball-and-socket joints and the round ends were balls on plane surfaces. The columns could bend equally in any direction. It was found that failure always took place in the direction of the least radius of gyration. The figures of Table XXVI give some idea of the relative values of the different endings *under the conditions of these experiments.* For a unit load of 25,000 pounds per square inch, for instance,

* *Trans. A.S.C.E.*, 1883, pp. 85–122.

$\frac{l}{r}$ is 80 for round ends, 129 for flat ends (by interpolation between 26,500 and 23,250), 119 for hinged ends, and 144 for fixed ends.

Problems

1. Using $E = 27{,}000{,}000$ lb. per sq. in. for wrought iron, find the ultimate unit loads for round-end columns for slenderness ratios of 160, 200, 300, and 400 by Euler's formula and compare with Table XXVI.
2. Take $\frac{l}{r} = 60$ for round-end columns in Table XXVI. Find the lengths for hinged, flat, and fixed ends.
3. Solve Problem 2 for $\frac{l}{r} = 100$ for round ends. *Ans.* 138; 160; 177.

If all the values for round ends from 40 to 200, inclusive, are taken, the corresponding values of $\frac{l}{r}$ which give the same unit load for the other end conditions are

	Hinged	Flat	Fixed
Maximum	1.45	1.69	1.87
Minimum	1.29	1.50	1.27
Mean of all	1.37	1.60	1.72

Only one value fell below 1.50 for fixed ends.

As far as these tests go, they indicate that a flat-end column 10 feet long, a fixed-end column 17.2 feet long, or a hinged-end column 13.7 feet long will carry the same load as a round-end column 10 feet long of the same cross section.

The conditions in actual structure may be very different from those of these carefully conducted experiments. Columns may be fixed to rather flexible beams. The eccentricity is likely to be greater than in these tests, and other factors may reduce the advantage which a hinged-end, flat-end, or fixed-end column has over a round-end column.

The American Society of Civil Engineers Column Committee recommends that three-fourths of the total length of a fixed-end column be used as l in the formulas for a round-end column; and that 0.85 of the length of a hinged-end column be used as l in the formulas for a round-end column. A flat-end column is equivalent to a fixed end. Lubricated moving columns of machinery should be treated as round-end columns.

CHAPTER XVI

WORKING FORMULAS FOR COLUMNS

172. Kinds of Formulas.—The secant formulas of Art. 166 are theoretically correct within the limits of the assumptions of beam theory, which are universally accepted. Careful tests, *in which the conditions of the theory are fully met*, amply verify these formulas. For round-end columns of uniform material, the results of experiments agree with theory within the limits with which the modulus of elasticity can be determined by direct compression. Moreover, the deflections at any point on a column agree with Equation (12) of Art. 166. Euler's formula, which is a special case of the secant formula, also agrees with experiments.

Euler's formula with a factor of safety may be used as a working equation for slenderness ratio greater than 150.

The secant formula applies to all lengths. It approaches Euler's formula as a limit for large slenderness ratios. Mathematically it is difficult to use unless curves are plotted similar to that of Fig. 242. On account of this difficulty, most engineers prefer some more convenient approximate equation. Such equations may be made which lie well inside the limits of the uncertainty as to the amount of eccentricity. The previous editions of this book, while giving the secant equations as presented in Chapter XV, have selected for engineering practice straight-line and Rankine equations which were officially recognized in building laws or were recommended by national engineering organizations. The validity of each of these equations was examined by comparison with the secant formulas and with experiments such as those of Tables XXIV, XXV, and XXVI. Some such equations have been founded on experiment, often inadequate, some on approximate theory, and a few on accurate theory.

These straight-line and Rankine equations will be presented in the present edition, since they are so largely used, but preference is given to working equations in the form of parabolas. The

Column Committee of the American Society of Civil Engineers has made the most valuable set of tests of full-size columns and has studied these tests with the secant formulas. Moreover, on the basis of all previous tests and a knowledge of the conditions in structures, the committee, as a piece of mature engineering judgment, has recommended lengths which may be assigned to hinged-end and fixed-end columns to adapt them to the equations of the round-end columns. From these the committee recommends working equations in the form of parabolas which agree

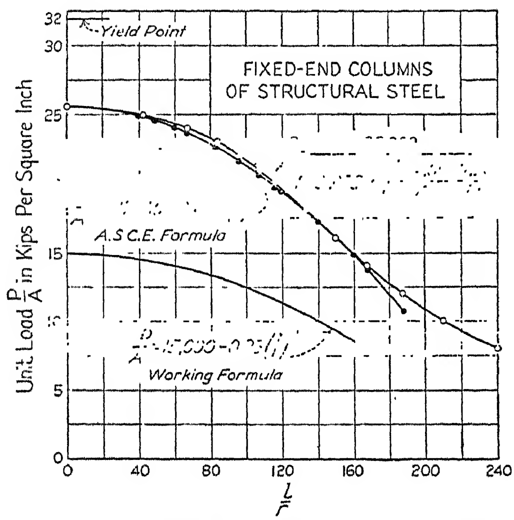


FIG. 247.—Ultimate and allowable unit loads.

closely with the secant equation over the entire range of slenderness ratios used in primary columns.

173. Fixed-end Structural-steel Columns.—The second column of Table XXVII gives slenderness ratios of round-end structural steel which correspond to unit loads of the first column. These values were used in the construction of the secant curve of Fig. 242. The American Society of Civil Engineers committee recommends three-fourths the length of a riveted column be used for l in the secant formula. The slenderness ratios under "fixed ends" in Table XXVII are the ratios of the second column multiplied by $\frac{4}{3}$ (or divided by $\frac{3}{4}$). The upper curve of Fig. 247, with points represented by hollow circles, was plotted from this column of figures. For fixed-end columns, for ultimate unit loads, the committee recommends the parabola

$$\frac{P}{A} = 25{,}600 - 0.425\left(\frac{l}{r}\right)^2. \qquad (1)$$

This begins with the secant curve at 25,600, is below the secant curve until $\frac{l}{r}$ is nearly 140, then crosses over, and is above the secant curve until $\frac{l}{r}$ is nearly 160 when it crosses again. (The portion above the secant cannot be shown on the small-scale drawing.)

TABLE XXVII.—RELATION OF ULTIMATE UNIT LOAD TO SLENDERNESS RATIO

Calculated by the secant formula with $E = 29{,}400{,}000$; eccentric ratio = 0.25.

$\frac{P}{A}$, pounds per square inch	Slenderness ratio, $\frac{l}{r}$			
	Round ends, free length = l	Hinged ends, free length = 0.85 l	Fixed ends, free length = 0.75 l	Round ends, Euler's
1,000	535.8	630.4	714.4	538.7
2,000	376.9	443.4	502.5	380.9
3,000	305.9	359.9	407.9	311.0
4,000	263.2	309.6	350.9	269.3
5,000	233.8	275.1	311.7	240.9
6,000	211.8	249.2	282.4	219.9
8,000	180.3	212.1	240.4	190.4
10,000	158.0	185.9	210.7	170.3
12,000	140.6	165.4	187.5	155.5
14,000	126.0	148.2	168.0	144.0
16,000	113.0	132.9	150.7	134.7
18,000	100.5	118.2	134.0	127.0
20,000	87.5	102.9	116.7	120.4
21,000	80.3	94.5	107.1	117.6
22,000	72.3	85.1	96.4	114.9
23,000	62.8	73.9	83.7	112.3
24,000	50.6	59.5	67.5	110.0
25,000	32.0	37.6	42.7	107.8
25,600	0	0	0	

Problems

1. Calculate the unit load when the slenderness ratio is 150.1 by Eq. (1). *Ans.* 16,025 lb./in.^{2}
2. Solve Problem 1 for slenderness ratio of 134. *Ans.* 17,969, 31 lb. below the secant curve.
3. Solve Problem 1 for a slenderness ratio of 83.7 at which the parabola seems farthest from the secant curve. Find the percentage of error. *Ans.* 22,623 lb./in.^{2}; 1.64 per cent.
4. Solve Problem 3 for the point at which the secant curve is at 24,000 lb. per sq. in. *Ans.* 1.4 per cent.

If Equation (1) is divided by 1.7, the result is

$$\frac{P}{A} = 15,059 - 0.25\left(\frac{l}{r}\right)^2 \tag{2}$$

For round numbers the American Society of Civil Engineers committee recommends

$$\frac{P}{A} = 15,000 - 0.25\left(\frac{l}{r}\right)^2 \quad \text{Formula XXXIII*}$$

for slenderness ratios from 0 to 160 as the working equation for structural steel columns or struts with ends fixed. The factor of safety of Formula XXXIII is really more than 1.7, since the unit stress has already been reduced from 32,000 to 25,600 to allow for possible eccentricity.

Problems

5. Find the unit safe load and the total load on an 8-in. 18.4-lb. standard I-beam for lengths of 5 ft. and 8 ft. *Ans.* P = 73,390 lb.; 62,665 lb.
6. Find the unit load and the total load on a 4-in. by 4-in. by ½-in. standard angle, 9 ft. 9 in. long, as a column with fixed ends. *Ans.* $\frac{P}{A}$ = 9,375 lb./in.^{2}; P = 35,156 lb.
7. Find the slenderness ratio and the ultimate load on a 5-in. by 3.5-in. by ⅝-in. standard angle as a column with fixed ends, 10 ft. long. *Ans.* P = 20,270 lb.
8. If a given length of a 5-in. by 3½-in. by 9/16-in. standard angle will carry 30,000 lb., what will the same length of a 10-in. 15.3-lb. channel carry? Solve by ratios.

174. Hinged-end Structural-steel Columns.—The American Society of Civil Engineers Column Committee recommends for

* Equations of this form are frequently called *Johnson's parabolic equations* after the late J. B. Johnson, who first proposed them, and who made many valuable contributions to experimental and applied strength of materials.

hinged-end steel columns, for slenderness ratios up to 160, the equation

$$\frac{P}{A} = 15{,}000 - \frac{1}{3}\left(\frac{l}{r}\right)^2. \qquad \text{Formula XXXIV}$$

In order to compare with the secant curves for ultimate load with eccentric ratio of 0.25, the formula is multiplied by $\frac{256}{150}$ which gives

$$\frac{P}{A} = 25{,}600 - 0.569\left(\frac{l}{r}\right)^2. \qquad (1)$$

From Table XXVII for 14,000 pounds per square inch, with 0.85 l as the free length of a hinged-end strut, the corresponding slenderness ratio in the third column of figures is found to be 148.2. When substituted in the parabola of Equation (1), this slenderness ratio gives 13,100 pounds per square inch as against 14,000 pounds per square inch from the secant equation.

Problems

1. Find the slenderness ratio in Table XXVII which corresponds to a unit load of 12,000 lb. per sq. in. for a hinged-end column. Calculate the ultimate load for this slenderness ratio from Eq. (1).

Ans. $\frac{P}{A} = 11{,}030$ lb./in.^{2}

2. Solve Problem 1 for unit stress of 16,000 lb. per sq. in.

Ans. $\frac{P}{A} = 15{,}550$ lb./in.^{2}

3. A plate-and-channel column, 20 ft. long, is made of two 10-in. 15.3-lb. channels, placed 6 in. apart back to back and loaded through pins at right angles to the webs of the channels. The two plates are each 12 in. by ¼ in. Find the allowable load as a hinged-end column.

Ans. $\frac{P}{A} = 14{,}000$ lb./in.^{2}; $P = 207{,}160$ lb.

4. The column of Problem 3 may be regarded as fixed with respect to axes perpendicular to the pins. Find the unit load by Formula XXXIII and find the total load if necessary.

Ans. $\frac{P}{A} = 13{,}870$ lb./in.^{2}; $P = ?$

(*This article may be omitted.*)

175. Euler's Extension of Parabola.—Formulas XXXIII and XXXIV are valid to $\frac{l}{r} = 160$. For greater slenderness ratios it is desirable to have an extension of the Euler type for each of these. Representing $\frac{P}{A}$ by y and

$\frac{l}{r}$ by x, the parabola becomes $y = S_u - k\,x^2$, in which k is a constant which may or may not be known, and Euler's equation is $y = \frac{C}{x^2}$. If x' and y' are the coördinates of the point of tangency of the two curves, then

$$S_u - k\,x'^2 = \frac{C}{x'^2}; \quad S_u x'^2 - k\,x'^4 = C. \tag{1}$$

At the point of tangency

$$\frac{dy}{dx} = -2\,k\,x' = -\frac{2\,C}{x'^3}; \tag{2}$$

$$k\,x'^4 = C. \tag{3}$$

From Equations (1) and (3),

$$S_u x'^2 = 2\,C; \qquad x'^2 = \frac{S_u}{2\,k}; \tag{4}$$

Since $y = \frac{C}{x^2}$,

$$y' = \frac{S_u}{2}. \tag{5}$$

For a fixed-end steel column (Formula XXXIII),

$$S_u = 15{,}000 \text{ and } k = \tfrac{1}{4}; \quad x'^2 = 30{,}000; \quad x' = 173.2;$$
$$2\,C = 15{,}000 \times 30{,}000; \quad C = 225{,}000{,}000.$$

Euler's extension is then

$$\frac{P}{A} = \frac{225{,}000{,}000}{\left(\frac{l}{r}\right)^2}. \tag{6}$$

This value of 225,000,000 in place of 290,000,000 would seem to represent a small safety factor of 1.3, but it must be remembered that l is the entire length of the fixed-end column. If $\frac{3}{4}\,l$ be regarded as the free length, the safety factor is increased to 2.2.

Problems

1. Derive the equation of Euler's extension for Formula XXXIV for a hinged-end steel column and the slenderness ratio at the point of tangency.

Ans. $\frac{P}{A} = \frac{168{,}750{,}000}{\left(\frac{l}{r}\right)^2}$.

2. A 3-in. by 1-in. bar is welded to rigid bodies at each end. Below what length will Formula XXXIII apply when the bar is subjected to compression? Find the total safe loads for lengths of 3 ft., 5 ft., and 8 ft.

Ans. 50 in.; 33,340 lb., 15,625 lb.?

3. Find the total safe load on a 6 × 5¾ 23-lb. wide flange section which is connected by pins at right angles to the web for lengths of 9 ft. and 22 ft. 6 in.

Example I

Derive a **parabolic** equation similar to Formula XXXIII for a round-end steel column by means of the equations of this article without reference to the secant formula. Also write Euler's extension.

The assumed eccentric ratio of 0.25 which was used in Art. 173 reduces the ultimate strength to 80 per cent of the yield point. By allowing the same reduction of E in Euler's formula,

$$\pi^2 E = 290{,}000{,}000 \times 0.8 = 232{,}000{,}000.$$

$$y' = \frac{S_u}{2} = 12{,}800 = \frac{232{,}000{,}000}{x'^2}. \tag{7}$$

$$x'^2 = 18{,}125; \; x' = \frac{l}{r} = 135.$$

$$k = \frac{S_u}{2\,x'^2} = 0.706;$$

$$\frac{P}{A} = 25{,}600 - 0.706\left(\frac{l}{r}\right)^2 \tag{8}$$

up to the slenderness ratio of 135.

For a working formula with a factor of safety of 1.7,

$$\frac{P}{A} = 15{,}000 - 0.42\left(\frac{l}{r}\right)^2. \tag{9}$$

For slenderness ratios above 135,

$$\frac{P}{A} = \frac{136{,}500{,}000}{\left(\frac{l}{r}\right)^2}. \tag{10}$$

Round-end columns are little used in structures. Slender pin-connected columns in airplanes should be treated as round-end columns. The connecting rod of an engine is a round-end column which is further complicated by transverse forces of the weight and the centrifugal force.

Problems

4. Calculate the total safe load on a 4-in. solid shaft as a round-end column for lengths of 5 ft., 10 ft., and 15 ft.
5. Calculate the safe load on a hollow shaft 4 in. outside diameter and 2 in. inside diameter as a column with round ends.
6. Compare Eqs. (7) and (8) with Table XXVII for round ends by secant equation.

Example II

Round-end spruce struts tested at the Bureau of Standards* had an average modulus of elasticity of 1,910,000 and ultimate strength of 5,200 lb. per sq. in., which was obtained by extending the graph of the columns back to zero slenderness ratio.

Derive a parabolic equation and Euler's extension on the assumption of eccentric ratio = 0.1.

$$\frac{5{,}200}{1.1} = 4{,}730 = S_u; \quad \frac{1{,}910{,}000\,\pi^2}{1.1} = 17{,}100{,}000.$$

$$2{,}365 = \frac{17{,}100{,}000}{\left(\frac{l}{r}\right)^2}; \quad \left(\frac{l}{r}\right)^2 = 7{,}230; \quad \frac{l}{r} = 85.$$

$$k = \frac{4{,}730}{2 \times 7{,}230} = 0.327;$$

$$\frac{P}{A} = 4{,}730 - 0.327\left(\frac{l}{r}\right)^2, \text{ to } \frac{l}{r} = 85.$$

$$\frac{P}{A} = \frac{17{,}100{,}000}{\left(\frac{l}{r}\right)^2}, \text{ for } \frac{l}{r} \text{ greater than 85.}$$

Problem

7. The average results of the tests were

$\frac{l}{r}$	25	50	75	100	125	150	175	200
$\frac{P}{A}$	4,676	4,049	3,109	1,824	1,206	785	629	439

Calculate $\frac{P}{A}$ for each slenderness ratio. Compare with experimental figures.

176. Straight-line Formulas.—The curves of Fig. 242 show that Euler's formula gives results which are very close to the correct values of the secant formula when $\frac{l}{r}$ is large. When the slenderness ratio is small, Euler's formula must not be used. Figure 248 shows one secant curve for an eccentric ratio of 0.25, as in Fig. 242, and another curve with an eccentric ratio of 0.1. For slenderness ratios of 200 or more, both secant curves are very close to Euler's curve. It is evident from these curves that the magnitude of the eccentricity makes very little difference when the slenderness ratio is large. For $\frac{l}{r} = 200$ or more,

* *Tech. Paper* No. 152, Bureau of Standards.

Euler's formula may be employed with a relatively small factor of safety for structural steel, and it may be used down to 150 with a larger safety factor. In structures, especially with fixed or square ends, the eccentricity is variable. The effect of eccentricity is shown by the area between the secant curves of Fig. 248. If the eccentric ratio ranges from 0.1 to 0.25, as shown in the figure, any curve which lies in the area between these secant curves may agree fairly with the results of tests. For this reason, straight-line curves have been largely used in practice. A curve

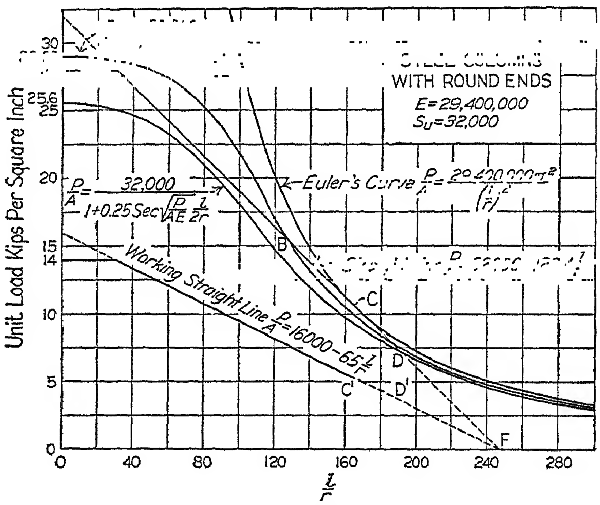


FIG. 248.—Straight-line, secant, and Euler's curves.

of this kind must pass through the ultimate strength for short blocks and approximate Euler's for slender struts. Some straight-line formulas have been derived from experiments and others have been made by drawing a straight line from the ultimate strength tangent to Euler's. Figure 248 shows a straight line through (0, 32,000) which is tangent to Euler's curve at C. A horizontal stopper, drawn at $\frac{P}{A} = 28{,}000$, replaces the upper part of the straight line.

The straight line crosses the upper secant curve at B and recrosses the lower secant curve at D. Usually the straight line is used to the point of tangency. It might well be extended to D. The lower straight line is a working curve which is valid to C' or better to D'.

A straight-line equation has the form

$$\frac{P}{A} = S_u - k\frac{l}{r}. \qquad \text{Formula XXXV}$$

If $\frac{P}{A} = y$ and $\frac{l}{r} = x$, this is recognized as the equation of a straight line with the Y intercept at S_u, and with a negative slope equal to k.

Problems

1. Plot Euler's curve for $E = 1{,}600{,}000$. Draw a tangent to this curve from (0, 4,800). Find graphically the abscissa and the ordinate of the point of tangency. Extend the straight line to the X axis. Calculate k. Draw a working curve with a factor of safety of 4, and a stopper at 1,000 lb. per sq. in.
2. In Problem 1 find the unit allowable load for slenderness ratios of 30, 50, 70, and 100.

177. Algebraic Derivation of the Straight-line Formulas.—While a straight-line formula may always be derived graphically by drawing Euler's curve and plotting the tangent, the methods of calculus are convenient and lead to a simple algebraic result. The problem is that of drawing a straight line tangent to a given curve through a given point which is not on the curve. Euler's formula may be written

$$y = \frac{a}{x^2}, \tag{1}$$

in which $y = \frac{P}{A}$, $x = \frac{l}{r}$, and $a = \pi^2 E$.

It is required to draw a tangent to the curve of Equation (1) which shall pass through the point $(0, S_u)$. The slope of this tangent is

$$\frac{dy}{dx} = -\frac{2a}{x_1^3}, \tag{2}$$

in which x_1 is the abscissa of the point of tangency. The equation of the tangent line is

$$y = -\frac{2a}{x_1^3}x + S_u, \tag{3}$$

in which x and y are the coördinates of any point on the line. Since the point of tangency (x_1, y_1) lies in the straight line of

Equation (3), these coördinates satisfy the equation of the line; hence

$$y_1 = -\frac{2a}{x_1^2} + S_u. \tag{4}$$

Since the point of tangency is on the curve, these coördinates also satisfy Equation (1); hence

$$y_1 = \frac{a}{x_1^2}. \tag{5}$$

From Equations (4) and (5) the coördinates of the point of tangency are found to be

$$y_1 = \frac{S_u}{3}, \qquad \text{Formula XXXVI}$$

$$x_1^2 = \frac{3a}{S_u}. \tag{6}$$

The value of x_1 from Equation (6) may be substituted in Equation (3) to get the desired straight-line equation. It is better, however, to use the easily remembered fact of Formula XXXVI that the ordinate of the point of contact is one-third the Y intercept of the straight line. When this ordinate is substituted in Euler's formula, the abscissa of the point of contact is found. The coördinates of the point of tangency and of the Y intercept together determine the equation of the straight line.

Example I

Derive a straight-line formula for steel which has a yield point of 32,000 and a modulus of elasticity of 29,400,000.

$$\frac{32,000}{3} = \frac{\pi^2\, 29,400,000}{\left(\frac{l}{r}\right)^2};$$

$$\left(\frac{l}{r}\right)^2 = 27,203; \quad \frac{l}{r} = 164.93; \tag{7}$$

$$k = \frac{2 \times 32,000}{3 \times 164.93} = 129.37. \tag{8}$$

$$\frac{P}{A} = 32,000 - 129.37\,\frac{l}{r} \tag{9}$$

gives the ultimate unit load. For a working formula with a factor of safety of 2 for a column with round ends,

$$\frac{P}{A} = 16{,}000 - 65\,\frac{l}{r}. \tag{10}$$

(Do not solve problems by Eq. (10), *unless designated.)*

Equation (9) is the upper straight line of Fig. 248.

Equation (10) is a working formula with a factor of safety of about 2 based on the upper straight line.

The Chicago building laws for structural steel require

$$\frac{P}{A} = 16{,}000 - 70\,\frac{l}{r}, \qquad \text{Formula XXXVII}$$

with a maximum (stopper) of 14,000 pounds per square inch, for round, hinged, or fixed ends. This formula was used by the American Railway Engineering Association and was copied by many cities. It is slightly more conservative than Equation (10), the validity of which is recognized from Fig. 248. Formula XXXVII may be used to $\frac{l}{r} = 180$.

If the free length of a fixed-end column is taken as three-fourths the total length, as recommended by the American Society of Civil Engineers Column Committee, k in Equation (10) becomes 50 instead of 65. A few years ago, the American Railway Engineering Association adopted

$$\frac{P}{A} = 15{,}000 - 50\,\frac{l}{r} \tag{11}$$

with a maximum of 12,500. This is an excellent formula for fixed-end and hinged-end columns. The specifications state that the slenderness ratio shall not exceed 100 for main compression members and shall not exceed 120 for wind and sway bracing.

The American Bridge Company used

$$\frac{P}{A} = 19{,}000 - 100\,\frac{l}{r} \tag{12}$$

with a maximum of 13,000 for slenderness ratios not greater than 120, and

$$\frac{P}{A} = 13{,}000 - 50\,\frac{l}{r} \tag{13}$$

for slenderness ratios between 120 and 200.

Since Equation (10) was derived from Equation (9) (which is tangent to Euler's) by dividing by the factor 2, Euler's extension of Equation (10) may be derived by dividing $\pi^2 E$ by the same factor, or more accurately by the ratio of $\frac{129.37}{65}$. When a straight-line equation has been derived from tests, instead of from a tangent to Euler's curve, it is necessary to use the principle of Formula XXXVI to find the ordinate of the point of tangency.

Example II

Find Euler's extension for the American Bridge Company formula [Eq. (12)].

$$\frac{19{,}000}{3} = 19{,}000 - 100\,\frac{l}{r}; \quad \frac{l}{r} = \frac{380}{3} = 127,$$

in which $\frac{l}{r}$ is the abscissa of the point of tangency at which $\frac{P}{A} = 6{,}333$ lb. per sq. in. If E_w is the modulus of elasticity in the working Euler's formula,

$$\frac{\pi^2 E_w}{\left(\frac{380}{3}\right)^2} = \frac{19{,}000}{3}; \quad \pi^2 E_w = \frac{4 \times 19^3 \times 10^5}{27} = 1{,}016 \times 10^5.$$

$$\frac{P}{A} = \frac{101{,}600{,}000}{\left(\frac{l}{r}\right)^2} \text{ or } \frac{100{,}000{,}000}{\left(\frac{l}{r}\right)^2}. \tag{14}$$

Equation (12), extended to $\frac{l}{r} = 127$, and Eq. (14) for any greater slenderness make an excellent conservative combination for all kinds of ends. (The American Bridge Company has discarded this equation in favor of the American Institute of Steel Construction equation.)

Since the point of tangency to Euler's curve is at $\frac{P}{A} = \frac{S_u}{3}$, when any straight-line equation gives a unit load much below this figure, an Euler's extension should be used instead.

Problems

1. A 12-in. 31.8-lb. standard I-beam is used as a column 10 ft. long. Find the total safe load by Chicago building laws, the American Bridge Company formula, and Eq. (11) if applicable.
   Ans. $P = 71{,}140$ lb.; 66,120 lb.
2. Solve Problem 1 for a length of 8 ft.
   Ans. $P = 86{,}550$ lb.; 87,930 lb.; 94,900 lb.
3. Solve Problem 1 for a length of 16 ft. by a straight-line equation which applies. *Ans.* 32,370 lb. by Eq. (13).
4. Solve Problem 3 by the first form of Euler's extension of the American Bridge Company equation. *Ans.* $P = 2{,}811 \times 9.26 = 26{,}030$ lb.

5. A 10 × 10 wide-flange (WF or CB) section, 60 lb. per ft., is used as a column 20 ft. long. Find the total safe load by suitable straight-line equations.
   Ans. P = 167,100 lb. by Formula XXXVII; 182,440 lb. by Eq. (11); 170,630 lb. by Eq. (12).
6. Solve Problem 7 for a length of 30 ft.
   Ans. P = 109,400 lb. by Formula XXXVI; 105,890 lb. by Eq. (13).
7. Solve Problem 8 by Euler's formula for the ultimate load as a round-end column and divide by a safety factor of 2.5.
8. Find the total safe load by Chicago building laws on a standard 5-in. pipe as a column 12 ft. in length. *Ans.* P = 45,720 lb.

178. Rankine's Formula.—Rankine's formula, sometimes called the *Gordon Rankine formula,* has long been the British favorite, although equations based on the secant formula are now gaining ground. It was the principal formula in America until about thirty years ago when the straight-line formulas largely displaced it. Recently, under the leadership of the American Institute of Steel Construction, one formula of this kind has gained the ascendancy. It is an *empirical formula,* which gives the unit load equal to the ultimate strength for a short block and approaches Euler's curve for a very long column. The formula is of the form

$$\frac{P}{A} = \frac{S_u}{1 + q\left(\frac{l}{r}\right)^2} \qquad \text{Formula XXXVIII}$$

in which S_u is the ultimate unit load in compression on a short block, and q is a coefficient, the value of which may be determined experimentally or mathematically from the condition that the curve approaches Euler's for a long column. The allowable unit load is obtained by dividing the numerator by the safety factor, which is the same as taking the allowable compressive stress instead of the ultimate strength as S_u.

The value of q which is derived from the condition that the unit load must approach Euler's value as a limit is called *Ritter's rational constant.* When $\frac{l}{r}$ is zero in Rankine's formula, the denominator is unity and $\frac{P}{A} = S_u$. Rankine's formula, therefore, satisfies one condition. To make it satisfy the other condition, the value of q must be so chosen that the unit load shall

be the same in Rankine's and in Euler's formulas for large values of the slenderness ratio.

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{S_u}{1 + q\left(\frac{l}{r}\right)^2}. \tag{1}$$

For large values of $\frac{l}{r}$ the second term in the denominator of Rankine's formula is so large relatively that the first term (unity) may be dropped. Then

$$\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{S_u}{q\left(\frac{l}{r}\right)^2}; \tag{2}$$

$$q = \frac{S_u}{\pi^2 E}. \tag{3}$$

This value of q is *Ritter's rational constant.*

Problems

1. Find the value of q for steel having a modulus of elasticity of 29,400,000 and an ultimate compressive strength of 32,000 lb. per sq. in.

Ans. $q = \frac{1}{9{,}080}$.

2. For the steel of Problem 1, find the unit load for slenderness ratios at intervals of 40 from 40 to 200.

Ans.	$\frac{l}{r}$	40	80	120	160	200
	$\frac{P}{A}$	27,200	18,770	12,370	8,380	5,920

3. Compare the last two results of Problem 2 with Euler's equation.
4. Find Ritter's constant for duralumin having a modulus of elasticity of 10.000,000 and an ultimate strength of 50,000 lb. per sq. in.

Ans. $q = \frac{1}{1{,}974}$.

5. Using the constant of Problem 4, find the ultimate load on a duralumin tube, 2.25 in. outside diameter, 0.093 in. thick, and 124.11 in. long.

Ans. $\frac{P}{A} = 3{,}460$ lb./in.^{2}; $P = 2{,}284$ lb.

Ritter's constant in the problems above has been calculated for round ends. The two lower curves of Fig. 249 which extend

to $\frac{l}{r} = 300$ give a comparison with the secant curve for which the eccentric ratio is 0.25. The Rankine curve with Ritter's constant gives lower values of the unit stress for values of $\frac{l}{r}$ above 60. Experiments give the same result. If a Rankine curve gives the correct unit load for long columns, it is unnecessarily safe for usual columns. If approximately correct for columns with slenderness ratios below 120, it is unsafe for longer columns.

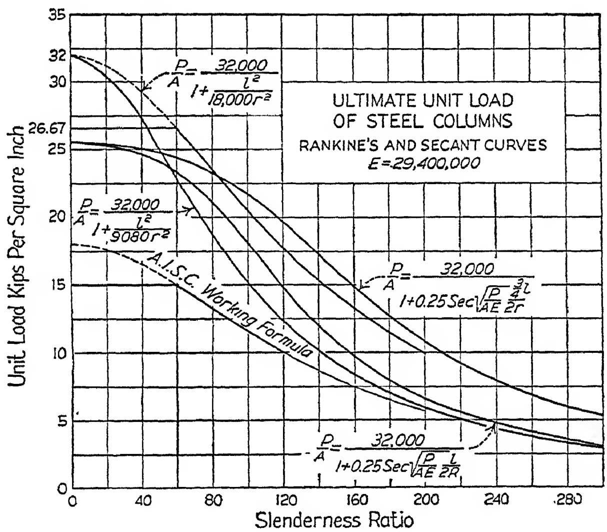


FIG. 249.—Comparison of A.I.S.C. with other curves.

The American Institute of Steel Construction gives the formula

$$\frac{P}{A} = \frac{18{,}000}{1 + \dfrac{l^2}{18{,}000\, r^2}} \qquad \text{Formula XXXIX}$$

with a maximum (stopper) of 15,000 pounds per square inch. The specifications state that this equation may be used with $\frac{l}{r}$ up to 120 for main members and up to 200 for secondary members. For this reason the curve is drawn light from $\frac{l}{r} = 120$ to $\frac{l}{r} = 200$.

Problems

6. Find $\frac{P}{A}$ by the A.I.S.C. formula for values of $\frac{l}{r}$ at intervals of 20 from 20 to 100. Compare with handbook.
7. Solve Problem 1 of Art. 177 by the A.I.S.C. formula.

   Ans. $\frac{P}{A} = 10{,}044.$ $P = 93{,}000$ lb.
8. Solve Problem 2 of Art. 177 by the A.I.S.C. formula.

Figure 249 gives the curve of the equation

$$\frac{P}{A} = \frac{32{,}000}{1 + \frac{l^2}{18{,}000\, r^2}}. \tag{4}$$

This is the American Institute of Steel Construction equation multiplied by $^{16}/_{9}$ to give the ultimate load and correspond with the secant curves of Figs. 241 and 248. A comparison with the secant curve shows that this Rankine formula would be unsafe for round ends such as the connecting rod of an engine. Round-end columns are not used in structures. A comparison with a curve for hinged ends (not drawn in Fig. 249) will show that the American Institute of Steel Construction formula is close at slenderness ratios of about 120 but is always higher. Figure 249 shows the secant curve for fixed ends, using three-fourths of the entire column as the free length, as recommended by the American Society of Civil Engineers Column Committee. The American Institute of Steel Construction curve is slightly above the secant curve for values of $\frac{l}{r}$ below 84, and safely on the other side for the remainder of its length. It would seem that the safety factor of the American Institute of Steel Construction formula for pin-end columns is rather low.

Problems

9. Plot the curve of the A.I.S.C. formula for ultimate strength of 32,000 lb. per sq. in. [Eq. (4)], and plot the secant curve for hinged ends from Table XXVII for comparison.
10. On the figure of Problem 9, plot the curve of Eq. (1) of Art. 174.
11. Calculate the unit load and the total safe load on a 4-in. by 3-in. by $\frac{1}{2}$-in. standard angle as a compression member 6 ft. long. Solve by the A.I.S.C. formula, Chicago building laws, and the A.S.C.E. formula for fixed ends *Ans.* $P = 34{,}350$ lb.; 26,406 lb.; 38,470 lb.

12. Solve Problem 11 for a length of 8 ft.

Ans. P = 26,000 lb.; 30,470 lb.; 18,875 lb.

13. Find the total safe load on a 12×12 65 lb. wide-flange section as a column 25 ft. 2 in. long by A.I.S.C. formula and Chicago building laws. *Ans.* P = 221 kips; 172 kips.

14. A 4-in. by 4-in. 10.5-lb. T-section, with ends welded to the beams, is 10 ft. long. Find the total safe load by A.I.S.C. and A.S.C.E. equations.

Ans. P = 25,770 lb.; 30,200 lb.

15. Solve Problem 14 for a length of 13 ft. 10 in.

16. Solve Problem 14 for a length of 20 ft. by a suitable equation.

Ans. P = 6,530 by Euler's equation with safety factor 2.

17. A 6-in. by 3½-in. 29.4-lb. Z-bar is used as a compression member 10 ft. 6 in. long. Find the total safe load by Chicago building laws and A.I.S.C. formula. *Ans.* P = 44,100 lb.; 66,280 lb.

18. Find the total safe load on a 12×8 40-lb. wide-flange beam, 16 ft. long with fixed ends, by Chicago building laws, A.I.S.C. formula, and A.S.C.E. formula. *Ans.* P = 137,200 lb. by A.I.S.C. formula.

19. A column with cover plates is made of one wide-flange beam, 14×16. core —CB 146–320 lb. or 14 WF 320 core section, and two 1-in. by 18-in. plates which are riveted or welded to the flanges (see handbook). Taking the properties of the beam from the handbook, calculate the moment of inertia and radius of gyration of the section for each principal axis.

Ans. A = 130.12 sq. in.; axis perpendicular to web, I = 6999.5 in.^{4}; r^2 = 53.792; r = 7.334 in.; axis parallel to web, I = 2,607.1 in.^{4}; r^2 = 20.0361; r = 4.476 in.

20. Find the total safe load of the column of Problem 19 for a length of 40 ft. by the A.I.S.C. formula and compare with the handbook.

179. Timber Columns.—The building laws of Chicago specify the straight-line equation

$$\frac{P}{A} = C\left(1 - \frac{l}{80\,D}\right), \tag{1}$$

in which l is the length in inches, D is the least dimension in inches, and C is the allowable compressive stress parallel to the grain of a short block of the material. In the problems which follow take C from the handbook for locations which are continuously dry.

Problems

1. Find the total safe load on a 6-in. by 6-in. select oak post, 12 ft. long.

Ans. $P = 36 \times 700 = 25{,}200$ lb.

2. Solve Problem 1 if the post is made of common-grade, dense southern yellow pine. *Ans.* P = 25,830 lb.

3. Find the dimensions of a square post, 15 ft. high, to carry a load of 60,000 lb. The material is common grade of dense coast-region Douglas fir.
Ans. 8.85 inches square.
4. A rectangular post of select redwood is 12 in. wide and 12 ft. long. What must be its other dimension to carry 100,000 lb.?
Ans. Not less than 10.13 in.
5. Solve Problem 4 for a load of 120,000 lb.

Relatively long timber struts, such as were used in the early biplanes, may be designed by Euler's formula. Experiments have shown that the ultimate strength of western-spruce* struts, tested with round ends, is given closely by the equation

$$\frac{P}{A} = \frac{16,000,000}{\left(\frac{l}{r}\right)^2}, \tag{2}$$

for slenderness ratios greater than 100.

Problems

6. Find the total safe load with a safety factor of 4 for a post of longleaf yellow pine which is 6 in. square and 20 ft. long, calculated as a strut with round ends, if $E = 1,200,000$. Use Formula XXXI. *Ans.* $P = 5,552$ lb.
7. Find the total safe load with a safety factor of 3 for a cylindrical post which is 4 in. in diameter and 12 ft. high, if $E = 1,000,000$ lb. per sq. in.
8. Solve Problem 7 if the post is hollow with inside diameter 2 in.

(*The remainder of Art.* 179 *may be omitted.*)

The formula of the Forest Products Laboratory, U. S. Department of Agriculture, for timber columns of rectangular section, which has been adopted by the American Society for Testing Materials,† is

$$\frac{P}{A} = S\left(1 - \frac{1}{3}\left(\frac{l}{K\,d}\right)^4\right), \tag{3}$$

in which S is the safe unit stress in compression parallel to the grain for short columns, l is the unsupported length in inches, d is the least transverse dimension in inches, and K is the value of $\frac{l}{d}$ at the point at which the curve of Equation (3) becomes tangent to Euler's curve with a safety factor of 3. If $\frac{P}{A} = y$ and $\frac{l}{d} = x$,

$$y = S - \frac{S\,x^4}{3\,K^4}; \tag{4}$$

$$\frac{dy}{dx} = -\frac{4\,S\,x^3}{3\,K^4}. \tag{5}$$

* *Tech. Paper* No. 152, Bureau of Standards, p. 20.
† A.S.T.M. Standards, 1930, Part II, Non Metallic Materials, p. 798.

$$y = \frac{\pi^2 E}{3\left(\frac{l}{r}\right)^2} = \frac{\pi^2 E}{36\left(\frac{l}{d}\right)^2} = \frac{\pi^2 E}{36\,x^2}; \tag{6}$$

$$\frac{dy}{dx} = -\frac{\pi^2 E}{18\,x^3}. \tag{7}$$

$$S - \frac{S\,x^4}{3\,K^4} = \frac{\pi^2 E}{36\,x^2};\quad 36\,S\,x^2 - \frac{12\,S\,x^6}{K^4} = \pi^2\,E. \tag{8}$$

$$\frac{4\,S\,x^3}{3\,K^4} = \frac{\pi^2 E}{18\,x^3};\quad \frac{12\,S\,x^6}{K^4} = \frac{\pi^2 E}{2}. \tag{9}$$

From Equations (8) and (9),

$$36\,S\,x^2 = \frac{3\,\pi^2\,E}{2};$$

$$x^2 = \frac{\pi^2 E}{24\,S};\quad x' = \frac{\pi}{2}\sqrt{\frac{E}{6\,S}};$$

$$y = \frac{\pi^2 E}{36\,x^2};\quad y' = \frac{2\,S}{3},$$

in which x' and y' are the coördinates of the point of tangency. When $y' = \frac{2\,S}{3}$ is substituted in Equation (4), $K = x'$.

Problems

(*Refer to tables in A.I.S.C. Handbook or Carnegie Pocket Companion.*)

9. Find the constant K for select redwood, continuously dry, if $E = 1{,}200{,}000$. *Ans.* $K = 22.2$.

10. A.S.T.M. tables give $K = 27.3$ for common-grade southern yellow pine, continuously dry. Find E. *Ans.* $E = 1{,}600{,}000$.

11. If $K = 24.8$ for select oak, continuously dry, what is E? *Ans.* $E = 1{,}500{,}000$.

12. If $K = 22.6$ for dense yellow pine, what is E? *Ans.* $E = 1{,}600{,}000$.

13. If $K = 25.3$ for common-grade dense Douglas fir, what is E? *Ans.* $E = 1{,}600{,}000$.

14. Find the total safe load on a 6-in. by 6-in. post 10 ft. long, made of select redwood for a location which is continuously dry.

$$\textit{Ans.}\ \frac{P}{A} = 1{,}000\left(1 - \frac{20^4}{3 \times 22.2^4}\right) = 780\ \text{lb./in.}^2; P = 28{,}080\ \text{lb.}$$

15. Solve Problem 14 for a length of 15 ft.

$$\textit{Ans.}\ P = \frac{36 \times 9.87 \times 1{,}200{,}000}{36 \times 30 \times 30} = 13{,}160\ \text{lb.}$$

16. Find the total safe load on a 10-in. by 12-in. post of select redwood 18 ft. 6 in. long by Eq. (3), by Eq. (6).

17. Solve Problems 14 and 16 by Chicago building laws. *Ans.* 27,000 lb.; 86,700 lb.

For solid cylindrical columns of radius d the radius of gyration is $\frac{d}{4}$, while the radius of gyration of a rectangular section is $\frac{d}{\sqrt{12}} = \frac{d\sqrt{3}}{6} = 0.2887\,d$.

The ratio of the radius of gyration of a circle of diameter d to the radius of gyration of a rectangle of side d is $0.2887 \div \frac{1}{4} = 1.155$. The value of $\frac{l}{d}$ for a cylindrical column must be multiplied by 1.155 to give a slenderness ratio corresponding to that of a rectangular post.

Problems

18. Find the total safe load on a cylindrical post of select dry oak 10 in. in diameter and 15 ft. long.

$$\textit{Ans. } P = 78.54 \times 1{,}000 \left(1 - \frac{(18 \times 1.155)^4}{3 \times 24.8^4}\right) = 65{,}600 \text{ lb.}$$

19. If the post of Problem 18 is hollow with inside diameter 6 in., what is the factor by which $\frac{180}{10}$ must be multiplied to get the $\frac{l}{d}$ to be used in Eq. (3)? *Ans.* $0.2887 \div \frac{1}{4}\sqrt{{}^{34}\!/_{25}} = 0.99$.

180. Cast-iron and Duralumin Columns.—Cast-iron columns are seldom used in structures, although there may be locations where the ability of cast iron to resist corrosion may make it desirable. Modern methods of centrifugal casting produce pipe of uniform thickness and quality which may be used with confidence. One established equation for cast iron is

$$\frac{P}{A} = 9{,}000 - 40\frac{l}{r}, \tag{1}$$

with a maximum slenderness ratio of 70. This is conservative and is recommended as a working formula.

An extensive series of experiments on duralumin, made by S. W. Thompson at McCook Field* gave an average modulus of elasticity of 10,987,000, an average proportional elastic limit in compression of 29,790, and an average yield point in compression of 31,730 pounds per square inch. The average crushing strength of 3-inch specimens was 55,580 pounds per square inch. The columns were tested on knife-edge supports, which were carefully centered to give the round-end condition with very small eccentricity. A straight line drawn through the experimental points to the axis of zero slenderness ratio gives approximately 48,000 pounds per square inch as the ultimate compressive stress of short columns. From these data the straight-line equation for the ultimate strength of round-end columns becomes

* Column, Crushing, and Torsional Strength of Duralumin Tubing, *Air Service Information Circ.*, vol. 5, No. 470, July 1, 1924.

$$\frac{P}{A} = 48{,}000 - 389\,\frac{l}{r}, \tag{2}$$

which is tangent to Euler's curve at $\frac{l}{r} = 82.3$. Mr. Thompson plotted the straight line

$$\frac{P}{A} \doteq 48{,}000 - 400\,\frac{l}{r} \tag{3}$$

with Euler's extension and found this curve to fit the experiments very closely.

The structural handbook of the Aluminum Company of America gives for round-end columns made of alloys 17ST and 25ST the working formula

$$\frac{P}{A} = 15{,}000 - 123\,\frac{l}{r}, \tag{4}$$

and

$$\frac{P}{A} = \frac{33{,}000{,}000}{\left(\frac{l}{r}\right)^2} \tag{5}$$

as Euler's extension.

If Equation (2) is divided by a safety factor of 3.2, the result is almost identical with Equation (4). Equation (5) is Euler's equation with a safety factor of 3.2 for material with a modulus of elasticity of 10,700,000. Equation (4) is valid to $\frac{l}{r} = 82$.

The Aluminum Company handbook gives

$$\frac{P}{A} = 15{,}000 - 61.5\,\frac{l}{r} \tag{6}$$

for fixed-end columns with slenderness ratios below 165, and

$$\frac{P}{A} = \frac{132{,}000{,}000}{\left(\frac{l}{r}\right)^2} \tag{7}$$

for more slender columns. Equations (6) and (7) assume that the ends of the columns are absolutely fixed, and that the free length is one-half the total length. The American Society of

Civil Engineers' Column Committee takes $\frac{3\,l}{4}$ as the free length of a "fixed-end" column.

Problems

1. A tube tested at McCook Field had an outside diameter of 2.250 in. and a thickness of 0.093 in. Its length was 124.11 in. Find the slenderness ratio. Calculate the ultimate unit load by Euler's formula, using E = 10,987,000. Compare with the results of the experiment which gave an ultimate total load of 2,470 lb. Compare Problem 5 (Art. 178).

   Ans. $\frac{l}{r} = 162.59$: $\frac{P}{A} = ?$

2. A piece of the rod of Problem 1 was 32.01 in. long. The ultimate total load was 22,480 lb. Find the ultimate unit load from the test and from Eq. (2).

   Ans. $\frac{P}{A} = 35{,}670$ from test; $\frac{P}{A} = 31{,}690$ from equation.

181. Selection of a Column for a Given Load.—The problem of designing or selecting a column of a given length to carry a given load varies with the form of the section. If the sections which are considered are all similar figures, the radius of gyration varies as the first power and the area varies as the second power of any dimension. For a circle of radius a, for instance, $r = \frac{a}{2}$ and $A = \pi a^2$. For a square of side b, $r = \frac{b}{\sqrt{12}}$ and $A = b^2$. A problem of this class may be solved algebraically for the unknown dimension. Euler's equation gives the fourth power of this unknown quantity (since the moment of inertia varies as the fourth power). The required result is obtained by extracting the square root of a square root. A straight-line formula gives a quadratic equation. Rankine's formula gives a quadratic equation in terms of the square of the unknown dimension. Any one of these equations may be easily solved.

Example I

A square steel bar, as a column 15 ft. long, carries a load of 15,000 lb. Find its dimensions as a round-end column with a factor of safety of 3, using E = 29,400,000.

If the factor of safety is applied to the total load, the calculation is made for a total load of 45,000 lb. It is evident that the required area will not be greater than 10 sq. in., which makes the radius of gyration small enough to use Euler's formula.

$$\frac{45{,}000}{b^2} = \frac{\pi^2 \times 29{,}400{,}000}{\dfrac{180 \times 180 \times 12}{b^2}};$$

$$b^4 = \frac{45{,}000 \times 180 \times 180 \times 12}{\pi^2 \times 29{,}400{,}000} = \frac{18^2 \times 90}{\pi^2 \times 49};$$

$$b^2 = \frac{54\sqrt{10}}{7\,\pi} = 7.765 \text{ in.}^2$$

$$b = 2.787; \frac{P}{A} = \frac{15{,}000}{7.765} = 1.932 \text{ lb. per sq. in.}$$

Example II

Solve Example I for a solid circular rod of radius r.

Ans. $r^4 = 6.3976$ in.^{4}; $r^2 = 2.5293$ in.^{2}; $r = 1.5904$ in.; $\frac{P}{A} = 1{,}889$ lb./in.^{2}; slenderness ratio $= 226.3$.

Example III

By Chicago building laws, find the diameter of a steel cylinder, 5 ft. long, to carry 60,000 lb.

$$16{,}000 - \frac{70 \times 60 \times 4}{d} = \frac{60{,}000 \times 4}{\pi\, d^2}.$$

$$d = 2.772 \text{ in.; slenderness ratio} = 86.6; \frac{P}{A} = 9{,}940 \text{ lb./in.}^2$$

Problems

1. Solve Example III for a load of 80,000 lb. Check.
2. Solve Example III for a hollow cylinder with outside diameter twice the inside diameter.
3. Solve Example III by Formula XXXIII of Art. 173. Check.

   Ans. $d^2 = 6.0528$; $d = 2.4602$; $\frac{P}{A} = 12{,}620$ lb./in.^{2}
4. Solve Example III by the A.I.S.C. equation. Check.

$$\frac{240{,}000}{\pi\, d^2} = \frac{18{,}000}{1 + \dfrac{16 \times 3600}{18{,}000\, d^2}}; \quad \frac{240}{\pi\, d^2} = \frac{18}{1 + \dfrac{3.2}{d^2}};$$

$$3\,\pi\, d^4 - 40\, d^2 - 128 = 0.$$

   Ans. $d^2 = 6.374$ in.^{2}; $d = 2.525$ in. $\frac{P}{A} = 11{,}980$ lb./in.^{2}
5. Solve Example III by the A.R.E.A. formula. *Ans.* $d = 2.692$ in.
6. Check Problem 5 by substitution in the formula which was used.

Since the sections of rolled shapes of different sizes are not similar figures, the selection of a column must be made by trial and error. The handbooks give the strength of columns of various shapes and of fabricated sections, which have been

calculated by the American Institute of Steel Construction formula. When another formula is specified, the approximate size may be selected from the table, and the calculation completed by trial.

Example IV

Select a wide-flange section for a column 20 ft. long to carry 240,000 lb. by Chicago building laws.

A 12 × 12 65-lb. WF (or CB) section carries 255 kips. Its least radius of gyration is 3.02 in., which substituted in the straight-line equation gives

$$\frac{P}{A} = 10{,}438 \text{ lb. per sq. in.; } 19.11 \times 10.438 = 200{,}000, \text{ approximately.}$$

The 12 × 12 79-lb. section carries a load about 20 per cent greater.

$$\frac{P}{23.22} = 16{,}000 - \frac{16{,}800}{3.05}; \quad P = 243.6 \text{ kips.}$$

Problems

7. Solve Example IV by the A.R.E.A. formula [Eq. (11) of Art. 177].

Ans. 12 × 12 79-lb. section.

8. Solve Example IV by the A.S.C.E. formula for fixed ends.

Ans. A 12 × 12 65-lb. section carries 255 kips.

9. Select a wide-flange section 30 ft. long, to carry 400,000 lb. by A.R.E.A. formula. *Ans.* A 12 × 12 161-lb. section carries 444 kips.

10. Solve Problem 3 for a length of 25 ft.

11. Solve Problem 3 by Chicago building laws.

12. Select a standard steel pipe, 20 ft. long, to carry 100,000 lb. by Chicago building laws and by A.R.E.A. formula.

Ans. 10 in.; 0.279 in. thick.

182. I-Beam Failure by Buckling the Compression Flange.—The compression flange of a beam may fail as a column by lateral deflection. In the calculation of this failure, the bending stress in the outer fibers S_c is taken as the unit load $\frac{P}{A}$ of the column equations. Unless the moment is constant, the unit stress in a simply-supported beam increases from the end to the middle, and the compression flange of one-half the beam is equivalent to a column which is fixed at one end, free at the other, and carries a distributed load. If the load on the beam is concentrated at the middle, this compressive force, which is transmitted from the tensile portion of the section by horizontal shear in the web, increases uniformly from the end to the middle. For a uniformly distributed load on the beam, the compressive force in the flange changes more slowly at the middle. For every loading, except

that which causes a constant bending moment, the compressive stress in the flange, which takes the place of the unit load in column formula, is not constant at its maximum value. For this reason, a column formula to take care of the deflection of the compression flange may be less conservative than would be required for uniform direct compression.

The American Railway Engineering Association Specifications of May, 1931, state that the stress per square inch in the compression flange of an I-beam shall not exceed

$$S_c = 16{,}000 - 150\,\frac{l}{b}, \tag{1}$$

in which l is the length of the unsupported flange, between lateral connections or knee braces, and b is the flange width. Since the flange is regarded as a rectangular area, its radius of gyration is $\dfrac{b}{\sqrt{12}}$ and $150\,\dfrac{l}{b}$ is equivalent to $43.3\,\dfrac{l}{r}\cdot$ The fact that the compressive force on the flange does not have its maximum value over the entire length accounts fully for the difference between Equation (1) and the straight-line formulas of the preceding article.

The American Institute of Steel Construction recommends

$$S_c = \frac{20{,}000}{1 + \dfrac{l^2}{2{,}000\,b^2}} \tag{2}$$

with a maximum of 18,000 pounds per square inch for unsupported flanges. The denominator $2{,}000\,b^2$ is equivalent to $24{,}000\,r^2$.

Problems

(Neglect the weight of the beam)

1. Calculate the maximum distance between lateral supports for a 15-in. 60.8-lb. standard I-beam if the maximum fiber stress is 15,000 lb. per sq. in. Solve by Eq. (1) and by Eq. (2). *Ans.* 40 in.; 12.9 ft.

2. Solve Problem 1 for a maximum stress of 13,500 lb. per sq. in.

3. Find the total safe load, uniformly distributed, on an 18-in. 54.7-lb. standard I-beam which is 15 ft. long and is supported at the ends with no lateral supports. Solve by both equations above.
Ans. W = 45,180, A.R.E.A.; W = 54,190, A.I.S.C.

4. Find a standard I-beam for a span of 25 ft. to carry a load of 10,000 lb. 10 ft. from each end with a maximum stress of 14,000 lb. per sq. in. How many stiffeners will be needed? Solve by A.R.E.A. formula.
Ans. 18 in. 54.7 lb. or 15 in. 70 lb.; three stiffeners.
5. Solve Problem 4 by A.I.S.C. equation. *Ans.* Two stiffeners required.
6. A 10-in. 40-lb. standard I-beam is used as a cantilever 12 ft. long to carry a distributed load. What is the maximum load if the lower flange has no lateral supports?

183. Column Failure by Flange Buckling at Edge.—A column as a whole may be sufficiently rigid to carry the required load but may begin to fail by lateral buckling of the edge of a thin flange.

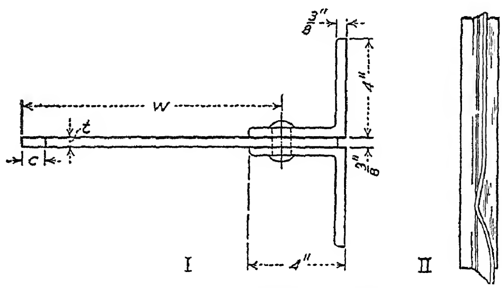


FIG. 250.—Flange buckling.

Figure 250, I, shows a fabricated T-section which was used by the American Society of Civil Engineers Committee to study this kind of failure.*

A 3⁄8-inch plate is riveted to two 4-inch by 4-inch angles. For most of the experiments these angles were 3⁄8 inch. Since these bent, the free width w of the plate (which will hereafter be called the flange) is taken from the center of the rivet holes. When 3⁄4-inch angles were used, they were found to be so rigid that the free width of the flange was measured from the toe to the angle. The free outstanding widths varied from 4.29 inches to 11.17 inches. The length of all test pieces was 75 inches.

If a small strip of width c is regarded as cut off from the edge of the flange, this strip would form a column with a slenderness ratio of 693, which would carry a very small load. The deflection of the strip is restrained by the remainder of the plate, which acts as a horizontal cantilever fixed at the angles. If the flange is too wide, the load carried by this strip at the edge fails to offset the damage done by its bending moment, and the ultimate load

* *Trans.* A.S.C.E.; vol. 98, Part III, 1933, p. 1435.

on the section is not increased by the increased area. If $\frac{w}{t}$ is the ratio of the outstanding width to the thickness of the flange, these tests showed that no increase in total load was secured when this ratio was increased from 15 to 20. From these tests, it is evident that a free flange width greater than fifteen times the thickness represents a waste of material

Figure 250, II, shows how a flange may buckle under load as a column. The free edges of the flanges of a wide-flange beam or H-beam may buckle in a similar way. The columns of Table XXV were made of one 10-inch plate and four 4-inch by 3-inch by $\frac{3}{8}$-inch angles. The ratio of the free flange length to the thickness was $\frac{29}{3}$. "All pin-end columns of slenderness ratio 25 and 50 failed by triple flexure with *buckling of the flanges.*" The square-end columns did not fail in the same way because the *ends* of the outer strips of the flanges were kept from lateral movement and from rotation by the compression heads of the machine, while the only restraint on a strip of a pin-end column was the stiffness of the flange.

Deflection of one flange of a column produces a torque about the axis of the column which tends to cause the other flanges to deflect in the same direction. The result is that the entire member is twisted. In a testing machine with the ends fixed, the maximum angle of twist is near the middle of the length. A column fixed at the bottom with a free-load at the top twists with the maximum displacement at the top. This sometimes happens to an oil-well derrick, regarded as a single-latticed column.

184. Web Crippling of Beams.—Most specifications designate 12,000 pounds per square inch as the allowable shearing stress in beams. This stress is computed by the approximate equation (Art. 142)

$$S_s = \frac{V}{t\,d}, \tag{1}$$

in which V is the total vertical shear, t is the thickness of the web, and d is the total depth. The product $t \times d$ is sometimes designated by A, meaning the area of the web regarded as extending the entire depth of the beam.

Relatively short beams reach their allowable shearing stress before reaching their allowable bending stress.

Example I

What is the minimum length of a 15-in. by 6-in. 65-lb. standard I-beam for which bending governs? The beam is uniformly loaded, simply-supported, and the allowable bending stress is 18,000 lb. per sq. in.

$$\text{Max. } M = 84.3 \times 18{,}000 = 1{,}517{,}400 \text{ in.-lb.}$$
$$\text{Max. } W = 0.672 \times 15 \times 12{,}000 \times 2 = 241{,}920 \text{ lb.}$$
$$\frac{W\,l}{8} = M; \quad l = 50.2 \text{ in.}$$

Bending governs for lengths greater than 50.2 in.; shear governs for smaller lengths.

Problems

1. Find the total safe load on a 12-in. by 5-in. 35-lb. standard I-beam which is 5 ft. long, simply-supported, and carries a concentrated load at the middle. *Ans.* $P = 45{,}360$ lb.
2. Solve Problem 1 if the load is 2 ft. from one end.

Figure 251 applies to another form of web crippling.* It was shown in Art. 22 that vertical shear causes compressive and tensile stresses, which are a maximum at 45 degrees with the vertical, where each is equal to the applied shearing stress. The web of an I-beam subjected to vertical shear may be regarded as made up of a series of parallel columns with fixed ends, such as $F\,G$ of Fig. 251. Each column may be assumed to be 1 inch wide. Its thickness is t the thickness of the web. If c is the vertical distance between the flanges, the length of this column is $\sqrt{2}\,c$. Since the radius of gyration of a rectangular section is $\frac{t}{\sqrt{12}}$,

$$\text{Slenderness ratio} = \frac{\sqrt{2}\,c}{\frac{t}{\sqrt{12}}} = \frac{2\,c\,\sqrt{6}}{t}. \tag{2}$$

It is customary to regard the web as perfectly fixed at the ends with a load of very little eccentricity. Under these conditions, the free length is one-half of l and the equivalent slenderness ratio

* *Bull.* No. 86, University of Illinois Engineering Experiment Station on Strength of Webs in I-beams and Girders, by Profs. H. F. MOORE and W. M. WILSON, is an extensive theoretical and experimental study. Equation (3) is in the form given in this bulletin.

Web Buckling in Steel Beams, by Prof. I. M. LYSE and H. J. GODFREY, *Proc. A.S.C.E.*, February, 1934.

is $\frac{\sqrt{6}\,c}{t}$. Euler's formula for fixed ends then becomes

$$\frac{P}{A} = s_c = s_s = \frac{\pi^2 E}{6\left(\frac{c}{t}\right)^2} = \frac{1.64\,E}{\left(\frac{c}{t}\right)^2}, \tag{3}$$

which gives the ultimate strength of the web in resistance to diagonal buckling.

Many formulas for diagonal buckling have been given in handbooks and specifications, which differ greatly as to the allowance which should be made for this possibility of failure.* The

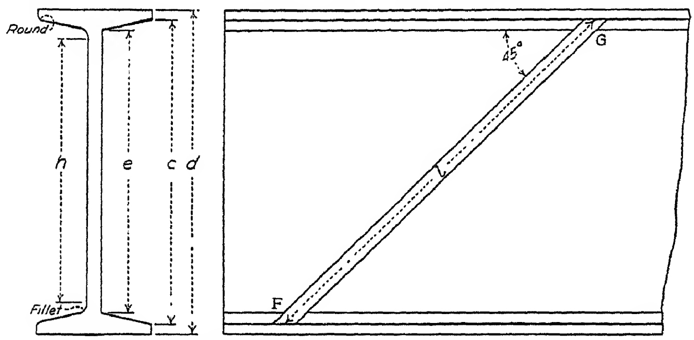


FIG. 251.—I-beam web as a diagonal column.

American Institute of Steel Construction formula, now largely specified, is

$$s_s = s_c = \frac{18{,}000}{1 + \frac{c^2}{7{,}200\,t^2}}. \tag{4}$$

This is an equation of the Rankine form, although the constant is very different from any which Rankine used, since the term

$$\frac{c^2}{7{,}200\,t^2} \text{ is equivalent to } \frac{l^2}{172{,}800\,r^2}.$$

From the American Institute of Steel Construction handbook of 1927, this formula is found to have been based on the following: (1) a unit compressive stress of 18,000 pounds per square inch which had been specified as the allowable bending stress in steel beams; (2) a specified shearing stress of 12,000 pounds per square

* See paper by R. FLEMING in *Engineering News*, April 6, 1916, and paper by HENRY KERCHER in *Engineering News*, May 4, 1916.

inch; (3) the experimental fact that failure does not occur by diagonal buckling unless $\frac{c}{t}$ is greater than 60. A formula of the Rankine type was desired which satisfies $s_c = 18{,}000$ when $\frac{c}{t} = 0$, $s_s = s_c = 12{,}000$ when $\frac{c}{t} = 60$. The safety factor of the first condition is about 2. Substitution in Equation (3) gives

$$s_s = s_c = \frac{1.64 \times 29{,}400{,}000}{36{,}000} = 13{,}390 \text{ lb. per sq. in.}$$

as the ultimate strength when $\frac{c}{t} = 60$. The safety factor of the second condition is about 1.1. When $\frac{c}{t} = 100$, Equation (4) gives 7,530 pounds per square inch as the *allowable* shearing stress and Equation (3) gives 4,800 pounds per square inch as the *ultimate* shearing strength.

For $\frac{c}{t} = 160$, the handbooks give 3,950 pounds per square inch as the allowable stress, while Equation (3) gives 1,880 as the ultimate stress.

The value given to c in these equations varies with the handbooks. The American Institute of Steel Construction takes the maximum plane portion of the web between fillets (the distance h of Fig. 251) and the distance between the toes of the angles for plate girders (page 146, handbook). For I-beams, many take the distance e of Fig. 251 and page 134 of the "Pocket Companion." For wide-flange beam which have rectangular flanges, e is the same as c.

A working formula may be obtained from Equation (3) by dividing by a suitable safety factor. If 1.64 is taken as this factor,

$$s_s = \frac{E}{\left(\frac{c}{t}\right)^2}. \tag{5}$$

When $s_s = 12{,}000$ pounds per square inch and $E = 29{,}400{,}000$ for steel, Equation (5) gives $\frac{c}{t} = \sqrt{2{,}450} = 50$, nearly. Equation (5) may be used for the working shearing stress for values

of $\frac{c}{t}$ greater than 50. For smaller ratios of c to t, the allowable shearing stress is 12,000 pounds per square inch.

Problems

3. Find the allowable shearing stress on a 33 × 11½ 125-lb. wide-flange section by Eq. (5).

Ans. $c = 33 - 1.61 = 31.39$ in.; $\frac{c}{t} = 55.07$;

$s_s = 29{,}400{,}000 \div 3{,}032.7 = 9{,}695$ lb./in.²

4. Find the total load, uniformly distributed, on the beam of Problem 3. Find the maximum length which can carry this load.

Ans. $V = 182{,}360$ lb.; $W = 364{,}720$ lb.; $l = 152.1$ in.

5. Solve Problem 3 for a 30 × 10½ 108-lb. wide-flange section.

6. Find the total safe load for the beam of Problem 3. Compare with the handbook tables.

7. Solve Problems 3 and 4 for a 36 × 12 160-lb. wide-flange section.

The web of an I-beam or plate girder may fail as a column over a support or under a concentrated load. In Fig. 252, the

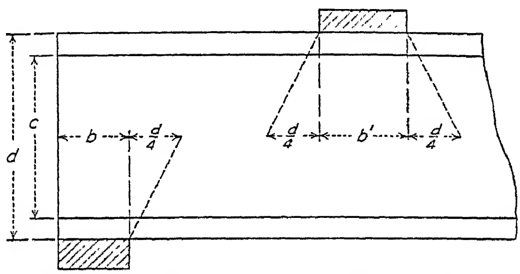


FIG. 252.—I-beam web as a vertical column.

end support is b wide. The material above this support acts as a vertical column of width b. However, the web at the right, which is not directly over the support, transmits a part of the load by diagonal compression and also serves to stiffen the material directly over the load. Experiments have shown that the effective width for the end reaction is $b + \frac{d}{4}$. For a concentrated load where the web is stiffened from both sides, the effective width is $b' + \frac{d}{2}$. The column formula used by the handbooks is Formula XXXIX of Art. 178 modified for absolutely fixed end. This is

$$\frac{P}{A} = \frac{18,000}{1 + \dfrac{d^2}{6,000\ t^2}} \tag{6}$$

with a maximum of 15,000 pounds per square inch.

Example II

Find the unit load for web buckling at support or concentrated load for a 16 × 7 40-lb. wide-flange beam. Find the maximum reaction at the end if the width of the support is 3.5 in. Compare with the handbooks.

$$\frac{P}{A} = \frac{18,000}{1 + \dfrac{256}{6,000 \times 0.09429}} = \frac{18,000}{1.4527} = 12,390;$$

$$R = 12,390 \times 0.307 \times 7.5 = 28,530 \text{ lb.}$$

Problems

8. Solve Example II for a 20-in. 65.4-lb. standard I-beam.

Ans. $\frac{P}{A} = 18,000 \div \frac{38}{30} = 14,210; \quad R = ?$

9. Solve Example II for a 20-in. 75-lb. standard I-beam.

10. The beam of Example II is 8 ft. long, center to center of supports. It carries a load 40 in. from the left support, which produces a maximum bending stress of 16,000 lb. per sq. in. Neglecting the weight of the beam, find the width of the load and the width of each end bearing.

Ans. $b' = 11.61 - 8 = 3.61$ in.

CHAPTER XVII

COMBINED STRESS

185. Resultant of Shearing and Tensile Stress.—Figure 253 represents a block of breadth dx, height dy, and length l, subjected to tensile stresses of intensity s_t perpendicular to the left and right vertical faces, to shearing stresses of intensity s_s parallel to these faces, and to shearing stresses of equal intensity in the top and bottom faces. The shear on the left face is upward and

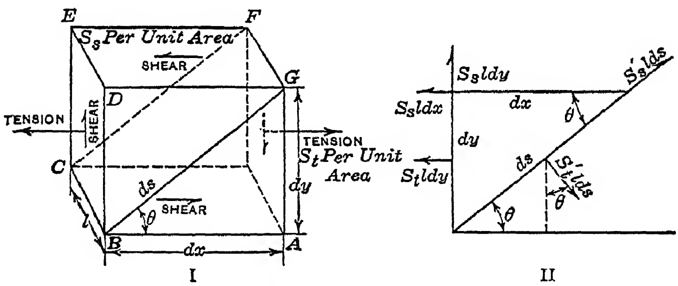


FIG. 253.—Combined shear and tension.

on the top face toward the left. It is desired to find the unit shearing stress parallel to the diagonal $B\,G$ or $C\,F$ and the unit tensile stress normal to the plane $B\,C\,F\,G$. The block may be considered as divided by the plane $B\,C\,F\,G$ into two equal triangular prisms. The prism which lies to the left of this plane will be taken as the free body in equilibrium. The forces which act on this free body are five in number:

Total tension $s_t l\,dy$, toward the left, applied at center of $B\,C\,E\,D$.

Total shear $s_s l\,dy$, upward, applied at center of $B\,C\,E\,D$.

Total shear $s_s l\,dx$, toward the left, applied at center of $D\,E\,F\,G$.

Total shear on $B\,C\,F\,G$, parallel to $B\,G$, applied at center of $B\,C\,F\,G$.

Total tension normal to $B\,C\,F\,G$ at its center.

The unknown unit shearing stress in the plane $B\,C\,F\,G$ will be represented by s_s' and the unknown unit tensile stress by s_t'. The total shear on this plane is then $s_s' l\,ds$, where ds is the length

of the diagonal BG. The total tension on the diagonal plane is $s_t' l\, ds$. The five forces which act on the wedge $BCEDFG$ are represented in a single plane in Fig. 253, II.

The magnitude of the unknown shearing stress s_s' may be found by resolving parallel to the line BG. If θ is the angle between $BCFG$ and the horizontal, the resolution parallel to BG, after dividing by l, is

$$s_t dy \cos\theta + s_s dx \cos\theta - s_s dy \sin\theta = s_s' ds. \qquad (1)$$

When Equation (1) is divided by ds, and $\frac{dx}{ds}$ and $\frac{dy}{ds}$ are expressed in terms of the cosine and sine of θ, the result is

$$s_s' = s_t \sin\theta\cos\theta + s_s(\cos^2\theta - \sin^2\theta); \qquad (2)$$

$$s_s' = \frac{s_t}{2}\sin 2\theta + s_s \cos 2\theta. \qquad (3)$$

The resolution normal to ds gives

$$s_t dy \sin\theta + s_s dx \sin\theta + s_s dy \cos\theta = s_t' ds; \qquad (4)$$

$$s_t' = s_t \sin^2\theta + 2s_s \sin\theta\cos\theta; \qquad (5)$$

$$s_t' = \frac{s_t}{2}(1 - \cos 2\theta) + s_s \sin 2\theta. \qquad (6)$$

These equations apply when the external shearing stresses on the block have the directions of Fig. 253. If the shear is reversed, some of the signs are changed.

Equation (5) may be derived by moments about the upper right corner of Fig. 253, II.

$$s_t' l\, ds \frac{ds}{2} = s_t l\, dy \frac{dy}{2} + s_s l\, dy\, dx; \qquad (7)$$

$$s_t' = s_t \sin^2\theta + 2s_s \sin\theta\cos\theta. \qquad (5)$$

Problems

1. A block is subjected to a horizontal tensile stress of 320 lb. per sq. in. and a horizontal and vertical shearing stress of 200 lb. per sq. in. in the directions shown in Fig. 253. Find the resultant unit shearing stress along the lower side of a plane which makes an angle of 30° with the horizontal toward the right. Solve by Eq. (2) and also by Eq. (3).
2. Solve Problem 1 for the unit compressive stress across the plane at 30° with the horizontal. Use Eq. (5) and also Eq. (6).
3. Solve Problem 1 for an angle of 25° by the equation which requires the least labor. *Ans.* $s_s' = 251.12$ lb./in.²

4. Solve Problem 2 for an angle of 25°.

Ans. $s_t = 57.15 + 153.20 = 210.35$ lb./in.^{2}

5. Given a tensile stress of 780 lb. per sq. in. and a shearing stress of 650 lb. per sq. in., find the resultant shearing and tensile stresses along and across a plane which makes $\tan^{-1} 5/12$ with the horizontal. Solve without tables.

Ans. $s_s' = \dfrac{3{,}600 + 5{,}950}{13} = 734.6$ lb./in.^{2}; $s_t' = 576.9$ lb./in.^{2}

6. Given a horizontal tensile stress of 600 lb. per sq. in. and a horizontal and vertical shearing stress of 400 lb. per sq. in. Find the resultant shearing stress along planes which make angles of 25°, 35°, and 50° with the horizontal.
7. Solve Problem 6 for the compressive stress.
8. Solve Problem 6 for an angle of 35° if the shearing stresses are opposite those of Fig. 253. Make a figure and solve from first principles, using numerical values instead of letters. *Ans.* $s_s' = 145.11$ lb./in.^{2}
9. Solve Problem 8 for the unit tensile stress across the 35° plane.

Ans. $s_t' = 178.48$ lb./in.^{2} compression.

10. Derive the expression for s_t' by moments about the lower left corner of Fig. 253, II.

186. Maximum Resultant Shearing Stress.—The direction which the plane $B\ C\ F\ G$ of Fig. 253 must have, in order that the unit shearing stress in it shall be a maximum, is found by differentiating the expression for s_s' of Equation (3) of Art. 185 with respect to θ:

$$\frac{d}{d\theta} s_s' = s_t \cos 2\theta - 2\, s_s \sin 2\theta = 0 \tag{1}$$

for maximum or minimum.

$$\tan 2\theta = \frac{s_t}{2\, s_s} = \frac{\frac{s_t}{2}}{s_s}. \tag{2}$$

The value of the maximum resultant unit shearing stress may be calculated by substituting in Equation (3) of the preceding article the values of $\cos 2\theta$ and $\sin 2\theta$ when $\tan 2\theta = \dfrac{\frac{s_t}{2}}{s_s}$. To find $\cos 2\theta$ and $\sin 2\theta$ a right triangle may be formed with s_s as the base and $\dfrac{s_t}{2}$ as the altitude (Fig. 254). The angle adjacent to the side s_s is 2θ, the hypotenuse is

$$\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}.$$

$$\cos 2\theta = \frac{s_s}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}, \quad \sin 2\theta = \frac{\frac{s_t}{2}}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}. \tag{3}$$

When these values of $\cos 2\theta$ and $\sin 2\theta$ are substituted in the expression for s_s' and a common factor is divided out, the result is

$$\text{Max. } s_s' = \pm \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}. \qquad \text{Formula XL}$$

A comparison of the equations with Fig. 254 shows that the maximum resultant shearing stress is the hypotenuse of a right

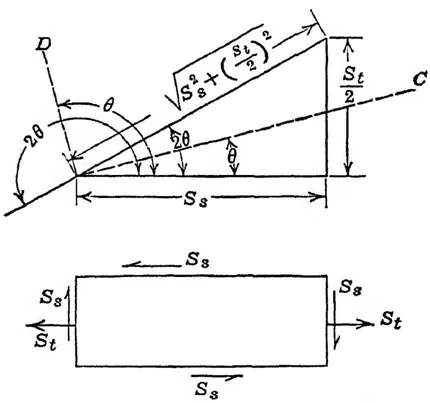


FIG. 254.—Maximum resultant shearing stress.

triangle of which the applied unit shearing stress is the base and one-half the applied unit tensile stress is the altitude. The angle between the maximum resultant shearing stress and the direction of the applied tension is one-half the angle which the hypotenuse of this triangle makes with the applied tension. The broken line through C in Fig. 254 gives the direction of one maximum shearing stress.

For any given tangent there are two angles which differ by 180 degrees; consequently there are two values of 2θ which are 180 degrees apart and two corresponding values of θ which are 90 degrees apart. These correspond to the two values of maximum shear at right angles to each other. The second maximum (or minimum) is along the broken line through D in Fig. 254.

Example

A part of a solid is subjected to a horizontal tensile stress of 400 lb. per sq. in. and a horizontal and vertical shearing stress of 100 lb. per sq. in. Find the direction and magnitude of the maximum resultant unit shearing stress.

$$\tan 2\theta = \frac{200}{100}; \quad 2\theta = 63° \, 26' \text{ or } 243° \, 26';$$
$$\theta = 31° \, 43' \text{ or } 121° \, 43'.$$
$$\text{Max. } s'_s = 100\sqrt{5} = \pm 223.6.$$

Figure 255, I, shows the applied tension and shear of this example. Figure 255, II, shows the maximum resultant shearing stresses which act on a portion of the body from the material outside this portion. The magnitude of each of these four shearing stresses is 223.6 lb. per sq. in.

Figure 255, II, shows the shearing stresses on *one* side of each plane. In Fig. 255, III, the shearing stresses are shown on both sides of the two planes

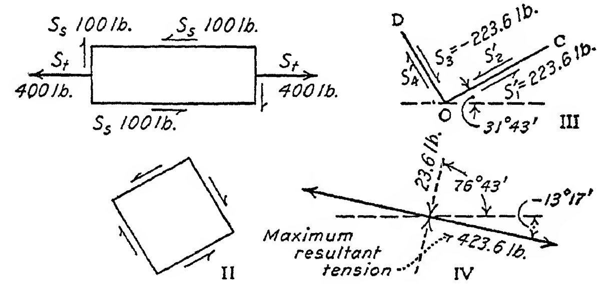


FIG. 255.

which intersect at O. In Fig. 253, s'_s is the shearing stress which the material to the right of the plane $C\,B\,F\,G$ exerts on the material to the left of this plane, and the equations of Art. 185 are based on this arrangement. In Fig. 255, III, s'_1 is the shearing stress which the material to the right of the plane $O\,C$ exerts on the material to the left of the plane. If the positive direction is from O toward C, s'_1 is the *maximum* unit shearing stress. At the plane $O\,D$, at right angles to $O\,C$, s'_3 is the stress which the material to the right of $O\,D$ exerts on the material to the left. Since s'_3 is opposite the positive direction of $O\,D$, it is regarded as negative. This stress, s'_3, is the *minimum* shearing stress of -223.6 lb. per sq. in.

Problems

1. A solid is subjected to a horizontal tensile stress of 1,000 lb. per sq. in. and a horizontal and vertical shearing stress of 1,200 lb. per sq. in., which is upward on the left side of any vertical section as in Fig. 253. Find the magnitude and direction of the maximum unit shearing stress.
Ans. Max. $s_s = 1{,}300$ lb./in.^{2} at 11° 19′.

2. In Problem 1, find the unit shearing stress at 10° with the horizontal and at 15° with the horizontal. *Ans.* $s_s = 1{,}298.6$ and 1.289.2 lb./in.^{2}

3. Solve Problem 2 for an angle of 100°. *Ans.* $s_s = -1{,}298.6$ lb./in.^{2}
4. Make a sketch similar to Fig. 255, II, for the maximum stress of Problem 1, and a sketch similar to Fig. 255, III, for the shearing stresses at 10° and 100° of Problems 2 and 3.
5. Find the maximum resultant shearing stress which is caused by a horizontal tensile stress of 600 lb. per sq. in. and vertical and horizontal shearing stress of 300 lb. per sq. in. Find the angle without writing.
6. A horizontal and vertical shearing stress of 600 lb. per sq. in. and an unknown horizontal tensile stress cause a maximum shearing stress of 680 lb. per sq. in. Find the unknown tensile stress.
7. A horizontal tensile stress of 800 lb. per sq. in. and an unknown horizontal and vertical shearing stress cause a maximum resultant at 40° with the horizontal. Solve for the unknown stresses.
8. Find the maximum resultant shearing stress which is caused by a horizontal tensile stress of 800 lb. per sq. in. and a horizontal and vertical shearing stress of 500 lb. per sq. in.
9. In Problem 8, find the angle at which the resultant shearing stress is zero. *Ans.* 64° 20′.

187. Maximum Resultant Tensile Stress.—From Equation (6) of Art. 185,

$$s_t' = \frac{s_t}{2}(1 - \cos 2\theta) + s_s \sin 2\theta. \tag{1}$$

$$\frac{d}{d\theta}(s_t') = s_t \sin 2\theta + 2\, s_s \cos 2\theta. \tag{2}$$

For the maximum and minimum s_t',

$$\tan 2\theta = -\frac{2\, s_s}{s_t} = -\frac{s_s}{\frac{s_t}{2}} \tag{3}$$

Comparison with Equation (2) of the preceding article shows that the double angle for maximum and minimum tensile stress is normal to corresponding direction for maximum shear, and, consequently, the directions of maximum and minimum tensions are at 45 degrees with the directions of maximum and minimum shear.

The double angle in the second quadrant (Fig. 256) gives

$$\sin 2\theta = \frac{s_s}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}, \quad \cos 2\theta = -\frac{\frac{s_t}{2}}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}},$$

When these values of the sine and cosine of 2θ are substituted in Equation (1), the result is

$$\text{Max. } s_t' = \frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2} = \frac{s_t}{2} + \text{max. } s_s'. \quad \text{Formula XLI}$$

For the double angle in the fourth quadrant, the sine of 2θ is negative and the cosine is positive. When these are substituted in Equation (1), the result is

$$\text{Min. } s_t' = \frac{s_t}{2} - \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2} = \frac{s_t}{2} - \text{max. } s_s'. \qquad (4)$$

Since the maximum unit shearing stress is always equal to or greater than one-half the unit tensile stress, the second term of Equation (4) is never less than the first term and the minimum stress is compressive.

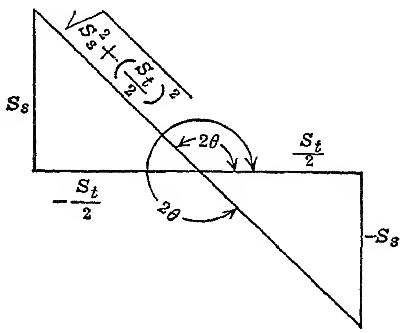


FIG. 256.—Double angle for maximum resultant tensile stress.

Example

Find the magnitude and direction of the maximum unit tensile stress caused by the stresses of the example of the preceding article.

$$\tan 2\theta = -\frac{100}{200} = -0.5; \quad 2\theta = 180° - 26° \, 34' = 153° \, 26';$$

$\theta = 76° \, 43'$ gives the angle of the plane *across which* the unit tensile stress is a maximum. Solving for this maximum tensile stress by Eq. (6) of Art. 185, $s_t' = 200 \times 1.8944 + 100 \times 0.4472 = 423.60$. By Formula XLI max. $s_t' = 200 + 223.6 = 423.6$. This maximum stress makes an angle of $76° \, 43' - 90° = -13° \, 17'$ with the horizontal. The second angle which has a tangent of -0.5 is $333° \, 26'$ or $-26° \, 34'$.

$$\begin{aligned} \cos 2\theta = 0.8944,\ \sin 2\theta &= -0.4472 \\ 0.1056 \times 200 &= 21.120 \\ -0.4472 \times 100 &= -44.72 \\ \hline \text{min. } s_t' &= -23.60 \end{aligned}$$

From Formula XLI, min. $s_t' = 200 - 223.6 = -23.6$ lb. per sq. in.

Problems

1. Find the magnitude and direction of the maximum and minimum tensile stress for Problem 1 of the preceding article.
Ans. Max. $s_t' = 1{,}800$ lb./in.^{2} *across* a plane at 56° 19′ with the horizontal; min. $s_t' = 800$ lb./in.^{2} compression in a plane at 56° 19′ with the horizontal.

2. Check Problem 1 by substitution in Eq. (6) of Art. 185.
3. Solve Problem 8 of the preceding article for the maximum resultant tensile stress.
4. The maximum resultant tensile stress which is caused by horizontal and vertical shearing stress and a tensile stress of 480 lb. per sq. in. is 700 lb. per sq. in. What is the direction of the maximum resultant shearing stress?
5. Find the maximum resultant shearing and tensile stresses which are caused by a horizontal tension of 300 lb. per sq. in. and a horizontal and vertical shear of 160 lb. per sq. in.
Ans. $\tan 2\theta = 0.9375$; $\theta_s = 21° 35'$ or $111° 35$; max. $s_s' = 219.32$ lb./in.^{2}; max. $s_t' = 369.32$ lb./in.^{2}; min. $s_t' = -69.32$ lb./in.^{2}

Figure 257, which applies to Problem 5, indicates a method for finding whether the maximum tensile stress is 45° above

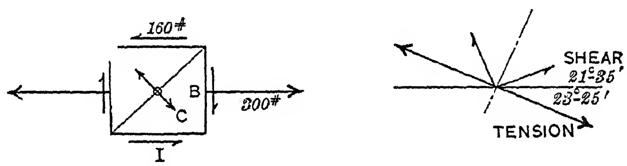


FIG. 257.—Direction of resultant tension.

or 45° below the direction of the maximum shearing stress. The tension which is caused by shear alone is 45° below the horizontal on the right side. The tension which results from this and the tensile stress of 300 pounds combined must lie between the two and is, therefore, below the horizontal. The maximum shearing stress is 21° 35′ above the horizontal. The maximum tensile stress is 21° 35′ − 45° 00′ or 23° 25′ below the horizontal. The minimum tensile stress, which is a compression of 69.32 pounds per square inch, lies in the direction of the broken line of Fig. 257.

188. The Resultant Stress in a Beam.—In a beam, the maximum resultant stress is due to a shearing stress which is a maximum at the neutral surface, and a tensile or compressive stress which is the greatest at the outer fibers. It is not usually necessary to calculate the maximum resultant tensile stress in a beam, since it is seldom greater than the bending stress in the outer fibers.

Problem

1. A 6-in. by 10-in. beam is supported at points 30 in. apart and carries a load of 20,000 lb. midway between the supports. Find the magnitude and direction of the maximum resultant tension, shear, and compression,

at sections 5 in. and 10 in. from the left support at points 0 in., 1 in., 2 in., 3 in., 4 in., and 5 in. from the neutral axis.

Table XXVIII, below, gives the results of the calculation for this problem. It will be noticed that the tension is at 45° with the horizontal at the neutral surface and is 250 lb. per sq. in. At 5 in. from the end the resultant tensile stress increases to 500 lb. per sq. in. in the outer fibers, and at 10 in. from the end it increases to 1,000 lb. per sq. in.

TABLE XXVIII.—RESULTANT SHEAR AND TENSION IN A BEAM

Distance below axis		Shear, pounds	Tension, pounds	Maximum shear		Maximum tension		Maximum compression	
				Pounds	Angle	Pounds	Angle	Pounds	Angle
At 5 inches from end	0	250	0	250.0	0° 0′	250.0	−45° 0′	250.0	45° 0′
	1	240	100	245.2	5° 53′	295.2	−39° 07′	195.2	50° 53′
	2	210	200	232.6	12° 44′	332.6	−32° 16′	132.6	57° 44′
	3	160	300	219.3	21° 35′	369.3	−23° 25′	69.3	66° 35′
	4	90	400	219.0	32° 53′	419.0	−12° 07′	19.0	77° 53′
	5	0	500	250.0	45° 0′	500.0	0° 0′	0	90° 0′
At 10 inches from end	0	250	0	250.0	0° 0′	250.0	−45° 0′	250.0	45° 0′
	1	240	200	260.2	11° 49′	360.2	−33° 11′	160.2	56° 49′
	2	210	400	290.0	21° 48′	490.0	−23° 12′	90.0	66° 48′
	3	160	600	341.0	30° 58′	641.0	−14° 2′	41.0	75° 58′
	4	90	800	410.0	38° 40′	810.0	− 6° 20′	10.0	83° 40′
	5	0	1,000	500.0	45° 0′	1,000.0	0° 0′	0	90° 0′

The shearing stress is 250 pounds per square inch at the neutral surface at both sections. At the outer fibers the shearing stress is entirely due to the tensile stress and is 250 pounds per square inch at the section 5 inches from the support and 500 pounds per square inch at the section 10 inches from the support.

The tension at the neutral surface, which is entirely due to shear, is 250 pounds per square inch and makes an angle of 45° with the length of the beam. The maximum tension in the outer fibers is the same as if there were no shear.

Since the shear is a maximum at the neutral surface, where the bending stress is zero, and the tension is a maximum in the outer fibers, where the shear is zero, the maximum resultant stress at any point in a beam is seldom greater than the stress in the outer fibers which is due to bending alone. In a short I-beam, the resultant tensile or compressive stress in the web may be greater than the stress in the outer fibers. For the

24 × 12 100-pound wide-flange section of the problem of Art. 142, the unit shearing stress at 10.5 inches from the neutral surface was found to be 0.0800 V. If V = 120,000 pounds, s_s = 9,600 pounds per square inch. If the bending stress in the outer fibers is 16,000 pounds per square inch, the tensile stress

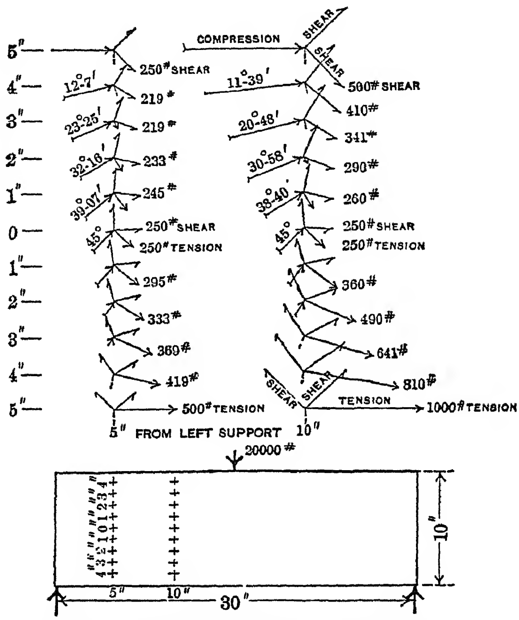


FIG. 258.—Resultant stress in a beam section.

10.5 inches below the neutral axis is 14,000 pounds per square inch.

$$\text{max. } s_t' = 7{,}000 + 11{,}880 = 18{,}880 \text{ pounds per square inch.}$$

For short, deep beams it is desirable to investigate for maximum resultant tension or compression in the web.

In a reinforced-concrete beam, the steel in tension is mathematically equivalent to a very wide flange of concrete. The unit shearing stress in the concrete which adjoins the reinforcement is large. Since the steel is near the outer surface, the bending stress in the concrete is also large. The resultant tensile stress in the concrete is relatively large, and such beams often begin to crack along surfaces at right angles to the direction of the maximum resultant tension.

Problems

2. A 6-in. by 8-in. beam of dense southern yellow pine is 5 ft. long between supports and carries 6,000 lb. 2 ft. from the left support. Find the maximum unit compressive stress. Find the unit compressive stress and the unit shearing stress 0.8 in. from the top at a section 20 in. from the left support. Find the maximum unit compressive stress at this point.

3. What is the average vertical shearing stress in any section to the left of the load for the beam of Problem 2? What is the shearing stress at the neutral surface? *Ans.* 75 lb./in.^{2}; 112.5 lb./in.^{2}

4. A 36 × 16½ 240-lb. wide-flange beam rests on supports 10 ft. apart and carries 600,000 lb. at the middle. Find the unit shearing stress at the neutral surface and at 3 in. from the outer fibers, at a section near the support. *Ans.* 11,775 lb./in.^{2} at neutral surface; 9,627 lb./in.^{2}

5. Calculate $\frac{V}{t\,d}$ for the beam of Problem 4 and compare with the mean of the two answers.

6. In Problem 4, at a section at which the maximum fiber stress is 14,400 lb. per sq. in., find the tensile stress 3 in. above the bottom and calculate the maximum resultant tensile and shearing stresses.
Ans. Max. $s_t' = 17{,}340$ lb./in.^{2}

189. Bending Combined with Torsion.—In a shaft subjected to bending moment, the maximum tensile stress is found in the fibers at the dangerous section which are most remote from the neutral surface. When subjected to torsion, all the outer fibers are at the maximum shearing stress. When the shaft is subjected to the combined effect of bending moment and torque, those fibers at the dangerous section which are farthest from the neutral surface are subjected to the combined effect of the maximum tensile or compressive stress and the maximum shearing stress which may be much larger than the results of Formulas IX and XVI.

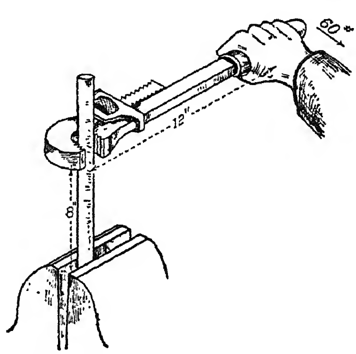


Fig. 259.—Torsion and bending.

Example

A 1-in. rod projects from a vise. A wrench, at right angles to the rod, grips it 8 in. from the vise. The wrench is turned by a force of 60 lb., perpendicular to the plane of the rod and wrench, which is applied to the wrench 12 in. from the axis of the rod. Find the maximum resultant shearing and tensile stress.

The bending moment at the vise is the same as if the force of 60 lb. were applied directly to the rod at 8 in. from the vise (Fig. 259).

$$M = 60 \times 8 = 480 \text{ in.-lb.};$$
$$S_t = \frac{480 \times 32}{\pi} = \frac{3,840 \times 4}{\pi} = 4,889 \text{ lb. per sq. in.}$$
$$T = 60 \times 12 = 720 \text{ in.-lb.}$$
$$S_s = \frac{720 \times 16}{\pi} = \frac{3,840 \times 3}{\pi} = 3,667 \text{ lb. per sq. in.}$$
$$\text{Max. } S_s' = \sqrt{3,667^2 + 2,444^2} = 4,407 \text{ lb. per sq. in.}$$
$$\text{Max. } S_t' = 2,444 + 4,407 = 6,851 \text{ lb. per sq. in.}$$

Since the section modulus used in torsion is twice that used in bending, and the force P is the same for both torque and bending moment, there is a large common factor which may be taken out to reduce the labor of computation. In this problem the factor is $\frac{3,840}{\pi}$ which is equal to 1,222.

$$\text{Max. } S_s' = 1,222\sqrt{3^2 + 2^2} = 1,222\sqrt{13} = 4,407 \text{ lb. per sq. in.}$$

Problems

1. A solid 3-in. shaft, which projects from a vise, is twisted by pipe wrench applied 40 in. from the vise. A force of 600 lb. at right angles to the plane of the shaft and the wrench is applied to the wrench 60 in. from the axis of the shaft. Find the maximum resultant shearing and tensile stress.
Ans. Max. $S_s' = 8,161$ lb./in.^{2}; max. $S_t' = 12,688$ lb./in.^{2}
2. A $3\frac{1}{2}$-in. standard pipe projects from a vise and is twisted by a force of 400 lb. applied 5 ft. from the axis of the pipe in a plane which is 4 ft. from the vise and perpendicular to the axis of the pipe. Find the maximum resultant shearing and tensile stress.
Ans. Max. $s_s = 6,419$ lb./in.^{2}; max. $s_t' = 10,429$ lb./in.^{2}
3. A solid 4-in. shaft is subjected to a compressive force of 80,000 lb. longitudinally and a torque of 6,000 ft.-lb. Find the maximum resultant shearing and compressive stress.
4. A 4-in. solid shaft transmits 200 hp. at 150 r.p.m. and is subjected to a compression of 40,000 lb. in the direction of its length. Find the maximum resultant compressive and shearing stress.
Ans. Max. $S_s' = 6,874$ lb./in.^{2}; max. $S_c' = 8,465$ lb./in.^{2}

190. Equivalent Moment and Torque.—For a circular section $J = 2I$, and when the outer radius is a,

$$S_t = \frac{Ma}{I}, \qquad \frac{S_t}{2} = \frac{Ma}{2I}; \tag{1}$$

$$S_s = \frac{Ta}{J} = \frac{Ta}{2I}; \tag{2}$$

$$\text{Max. } S_s' = \sqrt{S_s^2 + \left(\frac{S_t}{2}\right)^2} = \frac{a}{2I}\sqrt{T^2 + M^2} = \frac{a\sqrt{T^2 + M^2}}{J}. \tag{3}$$

The term $\sqrt{T^2 + M^2}$ may be regarded as the equivalent torque resulting from the combination of torsion and bending. In

the example of the preceding article $M = 480$ inch-pounds, $T = 720$ inch-pounds, and the equivalent torque is

$$240\sqrt{13} = 865.3$$

$$\text{Max. } S_s' = \frac{865.3 \times 16}{\pi} = 4{,}407 \text{ lb./in.}^2$$

$$\text{Max. } S_t' = \frac{Ma}{2I} + \frac{a\sqrt{T^2 + M^2}}{2I} = a\,\frac{(M + \sqrt{T^2 + M^2})}{2I}. \quad (4)$$

The term $\dfrac{M + \sqrt{T^2 + M^2}}{2}$ may be regarded as the equivalent bending moment.

Problems

1. A hollow shaft of 4 in. inside diameter and 6 in. outside diameter is subjected to a torque of 2,000 ft.-lb. and a bending moment of 1,500 ft.-lb. Find the equivalent maximum torque and moment and find the maximum unit shearing and tensile stress.
2. A 3-in. standard pipe is subjected to a bending moment of 13,792 in.-lb. and a torque of 20,688 in.-lb. Find the maximum tensile and shearing stresses. *Ans.* Max. $S_t' = 11{,}201$ lb./in.^{2}; max. $S_s' = 7{,}205$ lb./in.^{2}

191. Shear Combined with Tension in Two Directions.—Figure 260 represents a block subjected to shearing stress and a horizontal tension as in Fig. 253, together with a vertical tension of intensity s_v. To find the intensity of the resultant unit shearing and tensile stresses in any direction, the block is divided by a plane which makes an angle θ with the horizontal, and the wedge to the left of this plane is taken as the free body. The shearing stress at an angle θ with the horizontal is found by resolving parallel to this plane. If the length of the block perpendicular to the plane of the paper is taken as unity, the resolution is

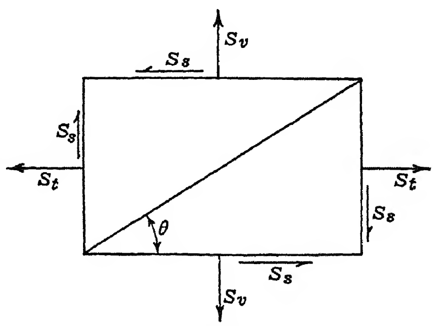


FIG. 260.—Tension in two directions combined with shear.

$$s_s' ds = s_t dy \cos\theta + s_s dx \cos\theta - s_s dy \sin\theta - s_v dx \sin\theta. \quad (1)$$

When Equation (1) is divided by ds, and $\cos\theta$ is substituted for $\dfrac{dx}{ds}$ and $\sin\theta$ is substituted for $\dfrac{dy}{ds}$, the result is

$$s'_s = (s_t - s_v) \sin \theta \cos \theta + s_s(\cos^2 \theta - \sin^2 \theta); \tag{2}$$

$$s'_s = \frac{s_t - s_v}{2} \sin 2\theta + s_s \cos 2\theta. \tag{3}$$

Equation (3) is the same as Equation (3) of Art. 185 except that $s_t - s_v$ replaces s_t.

To get the direction of the maximum unit shearing stress, Equation (3) is differentiated with respect to θ, and the derivative equated to zero.

$$(s_t - s_v) \cos 2\theta - 2s_s \sin 2\theta = 0; \tag{4}$$

$$\tan 2\theta = \frac{\frac{s_t - s_v}{2}}{s_s}. \tag{5}$$

When the values of the sine and cosine of θ from Equation (5) are substituted in Equation (3), the result is

$$\text{Max. } s'_s = \sqrt{s_s^2 + \left(\frac{s_t - s_v}{2}\right)^2}, \tag{6}$$

which is the same as Formula XL of Art. 186 with $s_t - s_v$ in place of s_t.

To find the resultant unit tensile stress across the inclined plane, a resolution is taken perpendicular to it. The equation of equilibrium is

$$s'_t\,ds = s_t dy \sin \theta + s_s dx \sin \theta + s_s dy \cos \theta + s_v dx \cos \theta; \tag{7}$$

$$s'_t = s_t \sin^2 \theta + s_v \cos^2 \theta + 2\, s_s \sin \theta \cos \theta; \tag{8}$$

$$s'_t = \frac{s_t}{2}(1 - \cos 2\theta) + \frac{s_v}{2}(1 + \cos 2\theta) + s_s \sin 2\theta \tag{9}$$

For the direction of maximum unit tensile stress,

$$\tan 2\theta = -\frac{s_s}{\frac{s_t - s_v}{2}}, \tag{10}$$

which is normal to the double angle for the maximum shearing stress.

$$\text{Max. } s'_t = \frac{s_t + s_v}{2} + \sqrt{s_s^2 + \left(\frac{s_t - s_v}{2}\right)^2}. \tag{11}$$

If the vertical stress is compressive, it may be regarded as negative tension. If s_c is this compressive stress,

$$\text{Max. } s_s' = \sqrt{s_s^2 + \left(\frac{s_t + s_c}{2}\right)^2}; \tag{12}$$

$$\text{Max. } s_t' = \frac{s_t - s_c}{2} + \text{max. } s_s'. \tag{13}$$

If s_s is zero, Equation (6) gives $\frac{s_t - s_v}{2}$ as the maximum unit shearing stress at 45 degrees with both s_t and s_v. There is, however, a greater unit shearing stress of magnitude $\frac{s_t}{2}$ in a plane parallel to s_v, at an angle of 45 degrees with the direction of the greater stress, s_t.

Equation (11) shows that when the shearing stress is zero, the maximum tensile stress is s_t.

Problems

1. A block is subjected to a horizontal tensile stress of 1,000 lb. per sq. in., a vertical tensile stress of 200 lb. per sq. in., and a horizontal and vertical shearing stress of 300 lb. per sq. in. which is upward at the left. Find the maximum unit shearing and tensile stress.
Ans. Max. $s_s' = 500$ lb./in.^{2}; max. $s_t' = 1{,}100$ lb./in.^{2}
2. Solve Problem 1 if the stress of 200 lb. per sq. in. is compressive.
3. A 1-in. rod projects upward from a vise, which grips the rod for a length of 1.5 in. The total pressure is 6,000 lb. directed east and west. A pipe wrench is attached to the rod 10 in. above the vise. With the wrench extending east, a horizontal force of 60 lb., directed south, is applied 12 in. from the axis of the rod. Find the resultant shearing and tensile stress at the south surface of the rod at the vise and also at the north surface. Assume that the horizontal compressive stress from the jaws is distributed uniformly over a 1.5-in. length and 1-in. thickness.
Ans. Max. $s_s' = 6{,}246$ lb./in.^{2}; max. $s_t' = 7{,}300$ lb./in.^{2}, north side. Max. $s_s' = 3{,}816$ lb./in.^{2}; max. $s_c' = 8{,}871$ lb./in.^{2}, south side.
4. A block is subjected to a horizontal tensile stress of 600 lb. per sq. in., and a vertical tensile stress of 200 lb. per sq. in., together with horizontal and vertical shearing stress of 300 lb. per sq. in. in the plane of the two tensile stresses. Find the maximum unit shearing stress.
Ans. Max. $s_s' = 360.6$ lb./in.^{2}
5. Solve Problem 4 if the unit shearing stress be only 100 lb./in.^{2}
Ans. Max. $s_s' = 223.6$ lb./in.^{2}

192. Theories of Failure.—In the preceding articles, maximum stresses have been calculated for various types of loading. It is a question, sometimes, what stress determines yield point or incipient failure. It was shown in Chapter I that stress in one direction causes deformation in the opposite sense in all directions in any

plane perpendicular to the direction of the applied force. If there is compressive force along the X axis producing unit *compression* δ, there will be unit *elongation* of $\mu\delta$ along the Y and Z axes. If at the same time, tensile force is applied along the Y axis, the total deformation along that axis is the elongation caused by the tension, together with the elongation caused by the compression along the X axis. It is a question whether the failure which may occur at right angles to the Y axis is influenced in any way by the additional deformation which was caused by the compression at right angles to it.

The *maximum-stress theory*, called also *Rankine's* theory, states that failure will begin when the resultant stress in any direction reaches the ultimate strength of the material in that direction, no matter what other stresses at right angles to this maximum stress may act at the same point. If there is no applied shearing stress, Equation (11) of the preceding article shows that the maximum resultant stress is s_t (provided s_t is numerically greater than s_v). The material will fail in tension when s_t reaches its tensile strength, no matter whether s_v is tension or compression, and no matter how large s_v may be, provided it is smaller than s_t. When s_t is tension and s_v is compression, or *vice versa*, the two combine to produce a large shearing stress at 45 degrees and the piece will fail by shear when this resultant shearing stress reaches the shearing strength of the material in that particular direction.

Cast iron is relatively weak in tension. Figure 97 shows the failure of a cast-iron bar which was tested in torsion. The fracture is at right angles to the line of the maximum resultant tensile stress which was caused by horizontal and vertical shear. When cast-iron bars are subjected to combined bending and twisting, the fracture still takes place at right angles to the maximum resultant tensile stress.

Figure 261 shows a reinforced concrete beam which was loaded at the third points. Failure began with a diagonal fracture at right angles to the maximum resultant tensile stress. Between the loads, the resultant tensile stress was horizontal, since the shear was practically zero, and the initial cracks were vertical.

While these illustrations show that failure takes place by maximum stress, they prove nothing in regard to the effect of stresses at right angles to the maximum.

The *maximum-strain theory*, sometimes called *St. Venant's* theory, assumes that a solid reaches its elastic limit when the unit

deformation reaches a given limit and that there is an ultimate unit deformation which cannot be exceeded without rupture, no matter in what way the stresses are applied which cause the deformation.

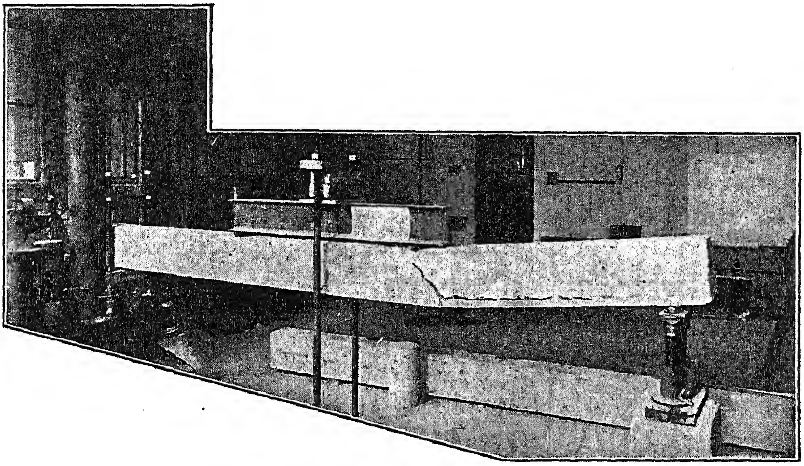


FIG. 261.—Failure of a reinforced-concrete beam.

Suppose a block (Fig. 262) is subjected to a direct tensile stress of s_t and to a compressive stress at right angles thereto of s_c, and suppose the material reaches its elastic limit in tension when the unit elongation is 0.001. According to the maximum-strain theory, if the unit elongation which is due to tension is 0.0008, and there is an additional unit elongation of 0.0002 in the same direction which is due to transverse compression, the combined unit elongation of 0.001 brings the material to the elastic limit.

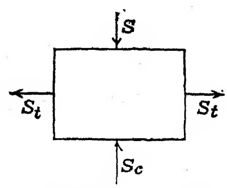


FIG. 262.—Tension and compression of right angles.

The tensile strength of some materials is much smaller than the compressive strength. If the ratio of the tensile strength to the compressive strength is less than Poisson's ratio for the material, a compressive load should cause failure by transverse tension. This is what seems to happen with porcelain and concrete. A porcelain rod, 1 inch in diameter and 16 inches long, supported a compressive load of 20,000 pounds per square inch and failed by splitting lengthwise. When porcelain is tested in tension, the heads of the specimen must be much larger than the minimum section, or the specimen will fail at the grips. Figure

263 shows the form of a series of bars of rectangular section. The pressure was transmitted to the heads from the grips through leather or lead sheets. Instead of failing at the minimum section, the bars failed along a curved surface AB at one of the heads. According to the maximum-strain theory, the unit deformation across this curved surface which was caused by tension and the unit deformation which was caused by compression were together greater than the unit deformation in the smaller sections which was caused by tension alone.

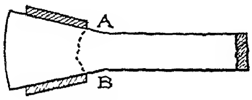


FIG. 263.

The behavior of porcelain under stress strongly supports the maximum-strain theory for *brittle materials.*

Figure 62 shows that concrete fails by splitting longitudinally. It is true that diagonal shear occurs at the ends, giving a pyramidal block, and the wedge action of this pyramid is given as an explanation of the splitting. However, when the compression heads of the testing machine are lubricated with a heavy grease, no pyramids are formed but the block splits the entire length.

The *maximum-shear theory,* frequently called the *Guest* theory* or the *Guest-Hancock*† theory, states that a material reaches its elastic limit in tension when the maximum resultant shearing stress reaches the elastic limit of the material in shear, and failure occurs when the maximum resultant shearing stress reaches the shearing strength of the material in shear.

It is evident that a bar in tension or compression will fail by shear provided it does not fail in some other way before the unit shearing stress (which, at 45 degrees, is one-half the unit tensile or compressive stress) reaches the ultimate shearing strength of the material. It is also evident that, when the unit shearing stress reaches the elastic limit in shear, there will be large linear deformations which will appear as the elastic limit in tension or compression. The point upon which there is *not agreement* is whether a solid ever reaches its elastic limit in tension or compression before reaching the elastic limit in shear, and whether failure is always by shear.

* See J. J. GUEST, On the Strength of Ductile Materials under Combined Stress, *Phil. Mag.*, July, 1900, pp. 69–132.

† E. L. HANCOCK, The Effect of Combined Stress on the Elastic Properties of Steel, *Proc. A. S. T. M.*, 1905, pp. 179–186; 1906, pp. 295–307.

The tests made by Guest and Hancock were upon ductile materials, and neither of them claimed that the maximum-shear theory applies to brittle solids.

The maximum unit shearing stress which is caused by tension or compression is at an angle of 45 degrees. Shear failure, however, frequently takes place at a much higher angle. The difference is due to friction. In Fig. 264, the component of the applied force parallel to $B\ C$ is $P \sin \theta$ and the component perpendicular to $B\ C$ is $P \cos \theta$. If A is the area of the normal cross section, the area of the plane through $B\ C$ normal to the plane of the paper is $A \sec \theta$. If s_s is the unit shearing strength, the shearing resistance is $s_s A \sec \theta$.

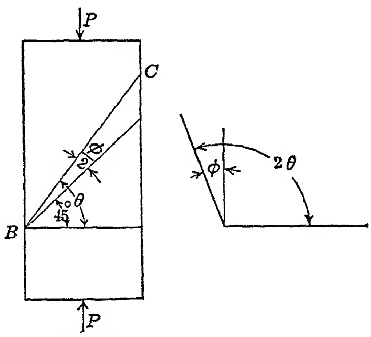


FIG. 264.—Shear failure caused by compression.

When a resolution is taken parallel to $B\ C$ of Fig. 264, the equation is

$$P \sin \theta = P \mu \cos \theta + s_s A \sec \theta, \tag{1}$$

in which μ is the coefficient of friction.

$$\frac{P}{s_s A} = \frac{\sec \theta}{\sin \theta - \mu \cos \theta}; \tag{2}$$

$$\frac{s_s A}{P} = \sin \theta \cos \theta - \mu \cos^2 \theta; \tag{3}$$

$$\frac{2\, s_s A}{P} = \sin 2\,\theta - \mu(1 + \cos 2\,\theta). \tag{4}$$

The load P is a minimum when the second member of Equation (4) is a maximum. When this is differentiated with respect to θ and equated to zero, the result is

$$\cos 2\theta + \mu \sin 2\theta = 0; \tag{5}$$

$$\cot 2\theta = \mu; \quad 2\,\theta = 90° + \text{arc tan } \mu;$$

$$\theta = 45° + \frac{\text{arc tan } \mu}{2}. \tag{6}$$

The angle which has a tangent equal to the coefficient of friction is called the *angle of friction* and is generally designated by ϕ.

$$\theta = 45° + \frac{\phi}{2}. \tag{7}$$

Failure in compression takes place along a plane which makes an angle of 45 degrees plus the angle of friction with the plane normal to the compressive force. Failure in tension takes place at 45 degrees. Timber failure by compression parallel to the grain takes place by buckling the fibers instead of sliding; Equation (7), therefore, does not apply.

The *maximum-energy* theory assumes that failure takes place when the energy per unit volume reaches some definite amount which is the energy per unit volume at which a bar of the material will fail in tension. The energy of tension or compression is $U = \frac{s^2}{2E}$; the energy of shear is $U = \frac{s_s^2}{2E_s}$. If Poisson's ratio is 0.25, $E_s = \frac{2E}{5}$, and the energy of unit volume in shear is $U = \frac{5s_s^2}{4E}$.

$$\text{Total } U = \frac{s_t^2}{2E} + \frac{5s_s^2}{4E} \tag{8}$$

for combined tension and shear.

Example

If 40,000 lb. per sq. in. is the elastic limit of steel, what is the maximum shearing stress which may be applied in the same plane as the tensile stress when the tensile stress is 30,000 lb. per sq. in.?

$$\frac{5s_s^2}{4} = \frac{40{,}000^2}{2} - \frac{30{,}000^2}{2};$$
$$5s_s^2 = 14 \times 10^8; \quad s_s^2 = 280 \times 10^6;$$
$$s_s = 16{,}730 \text{ lb. per sq. in.}$$

Solved by *maximum-stress* theory, this example gives

$$40{,}000 = 15{,}000 + \sqrt{s_s^2 + 15{,}000^2}$$
$$s_s = 20{,}000 \text{ lb. per sq. in.}$$

Solved by the *maximum-shear theory*, if the tensile strength is 40,000, the shearing strength is 20,000 lb. per sq. in.

$$20{,}000 = \sqrt{s_s^2 + 15{,}000^2}$$
$$s_s = 13{,}229 \text{ lb. per sq. in.}$$

According to the *maximum-strain theory*, the ultimate load is reached when the unit elongation in any direction is equal to the unit elongation which would be caused by the ultimate tensile stress of 40,000 pounds per square inch. When tension

s_t is in one direction and compression s_c is at right angles to this direction, unit deformation in the direction of the tension is

$$\frac{s_t}{E} + \frac{\mu s_c}{E} \tag{9}$$

in which s_t is the maximum resultant tensile stress and s_c is the maximum resultant compressive stress. When $\mu = 0.25$, Equation (9) for the example becomes

$$\frac{40{,}000}{E} = \frac{15{,}000 + \sqrt{s_s^2 + 15{,}000^2}}{E} + \frac{\sqrt{s_s^2 + 15{,}000^2} - 15{,}000}{4\,E};$$

$$115{,}000 = 5\sqrt{s_s^2 + 15{,}000^2};\quad 23{,}000^2 - 15{,}000^2 = s_s^2;$$
$$s_s^2 = 304 \times 10^6;\quad s_s = 17{,}436 \text{ lb. per sq. in.}$$

193. Fatigue of Metals.—There is a considerable area between the ascending and descending curves of a "loop" in Fig. 65. Since one coördinate is force and the other is displacement, this area represents work which is expended in stretching the bar and is not returned when the load is released.

Since energy is lost in a cycle of this kind, it is natural to expect that a great number of repetitions of stress would cause failure at a maximum stress lower than the ultimate strength of the material. The experiments of Wöhler and many other investigators show that this is true. Steel which is subjected to a complete reversal of stress will fail under a great number of repeated applications of a stress which is considerably below the proportional elastic limit. For instance, steel of 0.49 of 1 per cent carbon which was tested by Moore and Jasper* had an ultimate tensile strength of 91,500 pounds per square inch and a proportional elastic limit of 44,700 pounds per square inch. When this steel was tested under complete reversal of stress, the *endurance limit* was found to be 33,000 pounds per square inch. If the unit stress was changed from 33,000 pounds per square inch tension to 33,000 pounds per square inch compression, the specimen would stand an indefinite number of repetitions without failure. Two specimens of this material were each subjected to over one hundred million repetitions of a stress which varied from 33,100 pounds per square inch tension to 33,100 pounds per square inch compression. Neither of these failed. A third specimen was subjected to one billion repetitions

* H. F. MOORE and T. M. JASPER, *Bull.* No. 136, Engineering Experiment Station of the University of Illinois, pp. 23 and 31.

of a stress which varied from 33,000 pounds per square inch tension to the same compression without failure. The stress of 33,000 pounds per square inch is, therefore given as the *endurance limit* of this steel for complete reversal of stress. When the stress in this material varied from 34,000 to −34,000 pounds per square inch, each of the two specimens tested failed at about four million repetitions. At 37,000 pounds per square inch, the failure was at 1,225,000 repetitions; and at 50,000 pounds per square inch, the failure was at 42,000 repetitions.

If the stress is not completely reversed, the endurance limit is higher. The smaller the range between the maximum and minimum stress, the greater is the maximum stress which the material will endure for an indefinite time. The carbon steel above mentioned* has an endurance limit of 33,000 pounds per square inch for complete reversal of stress. The same material stood an indefinite number of repetitions when the stress ranged from 36,000 pounds tension to 21,600 pounds compression, from 47,000 pounds tension to zero, from 60,000 pounds tension to 12,000 pounds tension, or from 69,000 pounds tension to 34,500 pounds tension. These tests and many others show that the endurance is higher the smaller the range of stress.

194. Design for Variable Stress.—A number of methods have been proposed for designing members which are subjected to variable loads. This may be done by reducing the allowable unit stress or by adding an increment to the applied load.

Professor John Goodman† suggested a rule for this purpose. Goodman's "dynamic" rule is *add to the maximum load the difference between the maximum and minimum loads and treat the sum as a static load.* Under this rule, a load which varied from 12,000 pounds tension to 12,000 pounds compression would be equivalent to $12{,}000 + 24{,}000 = 36{,}000$ pounds tension or compression; a load which varied from 12,000 pounds tension to 6,000 pounds compression would be equivalent to

$$12{,}000 + 18{,}000 = 30{,}000 \text{ pounds tension;}$$

and a load which varied from 12,000 pounds tension to 4,000 pounds tension would be equivalent to 20,000 pounds tension.

* *Bull.* No. 136, Engineering Experiment Station of the University of Illinois, p. 77.

† GOODMAN, "Mechanics Applied to Engineering," p. 535.

Problems

1. If the maximum allowable unit stress for a given material is 15,000 lb. per sq. in., what is the required area of cross section when the applied load varies from 20,000 lb. to 30,000 lb.? *Ans.* 2.67 sq. in.

2. What is the cross section required to carry a load which varies from 30,000 lb. compression to 60,000 lb. tension if the allowable static unit stress is 12,000 lb. per sq. in. *Ans.* 12.5 sq. in.

Moore and Jasper have shown* that Goodman's "dynamic" rule errs on the side of safety. They suggest the following formula, which is the equation of a straight line,

$$S_r = S_{-1}\left(\frac{r+3}{2}\right),$$

in which r is the ratio of the minimum stress to the maximum stress, S_r is the endurance limit for the ratio r, S_{-1} is the endurance limit for complete reversal. When the stress is reversed, r is negative. For instance, if the stress changes from 8,000 pounds per square inch compression to 20,000 pounds per square inch tension, r is -0.4. To use this formula, of course, the endurance limit for complete reversal must be determined experimentally. Moore and Jasper state further:

> This formula can be used only up to the limit at which the maximum unit stress set up reaches the proportional elastic limit of the material, and for most steels this eliminates ratios of minimum stress to maximum stress greater than zero. Beyond the proportional elastic limit the static properties of the steel are the governing factors rather than the fatigue properties.

Problems

3. A load varies from 20,000 lb. compression to 40,000 lb. tension. If the endurance limit for complete reversal is 32,000 lb. per sq. in., what is it for this loading? *Ans.* $S_{-.5} = 32{,}000 \times 1.25 = 40{,}000$ lb./in.^{2}.

4. The endurance limit for complete reversal for 3.50 nickel steel tested at the University of Illinois was 60,000 lb. per sq. in. What should it be when the load changed from 100,000 lb. tension to 40,000 lb. compression.

Ans. 78,000 lb./in.^{2} One set of tests gave 83,000 lb./in.^{2} and another gave 81,000 lb./in.^{2}

195. Rapid Determination of Endurance Limit.—The determination of the endurance limit by subjecting each specimen to a

* *Bull.* No. 136, Engineering Experiment Station of the University of Illinois, pp. 82–89.

variable load for many million times is a long-drawn-out process. One short-time method recently developed consists in testing several specimens of the same material at different stresses in as many endurance machines through about 1,400,000 cycles, then removing the specimens and testing to failure in tension. When the ultimate tensile stresses are plotted as ordinates and the stresses in the endurance machines as abscissas, the endurance stress at which the tensile stress is a maximum is taken as the *endurance limit.**

One set of tests was made on eight specimens of cold-drawn steel. The preliminary endurance runs for four of these pieces were made at stresses which ranged from 42,000 to 48,000 pounds per square inch. When these were tested in tension, the ultimate strengths were found to *increase* gradually in the same order. The preliminary endurance runs for the last three pieces were made at stresses which ranged from 52,000 to 56,000 pounds per square inch. When these were tested in tension, the ultimate strengths were found to *decrease* gradually in the same order. The ultimate tensile strengths of the two pieces for which the endurance runs had been made at 48,000 and 52,000 pounds per square inch were nearly equal, while the tensile strength of the piece for which the endurance run had been made at 50,000 pounds per square inch was much greater than either of these. The maximum of 50,000 pounds was taken as the short-time value of the endurance limit. The long-time value for this material was 51,000 pounds per square inch.

This short-time endurance test is based on two opposing factors. When an endurance test load is started a little below the endurance limit and then raised after a few million cycles, work hardening increases the strength of the material, and the final endurance limit is higher than it would have been if the larger load had been applied from the beginning. On the other hand, stresses above the endurance limit start infinitesimal fractures which would cause failure after a great number of repetitions. The farther the stress is below the endurance limit, the *less* it *strengthens* by work hardening; the higher it is above the endurance limit, the *more* it *weakens* by initial fractures. Therefore, the greatest tensile strength after this short-time endurance test is nearest to the endurance limit.

* H. F. MOORE and H. B. WISHART, *Proc. A.S.T.M.*, 1933, vol. 33, Part II, p. 334.

Another short-time method depends upon the rise in temperature when the specimen is subjected to repeated stresses. It has been found that there is a sudden increase in rate of generation of heat, and, consequently, in the rate of rise in temperature, when the endurance limit is reached. This principle, which has been applied successfully by Putnam and Harsch, makes it possible to find the endurance limit of a test specimen in a relatively short time.*

196. Crystallization under Repeated Stress.—When steel fails under repeated applications of load, the fracture has a crystalline appearance. For this reason it was long thought that repeated stresses cause the formation of crystals in the steel. Microscopic† examination shows that all steel is crystalline and that crystals do not form at atmospheric temperatures. The crystalline appearance of the fracture is due to the fact that the fracture has taken place across the crystals of the steel.

* W. J. PUTNAM and J. W. HARSCH, *Bull.* No. 124, Engineering Experiment Station of the University of Illinois, pp. 119–127.

† For a complete list of papers on the Fatigue of Metals see *Bull.* No. 124, Engineering Experiment Station of the University of Illinois, pp. 168–178.

CHAPTER XVIII

ELASTIC ENERGY OF BENDING AND SHEAR

197. Energy of External Work.—The work done by a force is the product of the average force multiplied by the displacement of its point of application in the direction of the force. For elastic bodies, the average force is one-half the sum of the initial and final

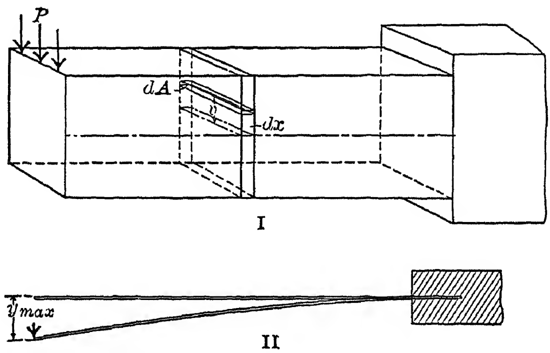


FIG. 265.—Work of deflection.

forces. When a beam is deflected by a force P, the average force is $\frac{P}{2}$. For a cantilever with a load on the free end (Fig. 265),

$$y_{\max} = \frac{P\,l^3}{3\,E\,I};$$

$$\text{External work} = U = \frac{P}{2} \times \frac{P\,l^3}{3\,E\,I} = \frac{P^2 l^3}{6\,E\,I}. \tag{1}$$

For a simply-supported beam which is loaded at the middle,

$$U = \frac{P}{2} \times \frac{P\,l^3}{48\,E\,I} = \frac{P^2 l^3}{96\,E\,I}. \tag{2}$$

When the elastic energy of a concentrated load is known, the deflection may be calculated from

$$\frac{P\,y}{2} = U, \quad y = \frac{2\,U}{P}. \tag{3}$$

Example

A simply-supported beam of length l carries a load P at a distance a from the left end and a distance b from the right end. Find the deflection under the load (see Fig. 265).

The portion of length a is a cantilever which is pushed upward by the reaction $\frac{P\,b}{l}$; the length b is another cantilever. From Eq. (1),

$$U = \frac{P^2b^2}{l^2} \times \frac{a^3}{6\,E\,I} + \frac{P^2a^2}{l^2} \times \frac{b^3}{6\,E\,I} = \frac{P^2\,b^2\,a^2}{6\,E\,I\,l^2}(a + b);$$

$$U = \frac{P^2b^2a^2}{6\,E\,I\,l} = \frac{P\,y}{2}; \quad y = \frac{P\,a^2b^2}{3\,E\,I\,l}, \text{ positive downward.}$$

Problems

1. How much work is done on a 3-in. by 4-in. cantilever, 6 ft. long, when a load of 100 lb. is placed on the free end if $E = 1{,}200{,}000$ lb. per sq. in.? How much additional work is done when a second load of 100 lb. is added?
Ans. $U = 32.4$ in.-lb.; $U = 97.2$ in.-lb.
2. In Problem 1, what was the average force when the first 100 lb. was applied? What was the average force when the second 100 lb. was applied?
3. A 6-in. by 8-in. beam, 16 ft. long, is supported at 6 ft. from the left end and held down at the right end. What is the total work when a load of 600 lb. is placed on the left end if $E = 1{,}200{,}000$ lb. per sq. in.? Calculate the work as two cantilevers.
Ans. $U = 72.9 + 121.5 = 194.4$ in.-lb.; $y = 0.648$ in.

When there is a uniformly distributed load of w pounds per unit length, the increment of load on a length dx is $w\,dx$, and the

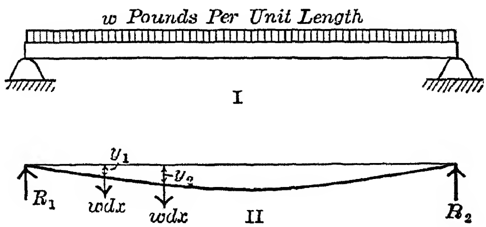


FIG. 266.—External work.

work done by this increment is $\frac{w\,y\,dx}{2}$, in which y is the deflection of the particular part of the beam upon which the increment rests. In Fig. 266, II, one increment $w\,dx$ is deflected a distance y_1, a second increment is deflected a distance y_2, etc.

The different values of y are determined from the equation of the elastic line. The total work is the sum of these increments of work.

$$\text{Total work} = \frac{w}{2}\int y\,dx, \tag{4}$$

with the ends of the beam as the limits.

The equation of the elastic line for a simply-supported uniformly loaded beam (from Art. 110) is

$$y = \frac{w}{24\,E\,I}(l^3x - 2\,l\,x^3 + x^4),$$

if y is taken positive downward.

$$dU = \frac{w^2}{48\,E\,I}(l^3x - 2\,l\,x^3 + x^4)\,dx; \tag{5}$$

$$U = \frac{w^2}{48\,E\,I}\left[\frac{l^3x^2}{2} - \frac{l\,x^4}{2} + \frac{x^5}{5}\right]_0^l = \frac{w^2l^5}{240\,E\,I}. \tag{6}$$

Problem

4. Find the total external work on a uniformly loaded cantilever.

Ans. $U = \dfrac{w^2l^5}{40\,E\,I}.$

198. Internal Work in a Beam.—The unit stress at a distance v from the neutral axis of a beam is $\dfrac{M\,v}{I}.$ The internal work or resilience per unit volume is $\dfrac{s^2}{2\,E}.$ Figure 265, I, shows an element of volume of cross section dA and length dx at a distance v from the neutral surface. The volume of this element is $dA\,dx$ and the internal energy is

$$dU = \frac{s^2}{2\,E}\,dA\,dx = \frac{M^2v^2}{2\,E\,I^2}\,dA\,dx. \tag{1}$$

$$\text{Total work in beam} = \int\!\!\int \frac{M^2}{2\,E\,I^2}\,v^2dA\,dx. \tag{2}$$

The integration of Equation (2) with respect to v as the variable gives the work done upon the volume of length dx between two vertical planes. Throughout this volume, x, M, and I are constant. The integral of v^2dA across the beam from the bottom to the top is I.

$$\text{Work} = \int \frac{M^2}{2\,E\,I}dx. \tag{3}$$

Equation (3) may be used to calculate the internal work in any beam. Unless M and I are constant, they must be expressed as functions of x before integrating.

For a *uniform beam under constant moment* M, Equation (3) becomes

$$U = \frac{M^2}{2\,E\,I}\int dx = \frac{M^2}{2\,E\,I}\Big[x\Big]_0^l = \frac{M^2 l}{2\,E\,I}. \tag{4}$$

For a *beam supported at the ends with uniformly distributed load*, $M = \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}$.

$$U = \frac{w^2}{8\,E\,I}\int (l^2x^2 - 2\,l\,x^3 + x^4)dx, \tag{5}$$

$$U = \frac{w^2}{8\,E\,I}\left[\frac{l^2x^3}{3} - \frac{l\,x^4}{2} + \frac{x^5}{5}\right]_0^l = \frac{w^2 l^5}{240\,E\,I} \tag{6}$$

which agrees with Equation (6) of Art. 197.

For a cantilever with a load on the free end, $M = -P\,x$. When the section is constant,

$$U = \frac{P^2}{2\,E\,I}\int x^2 dx = \frac{P^2}{6\,E\,I}\Big[x^3\Big]_0^l = \frac{P^2 l^3}{6\,E\,I}. \tag{7}$$

For a beam supported at the ends with a load P at a distance a from one end and at a distance b from the other (Fig. 165), the reaction at the end of the length a is $\frac{Pb}{l}$ and the moment in this length is $\frac{P\,b\,x}{l}$. The work in this part of the beam is

$$U = \frac{P^2b^2}{2\,E\,I\,l^2}\int_0^a x^2 dx = \frac{P^2b^2a^3}{6\,E\,I\,l^2}. \tag{8}$$

Similarly in the length b,

$$\text{Work} = \frac{P^2a^2b^3}{6\,E\,I\,l^2}. \tag{9}$$

For the entire length,

$$\text{Total work} = \frac{P^2a^2b^2(a+b)}{6\,E\,I\,l^2} = \frac{P^2a^2b^2}{6\,E\,I\,l}. \tag{10}$$

When the load is at the middle, $a = b = \frac{l}{2}$ and

$$\text{Total work} = \frac{P^2 l^3}{96\, E\, I}. \tag{11}$$

Problem

Find the total internal work in a uniformly loaded cantilever.

Ans. $U = \frac{w^2 l^5}{40\, E\, I}.$

199. Energy in Unit Volume.—Since the stress in any cross section of a beam increases from zero at the neutral axis to a maximum stress S in the outer fibers, the average energy per unit volume in any short portion is considerably smaller than $\frac{S^2}{2\, E}$. For all beams, except beams of constant strength or beams under constant moment, the stress in the outer fiber varies along the length and has its maximum value only at the dangerous section. For these reasons, the efficiency of a beam as a method of storing elastic energy depends on the form of the section and the arrangement of the loads. The most efficient beam is one in which the greatest portion of the material is brought to a relatively high stress.

For a beam of uniform section under constant moment,

$$U = \frac{M^2 l}{2\, E\, I}. \tag{1}$$

Equation (1) gives the total work in terms of the bending moment. It is often desirable to find the work in terms of the unit stress in the extreme fibers. If the neutral axis is midway between the extreme top and bottom fibers, $c = \frac{d}{2}$, and $M = \frac{2\, S\, I}{d}$. When this value of the moment is substituted in Equation (1), the result is

$$U = \frac{4\, S^2 I^2 l}{2\, E\, I\, d^2} = \frac{2\, S^2 I\, l}{E\, d^2}. \tag{2}$$

For a rectangular section,

$$U = \frac{2\, S^2 b\, d^3 l}{12\, E\, d^2} = \frac{S^2 b\, d\, l}{6\, E} = \frac{S^2}{6\, E} \times \text{volume}. \tag{3}$$

The average energy per unit of volume is $\frac{S^2}{6\,E}$, which is one-third as great as that in a block subjected to a uniform stress S.

Problems

1. Find the average energy per unit volume in a solid circular section subjected to a uniform bending moment.

   Ans. $\frac{S^2}{8E}$.

2. A steel bar 2 in. wide and ½ in. thick is 8 ft. long and rests on two supports 6 ft. apart and carries two equal loads on the ends. If E is 30,000,000 lb. per sq. in., what is the total elastic energy in the part between the supports when each load on the ends is 100 lb.? *Ans.* 82.9 in.-lb.

For a cantilever with uniformly distributed load,

$$U = \frac{w^2 l^5}{40\,E\,I} = \frac{W^2 l^3}{40\,E\,I}. \tag{4}$$

For a section which is symmetrical with respect to the neutral axis

$$\frac{W\,l}{2} = \frac{2\,S\,I}{d},$$

$$U = \frac{4\,S^2 I\,l}{10\,E\,d^2}. \tag{5}$$

For a rectangular section, for which $I = \frac{b\,d^3}{12}$,

$$\text{Total work} = \frac{S^2 b\,d\,l}{30\,E} = \frac{S^2}{30\,E} \times \text{volume}. \tag{6}$$

The total energy in a cantilever of rectangular section with uniformly distributed load is one-fifteenth as much as that in a block of the same volume with uniform compressive stress throughout.

200. Internal Work in a Shaft.—The unit shearing stress s_s produces a deformation of $\frac{s_s}{E_s}$ in planes at unit distance apart. The work of shear is the product of half the unit stress by the total deformation,

$$\text{Work per unit volume} = \frac{s_s}{2} \times \frac{s_s}{E_s} = \frac{s_s^2}{2\,E_s}, \tag{1}$$

In a solid circular shaft at a distance r from the axis, the unit shearing stress is $k\,r$ and

$$\text{Energy per unit volume} = \frac{k^2r^2}{2\,E_s}, \tag{2}$$

The element of volume of length l is $2\,\pi\,r\,l\,dr$ and

$$\text{Total energy} = \frac{\pi\,k^2l}{E_s}\int_0^a r^3dr = \frac{\pi\,l\,k^2a^4}{4\,E_s}, \tag{3}$$

in which a is the radius of the shaft. The maximum unit shearing stress in the outer surface is $S_s = k\,a$ and

$$\text{Total energy of shear} = \frac{S_s^2}{4\,E_s}\,\pi\,a^2l = \frac{S_s^2}{4\,E_s} \times \text{volume}. \tag{4}$$

Since the modulus of elasticity in shear is about two-fifths as great as the modulus in tension or compression, the total energy of a rod in torsion, for equal values of the unit stress, is one-fourth greater than that of the same rod in tension. However, since the elastic limit of steel and other similar materials in shear is somewhat smaller than in tension, the total energy which may be stored is approximately the same in both cases.

201. Work of Shear in a Rectangular Beam.—In a beam of rectangular section of breadth b and depth d subjected to a vertical shear V,

$$s_s = \frac{V}{I\,b}\int b\,v\,dv = \frac{V}{I}\left[\frac{v^2}{2}\right]_v^{\frac{d}{2}} = \frac{V}{8\,I}\,(d^2 - 4\,v^2); \tag{1}$$

$$\frac{s_s^2}{2\,E_s} = \frac{V^2}{128\,E_sI^2}\,(d^4 - 8\,d^2v^2 + 16\,v^4). \tag{2}$$

When the second member of Equation (2) is multiplied by the element of volume, which is $b\,dv\,dx$, and the product is integrated with respect to v with the limits of $-\frac{d}{2}$ and $\frac{d}{2}$, the result is

$$U = \int \frac{V^2b\,d^5}{128\,E_sI^2}\left(1 - \frac{2}{3} + \frac{1}{5}\right)dx = \int \frac{3\,V^2}{5\,E_sb\,d}\,dx. \tag{3}$$

When V is constant, the last term of Equation (3) for a beam of constant section is $U = \frac{3\,V^2l}{5\,E_sb\,d}$. For a cantilever beam with a load on the free end, $V = -P$ and

$$U = \frac{3\,P^2 l}{5\,E_s b\,d}. \tag{4}$$

To find the deflection which is caused by shear at the end of a cantilever with a load on the end,

$$\frac{P\,y}{2} = \frac{3\,P^2 l}{5\,E_s b\,d},$$

$$y = \frac{1.2\,P\,l}{E_s b\,d} = \frac{1.2\,V\,l}{E_s b\,d} = \frac{1.2\,s_s' l}{E_s} \tag{5}$$

in which s_s' is the average unit shearing stress.

The same relation holds for a beam supported at the ends with a concentrated load at the middle.

Problems

1. A 2-in. by 3-in. steel cantilever is 30 in. long and carries a load of 1,800 lb. on the free end. Find the deflection of bending and the deflection of shear if $E = 30{,}000{,}000$ and $E_s = 12{,}000{,}000$.

   Ans. $y_b = 0.12$ in.; $y_s = 0.0009$ in.

2. Solve Problem 1 for a length of 15 in. and a load of 3,600 lb.

   Ans. $y_b = 0.030$ in.; $y_s = 0.0009$ in.

202. Work of Two Loads.—When the expression

$$U = \int \frac{M^2}{2\,E\,I}\,dx$$

applies for two or more loads, the moments for the separate loads must be added together and then squared to give M^2 for integration. For a cantilever with a uniform load of w per unit length and a concentrated load P on the free end, the moment is $-P\,x - \dfrac{w\,x^2}{2}$.

$$U = \frac{1}{2\,E\,I}\int\left(P^2x^2 + P\,w\,x^3 + \frac{w^2x^4}{4}\right)dx =$$

$$\frac{1}{2\,E\,I}\left[\frac{P^2x^3}{3} + \frac{P\,w\,x^4}{4} + \frac{w^2x^5}{20}\right]_0^l; \tag{1}$$

$$U = \frac{P^2l^3}{6\,E\,I} + \frac{P\,w\,l^4}{8\,E\,I} + \frac{w^2l^5}{40\,E\,I}. \tag{2}$$

The first term of Equation (2) after the equality sign is the internal work which is done by the load P alone, as shown by

Equation (1) of Art. 197. The last term is the work which is done by the distributed load alone, as shown by Problem 4 of Art. 197. The intermediate term, which includes both P and w, must be the two loads together. If the load P is placed first on the beam, the average force during its own deflection is $\frac{P}{2}$ and the work is

$$U = \frac{P}{2} \times \frac{P\,l^3}{3\,E\,I} = \frac{P^2 l^3}{6\,E\,I}.$$

When the distributed load $w\,l\ (= W)$ is placed on the beam, the *additional* deflection at the *end* is $\frac{w\,l^4}{8\,E\,I}\left(=\frac{W\,l^3}{8\,E\,I}\right)$. Since the full load P is on the end of the beam while this deflection takes place, the work done by P is

$$U = P \times \frac{w\,l^4}{8\,E\,I} = \frac{P\,w\,l^4}{8\,E\,I} = \frac{P\,W\,l^3}{8\,E\,I}.$$

203. Maxwell's Theorem of Reciprocal Deflections.—Figure 267 represents a beam which rests at the ends on fixed supports or on horizontal planes so that no work is done by displacement at these supports. Figure 267, II, shows the elastic line of this beam when a load P is applied at the point A. The deflection of A from its original position y_p. If y_a represents the deflection at A which would be caused by unit load at A, the deflection caused by the load P is $P\,y_a$; hence

$$y_p = P\,y_a. \tag{1}$$

The load P at A causes a deflection at another point B, which may be called y_{pb}. If y_{ab} is the deflection at B which would result from a unit load at A, then the deflection caused by a load P at A would be $P\,y_{ab}$, and

$$y_{pb} = P\,y_{ab}. \tag{2}$$

Figure 267, III, shows the elastic line of this beam with a load Q at B and no load at A. The deflection at B is

$$y_q = Q\,y_b, \tag{3}$$

in which y_b is the deflection at B which would be caused by unit load at B. The deflection at A is

$$y_{qa} = Q\, y_{ba}, \tag{4}$$

in which y_{ba} is the deflection at A which would be caused by unit load at B.

Figure 267, IV, shows the elastic line for the beam when both P and Q are supported. If the load P is applied first, the average $\frac{P}{2}$ multiplied by y_p gives the work at A.

$$U_{pa} = \frac{P\, y_p}{2} = \frac{P^2 y_a}{2}. \tag{5}$$

Since there is no force at B, the deflection $y_{pb} = P\, y_{ab}$ represents no external work.

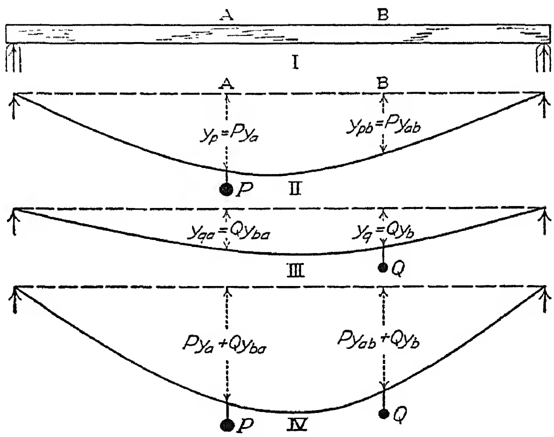


FIG. 267.—Reciprocal deflections.

When the load Q is applied at B, the additional deflection at A is $y_{qa} = Q\, y_{ba}$, which is the same as it would be if there were no load at A. Since the load P at A moves the distance y_{qa}, which equals $Q\, y_{ba}$, the additional work at A is

$$U_{qa} = P\, Q\, y_{ba}. \tag{6}$$

At B, when the load Q is applied, the work is

$$U_{qb} = \frac{Q\, y_q}{2} = \frac{Q^2 y_b}{2}. \tag{7}$$

$$\text{Total } U = U_{pa} + U_{qa} + U_{qb} = \frac{P^2 y_a}{2} + P\, Q\, y_{ba} + \frac{Q^2 y_b}{2}. \tag{8}$$

When the load Q is applied first, the first and last terms of the energy expression are the same as in Equation (8). When the

load P is applied, B moves a distance $P\,y_{ab}$ and takes the load Q that distance. The additional work at B is then

$$U_{pb} = Q\,P\,y_{ab}. \tag{9}$$

$$\text{Total } U = U_{pa} + U_{pb} + U_{qb} = \frac{P^2 y_a}{2} + Q\,P\,y_{ab} + \frac{Q^2 y_b}{2}. \tag{10}$$

Equation (8) gives the total energy when P is applied before Q; Equation (10) gives the total work when Q is applied before P. By the principle of *superposition* which is proved by experiment and theory, the deflection caused by several loads is the sum of the deflections caused by the loads acting separately (provided, of course, that the proportional limit is not exceeded); and the order of application of the loads is immaterial. Since Equations (8) and (10) represent the same total energy,

$$Q\,P\,y_{ab} = P\,Q\,y_{ba}; \quad y_{ba} = y_{ab}. \tag{11}$$

Equation (11) is *Maxwell's theorem of reciprocal deflection.* The deflection at a point A caused by a load at B is equal to the deflection at B caused by the same load at A.

There is nothing in the derivation of the equations which limits Maxwell's theorem to a simply-supported beam. It applies to any determinate or indeterminate frame.

(*This article may be omitted.*)

204. Castigliano's Theorem.—If Equation (8) or Equation (10) is differentiated with respect to the load P, the result is

$$\frac{\partial U}{\partial P} = P\,y_a + Q\,y_{ba}, \tag{1}$$

of which the first term $P\,y_a$ is the deflection at A caused by the load P, and the second term $Q\,y_{ba}$ is the deflection at A caused by the load Q. Castigliano's theorem states that the *derivative of the total elastic energy with respect to any concentrated load gives the total deflection at that load.* To get the deflection at any point at which there is no concentrated load, a "dummy" load P is assumed at that point. The last term of Equation (1) then gives the total deflection. If R is the reaction at a support of an indeterminate beam, $\frac{\partial U}{\partial R}$ is the deflection at that point. Since this deflection is zero at a fixed support, $\frac{\partial U}{\partial R} = 0$. This is the so-called method of "least work," which is really no work.

Example

Find the equation of the elastic line for a uniformly loaded cantilever.

If a dummy load P is placed at any distance a from the free end, the moment equation is

$$M = -\frac{w\,x^2}{2} - P(x - a); \tag{2}$$

$$M^2 = \frac{w^2x^4}{4} + P\,w(x^3 - a\,x^2) + P^2(x - a)^2. \tag{3}$$

Since the first term of Equation (3) $\frac{w^2x^4}{4}$ does not contain P, the derivative of the energy expression from this term with respect to P as the variable will be zero; hence this term need not be integrated. The last term $P^2(x - a)^2$ gives the deflection caused by P alone. Since P is a dummy force and the deflection caused by the distributed load is required, this term is not integrated. (If P were an actual load and the total deflection were desired under this load, then this last term would be integrated.)

$$U = \frac{P\,w}{2\,E\,I}\int(x^3 - a\,x^2)dx = \frac{P\,w}{2\,E\,I}\left[\frac{x^4}{4} - \frac{a\,x^3}{3}\right]_a^l;$$

$$U = \frac{P\,w}{2\,E\,I}\left(\frac{l^4}{4} - \frac{a\,l^3}{3} + \frac{a^4}{12}\right) = \frac{P\,w}{24\,E\,I}(3\,l^4 - 4a\,l^3 + a^4);$$

$$y = \frac{\partial U}{\partial P} = \frac{w}{24\,E\,I}(3\,l^4 - 4\,a\,l^3 + a^4), \text{ positive downward.}$$

The term which was integrated is the product of $\frac{w\,x^2}{2} \times P(x - a)$, of which $\frac{w\,x^2}{2}$ extends from 0 to l; $P(x - a)$ extends from a to l. The integral of the product has the limits a and l.

205. Elastic-energy Method.—If M_q represents the moment of any number of applied forces or couples and M_p represents the moment of a single concentrated force,

$$M^2 = (M_p + M_q)^2 = M_p^2 + 2\,M_pM_q + M_q^2; \tag{1}$$

$$U = \int\frac{M_p^2\,dx}{2\,E\,I} + \int\frac{M_pM_q\,dx}{E\,I} + \int\frac{M_q^2\,dx}{2\,E\,I}, \tag{2}$$

in which each term is equivalent to the corresponding term of Equation (8) or Equation (10) of Art. 203. In the "elastic-energy" method, the second term of Equation (2) is used exclusively. The force P is a dummy or auxiliary load. For a single concentrated load, which could be solved conveniently by the last term of Equation (2), it is customary to use the second term with the dummy force to avoid confusion. While the final

equations are practically the same as those obtained by Castigliano's methods, the derivation of the method is simpler and graphic integration is more convenient.*

$$\int \frac{M_p M_q \, dx}{E\,I} = P\, y_{qa}, \tag{3}$$

which is the work done by the force P when its point application at A is displaced a distance y_{qa} by forces which produce the moments M_q. From Equation (3) the deflection at A is

$$y = y_{qa} = \frac{1}{P}\int \frac{M_p M_q \, dx}{E\,I}. \tag{4}$$

For uniform beams,

$$E\,I\,y = \frac{\int M_p M_q dx}{P}. \tag{5}$$

If the force P is unity, and the beam is uniform,

$$E\,I\,y = \int M_p M_q dx. \tag{6}$$

Example

Derive the equation of the elastic line for a simply-supported, uniformly loaded beam.

In Fig. 268, the auxiliary force is upward at A at a distance a from the left support and a distance b from the right support. The reaction downward at the left end is $-\frac{P\,b}{l}$, and at the right end is $-\frac{P\,a}{l}$. Integration from each end to A gives less complicated expressions.

$$E\,I\,y = \frac{1}{P}\int\left(\frac{w\,l\,x}{2} - \frac{w\,x^2}{2}\right)\left(-\frac{P\,b\,x}{l}\right)dx + \frac{1}{P}\int\left(\frac{w\,l\,x}{2} - \frac{w\,x^2}{2}\right)\left(-\frac{P\,a\,x}{l}\right)dx; \tag{7}$$

$$E\,I\,y = -\frac{w\,b}{l}\left[\frac{l\,x^3}{6} - \frac{x^4}{8}\right]_0^a - \frac{w\,a}{l}\left[\frac{l\,x^3}{6} - \frac{x^4}{8}\right]_0^b; \tag{8}$$

$$E\,I\,y = -\frac{w\,a\,b}{6}(a^2 + b^2) + \frac{w\,a\,b}{8\,l}(a^3 + b^3); \tag{9}$$

$$E\,I\,y = -\frac{w\,a\,b}{24}(a^2 + 3\,a\,b + b^2). \tag{10}$$

* This method is largely used in Continental Europe. For a full discussion, which includes trusses as well as beams, see J. A. VAN DEN BROEK, "Elastic Energy Theory," John Wiley and Sons.

Problems

1. Derive the equation of the elastic line for a uniformly loaded cantilever.
2. Substitute $a = x$, $b = l - x$, in Eq. (10) and compare with Eq. (7) of Art. 93.

206. Graphic Integration.—The example of the preceding article shows that the method of elastic energy by algebraic

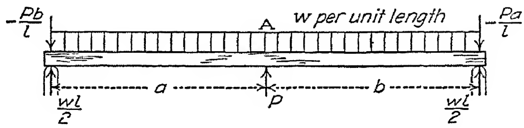


FIG. 268.—Auxiliary or dummy force.

integration offers no advantage over previous methods for deriving the equation of the elastic line of straight beams. Graphic integration, however, may be made less complicated.

If the M_p diagram is bounded by straight lines, and if the area of the M_q diagram (of any form whatever) is A_q, then

$$\int M_p M_q \, dx = A_q h_p, \tag{1}$$

in which h_p is the altitude of the M_p diagram under the center of gravity of the M_q diagram.*

Since M_p is the moment of a concentrated load and reactions, its equation is of the first degree and may be represented by $M_p = a + b\,x$.

$$\int M_p M_q dx = \int (a + b\,x)\, M_q dx = a \int M_q dx + b \int x\, M_q dx = a\, A_q + b\, \bar{x}\, A_q;$$

$$\int M_p M_q \, dx = (a + b\,\bar{x})\, A_q, \tag{2}$$

in which $\bar{x}$ is the abscissa of the center of gravity of A_q, and $a + b\,\bar{x}$ is the ordinate of the M_p diagram which is under the center of gravity of the M_q diagram.

Another improvement (*possibly new*) consists in reversing the M_p diagram, as shown in Fig. 269. Since the resultant moment from any set of forces is the same, no matter which way it is calculated, the result is the same whether the M_p and M_q diagrams go in the same or in opposite directions. If the dummy force is at a distance x from the left end, the dummy reaction

* From "Kräfteplan-Verfahren" by J. D. GEDO, p. 5.

at the right end is $-\frac{P\,x}{l}$ and the altitude of the negative dummy triangle is a simple function of x. Moreover, the areas $O\,A\,B$ and $O\,B\,C$ which accompany the positive dummy triangle are a triangle and a parabola, instead of a trapezoid and a truncated parabola.

Example I

Derive the equation of the elastic line for a simply-supported, uniformly loaded beam.

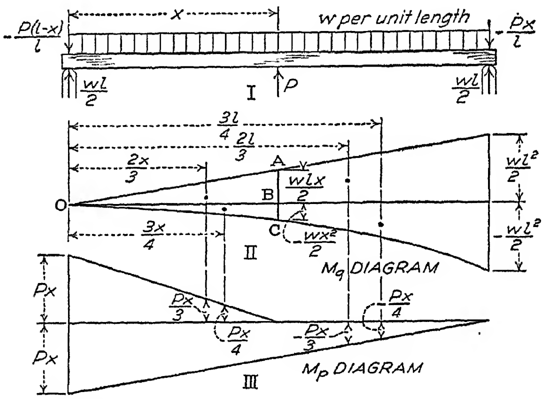


FIG. 269.—Moment diagrams for load and for dummy force.

Figure 269, I, shows the uniformly loaded beam, and Fig. 269, II, gives the M_q diagram. The dummy force P upward at a distance x from the left support causes a downward reaction of $-\frac{P\,x}{l}$ at the right support. The M_p diagram, starting from the right end, consists of a negative triangle of base l and altitude $-P\,x$ and a positive triangle of base x and altitude $P\,x$.

$$\int M_p M_q\,dx = \frac{w\,l^3}{4} \times \left(-\frac{P\,x}{3}\right) - \frac{w\,l^3}{6} \times \left(-\frac{P\,x}{4}\right) + \frac{w\,l\,x^2}{4} \times \frac{P\,x}{3} - \frac{w\,x^3}{6} \times \frac{Px}{4}; \quad (3)$$

$$\int M_p M_q\,dx = -\frac{P\,w\,l^3 x}{24} + \frac{P\,w\,l\,x^3}{12} - \frac{P\,w\,x^4}{24}; \quad (4)$$

$$E\,I\,y = \frac{1}{P}\int M_p M_q dx = -\frac{w\,x}{24}\,(l^3 - 2\,l\,x^2 + x^3). \quad (5)$$

Example II

Derive the equation of the elastic line for a simply-supported beam which carries a load Q at a distance b from the right support. By using Fig. 270 with $P = 1$,

$$E\,I\,y = \frac{Q\,b\,l}{2} \times \left(-\frac{x}{3}\right) - \frac{Q\,b^2}{2} \times \left(-\frac{b\,x}{3\,l}\right) + \frac{Q\,b\,x^2}{2\,l} \times \frac{x}{3}; \tag{6}$$

$$E\,I\,y = -\frac{Q\,b\,l\,x}{6} + \frac{Q\,b^3\,x}{6\,l} + \frac{Q\,b\,x^3}{6\,l} = -\frac{Q\,b\,x}{6\,l}(l^2 - x^2 - b^2). \tag{7}$$

If the load Q is to the left of P, replace b in Eq. (7) by a and measure x from the right end.

Problems

1. Construct Fig. 270 with P on the right of Q and derive the expression in terms of x and a which must be added to Eq. (7) to get the deflection. Compare Eq. (9) of Art. 101.

Ans. $-\dfrac{Q(x-a)^3}{6}$.

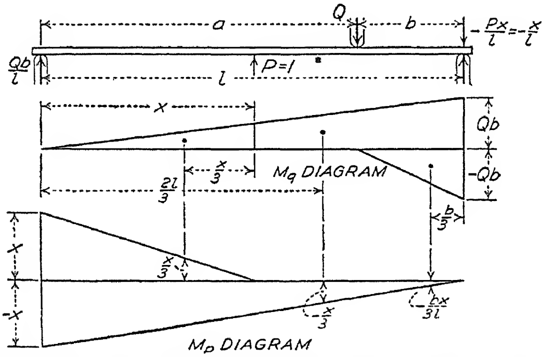


FIG. 270.—Concentrated load and dummy diagrams.

2. A simply-supported beam carries a uniformly distributed load over 0.6 l adjacent to the right end. Derive the equation of the elastic line for portion which is not loaded. Derive the equation of the elastic line for the loaded portion. *Ans.* $E\,I\,y = -0.0246\,w\,l^3x + 0.03\,w\,l\,x^3$.

207. Work of a Couple.—When a couple of moment M_t, in a plane perpendicular to the neutral surface, is applied to any section of a beam, the work done by the couple is the moment multiplied by the angular rotation in radians.

$$U = M_t\theta. \tag{1}$$

When the moment varies with the displacement, the work from zero displacement to θ is

$$U = \frac{M_t\theta}{2}. \tag{2}$$

When M_q and M_t act on the same beam, Equation (8) of Art. 203 becomes

$$U = \int \frac{M_t^2 dx}{2\,E\,I} + \int \frac{M_t M_q\,dx}{E\,I} + \int \frac{M_q^2 dx}{2\,E\,I}. \tag{3}$$

By using the second term of Equation (3) with Equation (1), since M_t is constant while the moment M_q is applied,

$$M_t\,\theta = \frac{1}{E\,I}\int M_t M_q\,dx; \tag{4}$$

$$E\,I\,\theta = \int \frac{M_t M_q\,dx}{M_t}; \tag{5}$$

If the magnitude of M_t is unity, Equation (5) becomes

$$E\,I\,\theta = \int M_t M_q dx. \tag{6}$$

Equations (4), (5), and (6) apply to beams of constant section only.

Example I

Find the slope at the end and at any section for a cantilever with a load on the free end (Fig. 271).

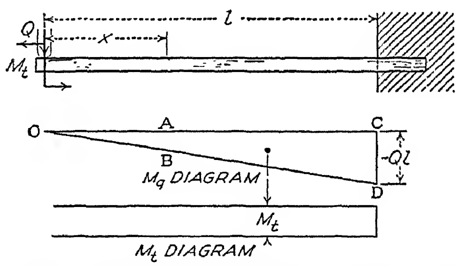


FIG. 271.—Beam with auxiliary moment at end.

The couple M_t at the free end is transmitted as a constant couple to the fixed end.

$$E\,I\,\theta = \int_a^l \frac{M_t\,Q\,x\,dx}{M_t} = -\left[\frac{M_t Q\,x^2}{2\,M_t}\right]_a^l = -\frac{Q(l^2 - a^2)}{2}. \tag{7}$$

By graphic integration, the slope at the end is

$$E\,I\,y = -\frac{Q\,l^2}{2} \times \frac{M_t}{M_t} = -\frac{Q\,l^2}{2} \tag{8}$$

For the slope at x, the trapezoid area is multiplied by $\frac{M_t}{M_t}$.

Problem

1. Find the slope of a uniformly loaded cantilever at the free end and at a distance x from the free end. Use Eq. (6) with algebraic and graphical integration.

Example II

Find the slope at any point on a uniformly loaded beam (Fig. 272). The dummy moment M_t at a distance x from the left end is transmitted to both

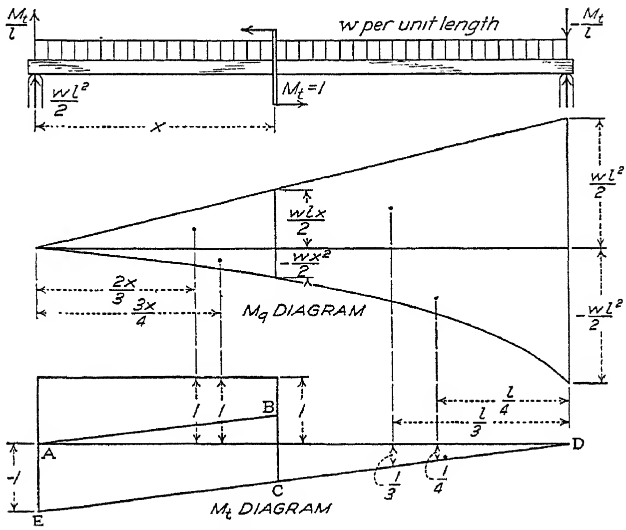


FIG. 272.—Auxiliary moment at any section.

ends and balanced by a downward force $\frac{M_t}{l}$ at the right support and an equal upward force at the left support, which together form a couple. With M_t equal to unity, the combined M_t diagram is $D\ C\ B\ A$. It is best to make the negative triangle of altitude -1 at the left end and the positive rectangle of altitude unity.

$$E\,I\,\theta = \frac{w\,l^3}{4} \times \left(-\frac{1}{3}\right) - \frac{w\,l^3}{6} \times \left(-\frac{1}{4}\right) + \frac{w\,l\,x^2}{4} \times 1 - \frac{w\,x^3}{6} \times 1; \tag{9}$$

$$E\,I\,\theta = -\frac{w\,l^3}{24} + \frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6}. \tag{10}$$

Problems

2. Construct the dummy triangle to find the slope at the left support of Fig. 272. Calculate the slope and check with Eq. (10).
3. Find the slope at the left end of a simply-supported beam which carries a load Q at a distance b from the right support. Use the M_q diagram of Fig. 270, and draw an M_t diagram.

4. Solve Problem 3 for the slope at any point to the left of the load. Use M_t diagram of Fig. 272.
5. Solve Problem 3 to find the slope to the right of the load.
6. A simply-supported beam carries a uniformly distributed load over a length of 0.6 l adjacent to the right end. Find the slope at the left end, at 0.4 l from the left end, and at the middle.
   Ans. $E\,I\,\theta = -0.0246\,w\,l^3;\ -0.0102\,w\,l^3;\ -0.0019\,w\,l^3.$
7. Find the deflection of the beam of Problem 6 for any value of x which is not greater than 0.4 l. From this derive an expression for the slope and check the first two answers of Problem 6.

208. Overhanging Beams.—Figure 273 shows a beam which has a span of length l and overhangs the left support a distance

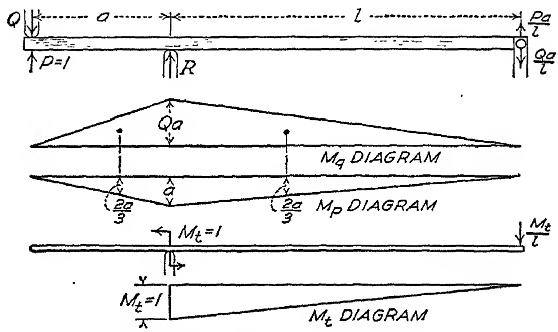


FIG. 273.—Slope over support.

a. When a load Q is placed on the left end, the upward reaction at the right support is reduced $\frac{Q\,a}{l}$. The beam may be regarded as made up of two cantilevers and the total work may be calculated by Equation (1) of Art. 197 without the use of a dummy force. Since all the external work is done by the load Q,

$$\frac{Q}{2}\,y_{\max} = \frac{Q^2a^3}{6\,E\,I} + \frac{Q^2\,a^2l}{6\,E\,I}; \tag{1}$$

$$y_{\max} = \frac{Q\,a^3}{3\,E\,I} + \frac{Q\,a^2l}{3\,E\,I}, \tag{2}$$

positive downward. The first term of the second member of Equation (2) is recognized as the deflection of a cantilever from the tangent at the fixed end.

If θ_1 is the change of slope at the left support when the load Q is applied, the work of bending over the support is equal to the internal work in the span of length l,

$$\frac{Q\,a}{2}\,\theta_1 = \frac{Q^2a^2l}{6\,E\,I}; \tag{3}$$

$$\theta_1 = \frac{Q\,a\,l}{3\,E\,I}. \tag{4}$$

The change of slope at the left support multiplied by the length a should give the last term of Equation (2).

Problems

1. Find the deflection at the left end of the beam of Fig. 273 by the elastic-energy method, using the M_q and M_p diagrams.

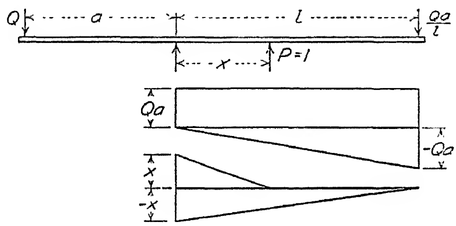


FIG. 274.—Diagrams for a single span.

2. Find the slope over the support of the beam of Fig. 273, using the M_q and M_t diagrams.

To find the deflection between the supports at a distance x from the left support, the M_p diagram is *usually* made for a simply-supported beam. The M_q diagram is not changed but is drawn for the single span alone, since the product of M_pM_q has no value unless both terms are included. Figure 274 shows these diagrams for the beam of Fig. 273. The M_p diagram is exactly like that of Fig. 269. With the M_q diagram as shown in Fig. 273, it would be convenient to reverse the M_p diagram and take the distance from the right support as x. In Fig. 274, the M_q diagram is drawn for two terms as the sum of a positive rectangle and a negative triangle. From this diagram,

$$E\,I\,y = Q\,a\,l \times \left(-\frac{x}{2}\right) + \frac{Q\,a\,l}{2} \times \left(-\frac{x}{3}\right) + Q\,a\,x \times \frac{x}{2} - \frac{Q\,a\,x^2}{2\,l} \times \frac{x}{3}; \tag{5}$$

$$E\,I\,y = -\frac{Q\,a\,l\,x}{3} + \frac{Q\,a\,x^2}{2} - \frac{Q\,a\,x^3}{6\,l}. \tag{6}$$

Problems

3. Knowing the slope at the left support from Eq. (4), derive Eq. (6) by area moments or successive integration.

4. Reverse the M_p diagram of Fig. 274 with x from the right support and find the equation of the elastic line for the span of Fig. 273. Check with Eq. (6).

209. Span Fixed at One End.—Figure 275 illustrates the indeterminate problem of a beam which is fixed at the right end, supported at a distance a from the left end, and uniformly loaded. Either the moment at the fixed end or the reaction of the support may be taken as the unknown. The dummy force is applied at the left support with the beam regarded as a cantilever. This gives an M_p triangle of altitude l when P is taken as unity. Since the deflection at the support is zero,

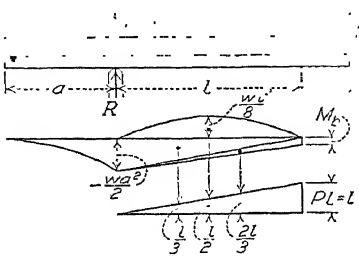


FIG. 275.—Fixed-end span as a cantilever for dummy force.

$$E\,I\,y = 0 = -\frac{w\,a^2l}{4} \times \frac{l}{3} + \frac{M_bl}{2} \times \frac{2\,l}{3} + \frac{W\,l^2}{12} \times \frac{l}{2}; \tag{1}$$

$$-\frac{w\,a^2l^2}{12} + \frac{M_bl^2}{3} + \frac{w\,l^4}{24} = 0; \tag{2}$$

$$M_b = -\frac{w\,l^2}{8} + \frac{w\,a^2}{4}. \tag{3}$$

Instead of the dummy force P at the left support a dummy moment M_t might be taken at the fixed end. The M_t diagram is a triangle like the M_p diagram and the calculation is the same.

Problems

1. Using Eq. (3) find the reaction at the support.
2. Write the moment diagram for the beam of Fig. 275 from the general moment equation $M_0 + V_0x - \frac{w\,x^2}{2}$. Draw the M_g diagram to represent each of these three terms and solve for the unknown shear.
3. A beam of length l is fixed at the right end, supported at the left end, and subjected to a load Q at a distance b from the right end and a distance a from the left end. Find the moment at the fixed end. Calculate the reaction of the support. Compare with Art. 123.

210. Two Spans, One End Fixed.—Figure 276 shows a uniformly loaded beam, 22 feet long, with the right end fixed, which is supported 4 feet from the left end and 10 feet from the left end. The moment over the second support and the moment at

fixed end are unknown. Figure 276, III, shows the beam resting on the second support and at the right end. The dummy force P at the location of the left support causes a force of $\frac{P}{2}$ at the right

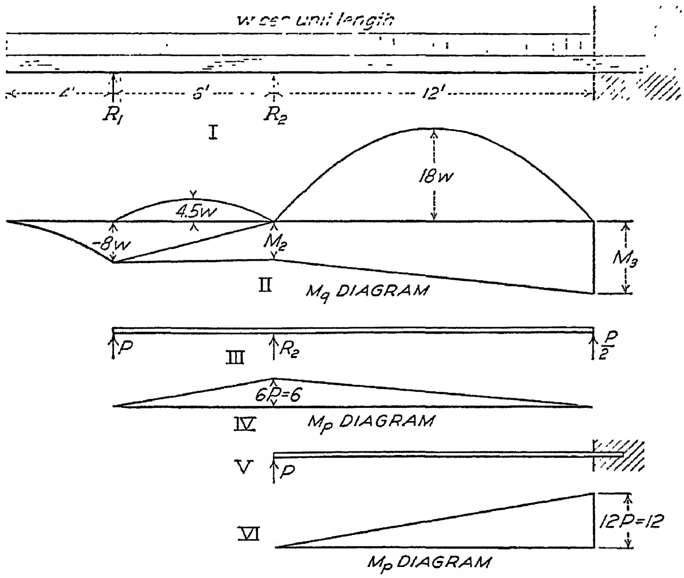


FIG. 276.—Dummy diagrams for two spans and for one span.

end. The dummy moment over the second support is 6 P which equals 6 if P is unity.

$$0 = -24\,w \times 2 + 3\,M_2 \times 4 + 18\,w \times 3 + 6M_2 \times 4 + 6\,M_3 \times 2 + 144\,w \times 3; \quad (1)$$

$$36\,M_2 + 12\,M_3 + 438\,w = 0; \quad (2)$$

$$6\,M_2 + 2\,M_3 + 73\,w = 0. \quad (3)$$

The 12-foot span is used alone for the second equation. For the dummy moment, the span is assumed to be fixed at the right end. The force P at the location of the second support (Fig. 276, V) makes the triangular diagram of Fig. 276, VI. The span might be regarded as simply-supported at the ends with a dummy moment M_1 at the fixed end, where the moment does no work. The final result is the same in each case.

$$0 = 6\,M_2 \times 4 + 6\,M_3 \times 8 + 144\,w \times 6; \quad (4)$$

$$M_2 + 2\,M_3 + 36\,w = 0. \quad (5)$$

$$M_2 = -7.4\,w; \quad M_3 = -14.3\,w.$$

To derive the equation of the elastic line for any span of an indeterminate beam *after* the unknown moments and reactions have been calculated, the dummy diagram is drawn for a span of a simply-supported beam. If either end is fixed, the span may be treated as a simply-supported beam or a cantilever. The theory of Art. 203 was derived for a span which is fully supported, but not necessarily indeterminate, without the dummy force; and the applications of the principles of this article must conform to these restrictions.

In Fig. 276, the M_q diagrams are made of the end-moment diagram and the simple-support combined diagram. These are most convenient when dealing with end reactions and slope at a support. When working inside the span to find the elastic line or the slope at any point, it is desirable to draw the terms in such way that no trapezoids or truncated parabolas will come next to the origin of coördinates. For the first span of Fig. 276 the general moment equation gives.

$$-8\,w + 6\,V_0 - 18\,w = -7.4\,w; \quad V_0 = 3.1\,w.$$

$$M_q = -8\,w + 3.1\,w\,x - \frac{w\,x^2}{2}. \tag{6}$$

Example I

Find the slope of the tangent at the left support of Fig. 276. The M_t diagram is a triangle of altitude 1 if M_1 is unity. Using the M_q formula of Eq. (6) Fig. 277,

$$E\,I\,\theta_1 = -48\,w \times \left(-\frac{1}{2}\right) + 55.8\,w \times \left(-\frac{1}{3}\right) - 36\,w \times \left(-\frac{1}{4}\right); \tag{7}$$

$$E\,I\,\theta_1 = 24\,w - 18.6\,w + 9\,w = 14.4\,w.$$

Problems

1. Solve Example I using the M_q diagram of Fig. 276.
2. Derive the equation of the elastic line for the 6-ft. span of Fig. 276, using the slope at the left support and the area-moment method with Fig. 277 (or use the method of integration between limits).

$$\textit{Ans.}\ E\,I\,y = 14.4\,w\,x - 4\,w\,x^2 + \frac{3.1\,w\,x^3}{6} - \frac{w\,x^4}{24}. \tag{8}$$

Example II

Derive the equation of the elastic line for the 6-ft. span of Fig. 276 by elastic energy without using the slope at the end.

From the M_q and M_p diagrams of Fig. 277,

$$E\,I\,y = -48\,w \times \left(-\frac{x}{2}\right) + 55.8\,w \times \left(-\frac{x}{3}\right) - 36\,w \times \left(-\frac{x}{4}\right) - 8\,w\,x \times \frac{x}{2} + \frac{3.1\,w\,x^2}{2} \times \frac{x}{3} - \frac{w\,x^3}{6} \times \frac{x}{4}. \qquad (9)$$

Example III

Find the slope at the left end of the 12-ft. span of Fig. 277. Regard the dummy beam as a cantilever fixed at the right end which makes the M_t a

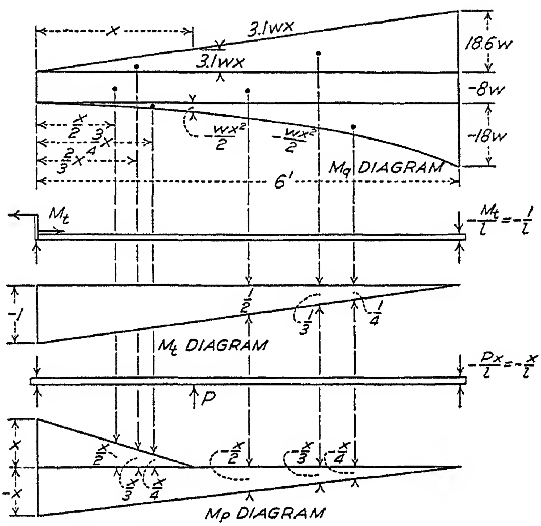


FIG. 277.—Diagrams for slope and for deflection.

rectangle of uniform height $-M_t = -1$. Use the M_q diagram of Fig. 276 and check with that of Fig. 278. Check also by means of the results of Example I and the area of the intervening moment diagram.

$$E\,I\,\theta_2 = -(7.4\,w + 14.3\,w)6 \times (-1) + 18\,w \times 8 \times (-1) = -13.8\,w.$$

Example IV

Find the slope at the left end of the 12-ft. span of Fig. 276 by means of the M_t diagram of Fig. 277 for a simply-supported span. First derive the M_q equation from the data of preceding figures or problems.

Example V

Derive the equation of the elastic line for the 12-ft. span of Fig. 276 using the dummy moment line of Fig. 278 for fixed end.

On Fig. 278, the positive M_q trapezoid is taken as two triangles. The expression for the rectangle of length $12 - x$ is easily determined. The entire parabola of length 12 is taken and the value for the parabola of length x is subtracted.

$$E\,I\,y = -7.4\,w \times (12 - x) \times \frac{12 - x}{2} + 65.1\,w \times \frac{12 - x}{2} \times \frac{2(12 - x)}{3} +$$

$$\frac{21.7\,w\,x}{4} \times \frac{12 - x}{2} \times \frac{12 - x}{3} - 288\,w \times (9 - x) - \left(-\frac{w\,x^3}{6}\right) \times \left(-\frac{x}{4}\right); \quad (10)$$

$$E\,I\,y = 18\,w(12 - x)^2 + \frac{21.7\,w\,x}{24}(12 - x)^2 - 288\,w(9 - x) - \frac{w\,x^4}{24}; \quad (11)$$

$$E\,I\,y = -13.8\,w\,x - 3.7\,w\,x^2 + \frac{21.7\,w\,x^3}{24} - \frac{w\,x^4}{24}. \quad (12)$$

Problems

3. Check Eq. (12) by substituting $x = 12$.
4. Solve Example V by area moments (or integration between limits), using the slope from Example III and the areas of length x of Fig. 278. Exam-

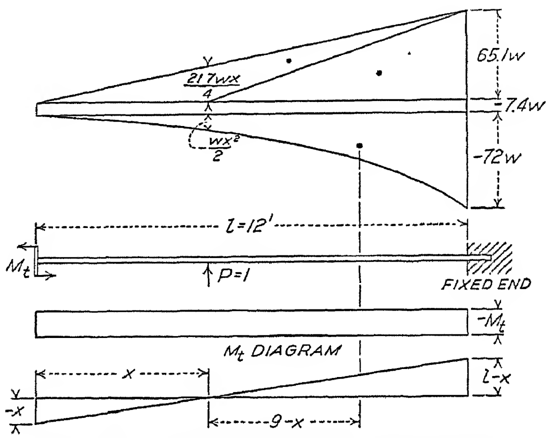


FIG. 278.—Diagrams for slope and deflection.

ple V is exactly the same as the deflection from the tangent at the right end by area moments.
5. Solve Example V, using the reversed M_p diagram for the span as a simply-supported beam, similar to that of Fig. 277. How does the difficulty of this method compare with that of Example V?

211. Both Ends Fixed.—Figure 279 shows a beam which is fixed at both ends, with a load Q at a distance a from the left end and b from the right end. The M_q diagram at the bottom is drawn with the simple-support portions separate. An M_t diagram is drawn with the right end fixed and the dummy moment at the left end. This is a rectangle. An M_p diagram is drawn with the left end fixed. This is a positive triangle. An equivalent triangle of different altitude would have been obtained if the beam were assumed to be simply-supported with a dummy moment at the left end. From the rectangular $M_t = -1$,

$$\left(\frac{M_1 l}{2} + \frac{M_2 l}{2} + \frac{Q\,b\,l}{2} - \frac{Q\,b^2}{2}\right) \times (-1) = 0; \qquad (1)$$

$$M_1 l + M_2 l + Q\,b\,l - Q\,b^2 = 0. \qquad (2)$$

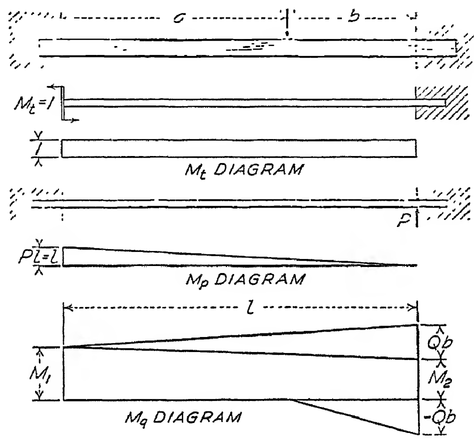


FIG. 279.—Diagrams for beam fixed at both ends.

From the triangular M_p,

$$\frac{M_1 l}{2} \times \frac{2\,l}{3} + \frac{M_2 l}{2} \times \frac{l}{3} + \frac{Q\,b\,l}{2} \times \frac{l}{3} - \frac{Q\,b^2}{2} \times \frac{b}{3} = 0; \qquad (3)$$

$$2\,M_1 l + M_2\,l + Q\,b\,l - \frac{Q\,b^3}{l} = 0; \qquad (4)$$

From Equations (2) and (4),

$$M_1 l = \frac{Q\,b^3}{l} - Q\,b^2; \qquad (5)$$

$$M_1 = -\frac{Q\,b^2}{l^2}(l - b) = -\frac{Q\,b^2 a}{l^2}. \qquad (6)$$

By symmetry, (7)

$$M_2 = -\frac{Q\,a^2 b}{l^2}. \qquad (8)$$

Problems

1. Calculate the shear at the left end of the beam of Fig. 279.

Ans. $V_1 = \frac{Q\,b^2}{l^3}(l + 2\,a)$. (9)

2. Find the maximum positive moment of the beam of Problem 1. Find the moment when the load is at the middle.

Ans. $M = \frac{2\,Q\,a^2 b^2}{l^3}$; $M = \frac{Q\,l}{8}$.

3. What is the moment at each end when the load is at the middle?

$$Ans.\ M_1 = M_2 = -\frac{Q\,l}{8}.$$

4. Find the deflection under the load for the beam of Fig. 279. Use the reversed M_p diagram for a simply-supported span, as shown in Fig. 277. Calculate the deflection when $a = b$.

$$Ans.\ y = -\frac{Q\,a^3b^3}{3\,E\,I\,l^3};\ y = -\frac{Q\,l^3}{192\,E\,I}.$$

212. Three Moments.—Figure 280 shows a beam which is continuous over three fixed points A, B, and C. The points B

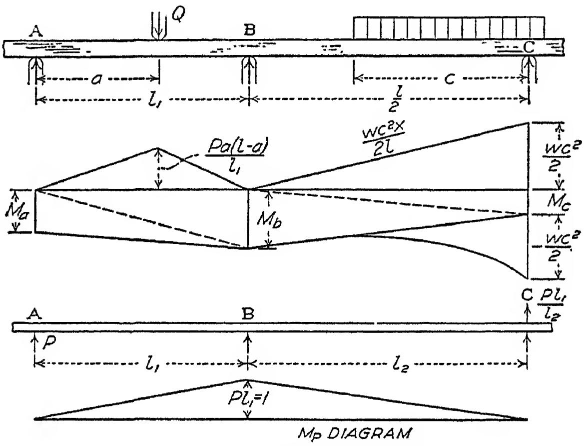


FIG. 280.—Three moments.

and C carry the beam as a simply-supported span. The dummy force P is applied to the overhanging end of length l_1 at the point A. The moment of P about B is balanced by the additional reaction of $\dfrac{P\,l_1}{l_2}$ at C. The M_p diagram has a maximum altitude of $P\,l_1$ at B. Since the $M_p\,M_q$ term is equated to zero, $P\,l_1$ may conveniently be equated to unity.

$$\frac{M_al_1}{2}\times\frac{1}{3}+\frac{M_bl_1}{2}\times\frac{2}{3}+\frac{P\,a(l-a)}{2}\times\frac{l+a}{3}+\frac{M_bl_2}{2}\times\frac{2}{3}+$$
$$\frac{M_cl_2}{2}\times\frac{1}{3}-\frac{w\,c^2l_2}{4}\times\frac{1}{3}-\frac{w\,c^3}{6}\times\frac{1}{4}=0;\quad(1)$$

$$M_al_1+2\,M_bl_1+2\,M_bl_2+M_cl_2+P\,a\,b^2+\frac{w\,c^2l_2}{2}-\frac{w\,c^3}{4}=0.\quad(2)$$

213. Closed Ring.—Figure 281 shows a closed ring, the radius of which is relatively large compared with the radius of the

cross section. This ring is supported at the top and subjected to a load Q at the bottom so that the resultant applied force acts along a diameter. For beams that are curved in any way, the method of *elastic energy* has great advantages.

In the treatment of deflection of curved beams, forces *outward* along a radius cause positive moment. The moment is *positive*

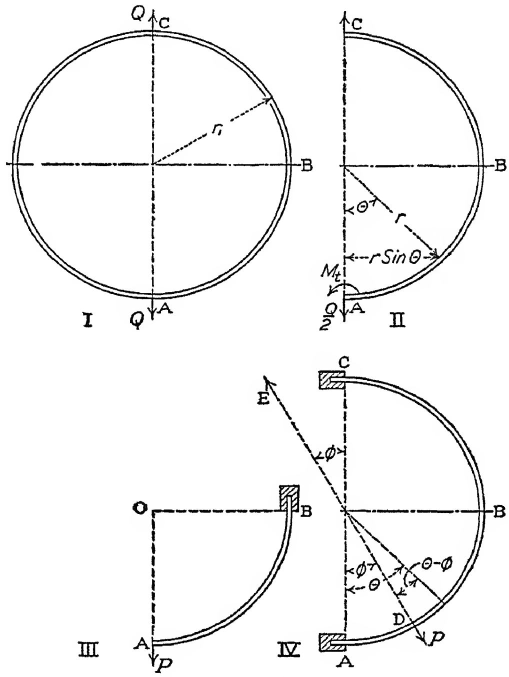


FIG. 281.—Closed ring with single load.

when the *outer* fibers are in *compression*. Considering the right half of the ring as the free body, there is no horizontal tension at A or C. No horizontal external forces act on this free body; hence the sum of the tensions at A and C is zero. From symmetry, the tension at A must have the same magnitude and sign as the tension at C. Since the sum is zero, each force must be zero. The vertical load on each half is $\frac{Q}{2}$.

$$M = M_a + \frac{Q\,r}{2} \sin \theta. \tag{1}$$

From symmetry it is evident that there is no change of slope at A, B, or C. A portion from A to B may be regarded as a free

body fixed at B. A dummy moment M_t applied at A is transmitted as a constant M_t from A to B (or from A to C). The element of length is $dl = r\,d\theta$.

$$E\,I\,\theta = \frac{\int M_t\left(M_a + \frac{Q\,r}{2}\sin\theta\right)r\,d\theta}{M_t}. \tag{2}$$

Since M_t is a constant, it may be canceled.

$$0 = \int_0^{\frac{\pi}{2}}\left(M_a + \frac{Q\,r}{2}\sin\theta\right)r\,d\theta = r\left[M_a\theta - \frac{Q\,r}{2}\cos\theta\right]_0^{\frac{\pi}{2}}; \tag{3}$$

$$\frac{M_a\pi}{2} + \frac{Q\,r}{2} = 0; \quad M_a = -\frac{Q\,r}{\pi} = -0.3183\,Q\,r. \tag{4}$$

From Equation (1),

$$M_b = -\frac{Q\,r}{\pi} + \frac{Q\,r}{2} = (-0.3183 + 0.50)Q\,r = 0.1817\,Q\,r. \tag{5}$$

To find the elongation of the vertical radius $O\,A$, the quadrant $A\,B$ of Fig. 281,II, is taken as a free body, which is fixed at B and subjected to the dummy load P at A. $M_p = P\,r\sin\theta$.

$$\frac{M_pM_q}{P} = Q\,r^2\sin\theta\left(-\frac{1}{\pi} + \frac{\sin\theta}{2}\right) = Q\,r^2\left(-\frac{\sin\theta}{\pi} + \frac{\sin^2\theta}{2}\right). \tag{6}$$

$$E\,I\,y = Q\,r^3\int\left(-\frac{\sin\theta}{\pi} + \frac{1}{4} - \frac{\cos 2\theta}{4}\right)d\theta; \tag{7}$$

$$E\,I\,y = Q\,r^3\left[\frac{\cos\theta}{\pi} + \frac{\theta}{4} - \frac{\sin 2\theta}{8}\right]_0^{\frac{\pi}{2}} = Q\,r^3\left(-\frac{1}{\pi} + \frac{\pi}{8}\right); \tag{8}$$

$$y = \frac{Q\,r^3}{E\,I}(-0.3183 + 0.3927) = 0.0744\,\frac{Q\,r^3}{E\,I}, \tag{9}$$

which is the elongation of the radius $O\,A$. The diameter $C\,A$ is lengthened $0.1488\,\frac{Q\,r^3}{E\,I}$.

The same result is obtained if the half circle $A\ C$ is treated as a free body. The M_p expression is not changed but the limits become 0 and π. The integral gives the elongation of the diameter.

To find the elongation of any diameter, such as $D\,E$, which makes an angle ϕ with the vertical, dummy forces of magnitude $2\,P$ are applied at the ends of this diameter. Each dummy force

is supposed to be broken into two equal forces of magnitude P. The portion $D\,C$ of the ring is supposed to be fixed at C. $M_p = P\,r\sin(\theta - \phi) = P\,r(\sin\theta\cos\phi - \cos\theta\sin\phi)$;

$$M_q = Q\,r\left(-\frac{1}{\pi} + \frac{\sin\theta}{2}\right);$$

$$\frac{M_pM_q}{P} = Q\,r^2\left(-\frac{\sin(\theta-\phi)}{\pi} + \frac{\sin^2\theta\cos\phi}{2} - \frac{\sin\theta\cos\theta\sin\phi}{2}\right); \qquad (10)$$

$$\frac{M_pM_q}{P} = Q\,r^2\left(-\frac{\sin(\theta-\phi)}{\pi} + \frac{\cos\phi}{4} - \frac{\cos 2\theta\cos\phi}{4} - \frac{\sin 2\theta\sin\phi}{4}\right); \qquad (11)$$

$$\frac{\int_\varphi^\pi M_pM_q r\,d\theta}{P} = Q\,r^3\left[\frac{\cos(\theta-\phi)}{\pi} + \frac{\theta\cos\phi}{4} - \frac{\sin 2\theta\cos\phi}{8} + \frac{\cos 2\theta\sin\phi}{8}\right]_\phi^\pi; \qquad (12)$$

$$\frac{\int_\varphi^\pi M_pM_q r\,d\theta}{P} = Q\,r^3\left(-\frac{\cos\phi}{\pi} - \frac{1}{\pi} + \frac{(\pi-\phi)\cos\phi}{4} + \frac{\sin 2\phi\cos\phi}{8} + \frac{\sin\phi}{8} - \frac{\cos 2\phi\sin\phi}{8}\right). \qquad (13)$$

$$\frac{\sin 2\phi\cos\phi}{8} - \frac{\cos 2\phi\sin\phi}{8} = \frac{\sin\phi}{8};$$

$$\frac{\int_\varphi^\pi M_pM_q r\,d\theta}{P} = Q\,r^3\left(-\frac{1}{\pi} - \frac{\cos\phi}{\pi} + \frac{(\pi-\phi)\cos\phi}{4} + \frac{\sin\phi}{4}\right). \qquad (14)$$

The portion of the right half of the ring from A to D is supposed to be fixed at A. The dummy moment is the moment of the second half of the force $2\,P$ acting at D.

$$M_p = P\sin(\phi - \theta).$$

$$\frac{\int_0^\varphi M_pM_q r\,d\theta}{P} = Q\,r^3\int_0^\varphi\left(-\frac{\sin(\phi-\theta)}{\pi} + \frac{\sin 2\theta\sin\phi}{4} - \frac{\cos\phi}{4} + \frac{\cos 2\theta\cos\phi}{4}\right)d\theta. \qquad (15)$$

$$= Q\,r^3\left[-\frac{\cos(\phi-\theta)}{\pi} - \frac{\cos 2\theta \sin\phi}{8} - \frac{\theta\cos\phi}{4} + \frac{\sin 2\theta\cos\phi}{8}\right]_0^{\varphi} \quad (16)$$

$$= Q\,r^3\left(-\frac{1}{\pi} + \frac{\cos\phi}{\pi} - \frac{\cos 2\phi\sin\phi}{8} + \frac{\sin\varphi}{8} - \frac{\phi\cos\phi}{4} + \frac{\sin 2\phi\cos\phi}{8}\right) \quad (17)$$

$$= Q\,r^3\left(-\frac{1}{\pi} + \frac{\cos\phi}{\pi} + \frac{\sin\phi}{4} - \frac{\phi\cos\phi}{4}\right). \quad (18)$$

When Equations (14) and (18) are added to get the total energy in the left half of the ring, the result is

$$E\,I\,y = Q\,r^3\left(-\frac{2}{\pi} + \frac{(\pi-2\phi)\cos\phi}{4} + \frac{\sin\phi}{2}\right). \quad (19)$$

Problems

1. Find the elongation of the diameter in the line of the load by means of Eq. (19).

Ans. Elongation $= \frac{0.1488\,Q\,r^3}{E\,I}$.

2. Find the elongation of a diameter at 90° with the line of the load.

Ans. Elongation $= -\frac{0.1366\,Q\,r^3}{E\,I}$.

3. Find the elongation at 45° with the line of the load.

Ans. Elongation $= -\frac{0.0054\,Q\,r^3}{E\,I}$.

4. A ring of circular section, 2 in. in diameter, has a radius of 10 in. Find the maximum positive moment and the maximum negative moment when this ring is subjected to a pull of 1,200 lb.

Ans. $M = -3{,}820$ in.-lb.; 2,180 in.-lb.

214. Closed Ring with Two Symmetrical Loads.—Figure 282 shows a closed ring, suspended at the top, which resists a force Q, normal to the circumference, at an angle β to the right of the vertical downward and an equal force at the same angle to the left of the vertical downward. Since the vertical diameter $A\,C$ divides the ring and the loads symmetrically, the half ring of Fig. 282,II, may be taken as the free body. This half ring may be regarded as fixed at the top and fixed, in so far as rotation is concerned, at the bottom. There is an unknown bending moment M_a and an unknown tension H at A.

$$0 \text{-------------} \pi \qquad \beta \text{---------} \pi$$

$$M_q = M_a - H\,r(1 - \cos\theta) + Q\,r \sin(\theta - \beta). \qquad (1)$$

Since the integrals required for finding the unknown quantities are equated to zero, any power of the constant radius which multiplies all terms may be dropped.

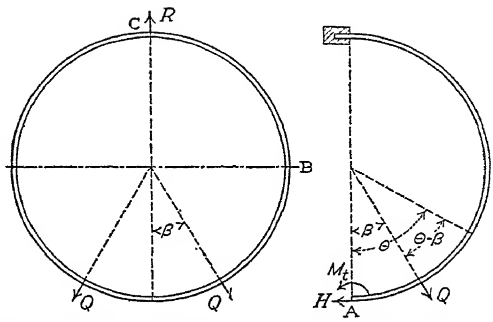


FIG. 282.—Closed ring with two loads symmetrically placed.

A dummy moment M_t at A is carried unchanged to the fixed end at C.

$$\frac{\int M_t M_q\,d\theta}{M_t} = \int_0^{\pi}\Big(M_a - H\,r(1 - \cos\theta)\Big)d\theta + Q\,r\int_{\beta}^{\pi}\sin(\theta - \beta)\,d\theta; \qquad (2)$$

$$0 = \Big[M_a\theta - H\,r\,\theta + H\,r\sin\theta\Big]_0^{\pi} - Q\,r\Big[\cos(\theta - \beta)\Big]_{\beta}^{\pi}; \qquad (3)$$

$$0 = M_a\pi - H\,r\,\pi + Q\,r\cos\beta + Q\,r. \qquad (4)$$

The horizontal force H at A causes no horizontal displacement. A dummy force P at A gives

$$M_p = P\,r(1 - \cos\theta).$$

$$\frac{M_p M_q}{P\,r} = M_a - M_a\cos\theta - H\,r(1 - 2\cos\theta + \cos^2\theta) + Q\,r\sin(\theta - \beta) - Q\,r\sin\theta\cos\theta\cos\beta + Q\,r\cos^2\theta\sin\beta; \qquad (5)$$

$$\frac{\int M_p M_q\,d\theta}{P\,r} = 0 = \int_0^{\pi}\left(M_a - M_a\cos\theta + H\,r\left(-\frac{3}{2} + 2\cos\theta - \frac{\cos 2\theta}{2}\right)d\theta\right) + Q\,r\int^{\pi}\left(\sin(\theta - \beta) - \frac{\sin 2\theta\cos\beta}{2} + \frac{\sin\beta}{2} + \frac{\cos 2\theta\sin\beta}{2}\right)d\theta; \qquad (6)$$

$$0 = \left[M_a\theta - M_a \sin\theta + H\,r\left(-\frac{3\,\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right)\right]_0^\pi$$
$$+ Q\,r\left[-\cos(\theta - \beta) + \frac{\cos 2\theta \cos\beta}{4} + \frac{\theta \sin\beta}{2} + \frac{\sin 2\theta \sin\beta}{4}\right]_\beta^\pi \quad (7)$$

$$0 = M_a\pi - \frac{3\,H\,r\,\pi}{2} + Q\,r\left(\cos\beta + 1 + \frac{\cos\beta}{4} - \frac{\cos 2\beta\cos\beta}{4} + \frac{(\pi - \beta)}{2}\sin\beta - \frac{\sin 2\beta \sin\beta}{4}\right). \quad (8)$$

When Equation (4) is subtracted from Equation (8),

$$0 = -\frac{H\,r\,\pi}{2} + Q\,r\left[\frac{(\pi - \beta)\sin\beta}{2} + \left(\frac{\cos\beta}{4} - \frac{\cos 2\beta\cos\beta}{4} - \frac{\sin 2\beta\sin\beta}{4} = 0\right)\right]; \quad (9)$$

$$H = \frac{\pi - \beta}{\pi} Q \sin\beta. \quad (10)$$

Equation (10) expresses the remarkably simple relation that the horizontal component of Q is divided in the inverse ratio to the arcs of the circle.

Problems

1. Find the horizontal tension at the bottom when the angle is 60°. Check by resolutions parallel to Q using an arc of 120° as the free body.

   Ans. $H = \dfrac{Q}{\sqrt{3}}$.

2. In Problem 1, find the horizontal tension at the top.

   Ans. $T = \dfrac{Q}{2\sqrt{3}}$.

3. Find H and T when the forces Q make angles of 45° with the vertical.

   Ans. 0.5303 Q; 0.1768 Q.

4. Solve Problem 2 by horizontal resolutions with the right half of Fig. 282 as the free body and H known from Problem 1.

From Equations (10) and (4),

$$M_a = \frac{\pi - \beta}{\pi} Q\,r\sin\beta - \frac{Q\,r}{\pi}(1 + \cos\beta). \quad (11)$$

5. Find M_a when $\beta = 60°$.

   Ans. $M_a = \dfrac{Q\,r}{\sqrt{3}} - \dfrac{3\,Q\,r}{2\pi} = 0.0996\,Q\,r$.

6. With M_a and H_a known, find M_c at the top when $\beta = 60°$. Check by finding M_f at the load Q.

   Ans. $M_c = M_f = -0.1888\,Q\,r$.

CHAPTER XIX

CURVED BEAMS AND HOOKS

215. Stresses in Curved Beams.—Figure 283 represents a portion of a curved beam between two planes $A\ B$ and $C\ D$, which are perpendicular to the plane of the paper and intersect at the center of curvature of the beam. The plane $A\ B$ at the left end of the portion is regarded as fixed, while the plane $C\ D$ at at the right end is rotated through an angle θ to the position $C'\ D'$ when the beam is bent. The unit stresses in a beam of

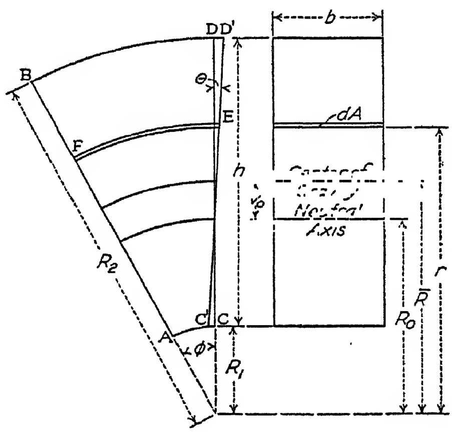


FIG. 283.—Curved beam of rectangular section.

this kind do not vary directly as the distance from the neutral surface, because the length of the elementary filaments are not the same. If the neutral axis were midway between C and D, the total elongation $D\ D'$ would equal the total compression $C\ C'$ but the *unit elongation* at the top would be smaller than the *unit compression* at the bottom because the original length $B\ D$ is greater than the original length $A\ C$.

Since the length of any filament, such as $E\ F$, is proportional to its distance from the center of curvature, the unit deformation and the unit stress vary as the quotient of the distance from the neutral axis divided by the distance from the center of curvature.

In Fig. 283, R_1 is the inside radius, R_2 is the outside radius, R_0 is the radius of the neutral surface, r is the radius of any filament, and v_0 is the distance of the neutral axis from the center of gravity of the cross section. The angle at the center of curvature subtended by the portion of the beam is ϕ, so that the original length of any filament is $r\phi$. The angle through which the plane $C\,D$ is turned when the beam is bent is θ. If the assumption that a cross section of a beam remains plane when the beam is bent *be valid for curved as well as for straight beams*, the deformation of a filament at a distance $r - R_0$ from the neutral axis is $(r - R_0)\theta$ and

$$\text{Unit deformation} = \frac{(r - R_0)\theta}{r\,\phi}. \tag{1}$$

$$\text{Unit stress} = s = \frac{E(r - R_0)\theta}{r\,\phi} = k\left(1 - \frac{R_0}{r}\right). \tag{2}$$

in which $k = \dfrac{E\,\theta}{\phi}$ is a constant for every part of the section at any particular loading.

At the innermost fibers

$$S_1 = k\left(1 - \frac{R_0}{R_1}\right). \tag{3}$$

At the extreme outer fibers

$$S_2 = k\left(1 - \frac{R_0}{R_2}\right). \tag{4}$$

The location of the neutral axis is found by means of the condition that the total stress across any section is zero.

$$\text{Stress on element of area } dA = k\left(1 - \frac{R_0}{r}\right)dA; \tag{5}$$

$$\text{Total stress} = k\int_{R_1}^{R_2}\left(1 - \frac{R_0}{r}\right)dA = 0. \tag{6}$$

$$A = R_0\int_{R_1}^{R_2}\frac{dA}{r}; \quad R_0 = \frac{A}{\displaystyle\int_{R_1}^{R_2}\frac{dA}{r}}. \tag{7}$$

The first step in the derivation of a formula for the stress in a curved beam of any section is the location of the neutral axis for that section by means of Equation (7), which does not depend

upon the constant k. With R_0 now known, the second step is the derivation of an expression for the resisting moment to be used for the elimination of k.

Since the resisting moment of a beam is a couple, the calculation of the moment of this couple may be made with respect to any axis whatever. The mathematics is greatly simplified if the *origin* of *moments* is taken at the center of curvature of the beam. With the *origin* of *coördinates* at this center of curvature, the moment arm of every element is r.

$$\text{Moment} = \int_{R_1}^{R_2} s\, r\, dA = k\int_{R_1}^{R_2} r\left(1 - \frac{R_0}{r}\right)dA = k\int_{R_1}^{R_2}(r - R_0)\, dA; \quad (8)$$

$$M = k\left(\int_{R_1}^{R_2} r\, dA - R_0\int_{R_1}^{R_2} dA\right) = k(\bar{R} - R_0)A, \quad (9)$$

in which $\bar{R}$ is the radius to the center of gravity of the cross section.

$$M = k\, v_0 A, \qquad \text{Formula XLII}$$

in which $v_0 = \bar{R} - R_0$ equals the distance of the center of gravity of the cross section from the neutral surface.

216. Curved Beams of Rectangular Section.—For a rectangular beam of breadth b and depth h, $dA = b\, dr$ and $A = b\, h$.

$$R_0 = \frac{b\, h}{b\displaystyle\int_{R_1}^{R_2}\frac{dr}{r}} = \frac{h}{\log_e \dfrac{R_2}{R_1}}; \quad (1)$$

$$v_0 = R_1 + \frac{h}{2} - R_0. \quad (2)$$

Example I

A curved beam of rectangular section is 3 in. wide and 4 in. high. The inner radius is 4 in. Locate the neutral axis and find the stress at the inner and outer surface.

$$R_2 = 8 \text{ in.}; \quad \frac{R_2}{R_1} = 2;$$

$$R_0 = \frac{4}{\log_e 2} = \frac{4}{0.69315} = 5.7707 \text{ in.}$$

$$v_0 = 6 - 5.7707 = 0.2293 \text{ in.}$$

$$\frac{v_0}{h} = 0.05732.$$

In a rectangular beam of depth equal to the radius of the inner surface, the neutral axis is shifted toward the center of curvature 0.05732 of the depth, or nearly 6 per cent.

To find the stress at any fiber of a rectangular beam in terms of the bending moment, the constant k is eliminated from Formula XLII and Eq. (2) of Art. 215.

$$s = k\left(1 - \frac{R_0}{r}\right); \quad M = k\, v_0 b\, h;$$

$$s = \frac{M\left(1 - \frac{R_0}{r}\right)}{v_0\, b\, h}. \tag{3}$$

To find the unit stress in the outer fibers,

$$S_2 = \frac{M\left(1 - \frac{5.7707}{8}\right)}{3 \times 4 \times 0.2293} = 0.10127\, M;$$

$$S_1 = \frac{M\left(\frac{5.7707}{4} - 1\right)}{2.7516} = 0.16089\, M.$$

If S is the unit stress in a straight beam of the same section

$$S = \frac{M}{8}; \quad \frac{S_2}{S} = 0.8102; \quad \frac{S_1}{S} = 1.2871.$$

It is convenient to express R_1 and R_2 in terms of h as a unit and to take b as unity. For any particular problem, the values of R_0 and v_0 may be multiplied by the height in inches, and the values of S_1 or S_2 may be divided by $b\, h^2$. Since v_0 is the difference between relatively large numbers, it is necessary to use five-place tables to insure three-place accuracy. If good tables of natural logarithms are not available, common logarithms may be used.

$$R_0 = \frac{1}{\log_e \frac{R_2}{R_1}} = \frac{1}{\frac{\log_{10} \frac{R_2}{R_1}}{0.43429448}} = \frac{0.43429448}{\log_{10} \frac{R_2}{R_1}}.$$

Example II

If $\frac{h}{R_1} = 2$, find the stress in the outer fibers in terms of the moment regarding h and b as unity.

$$R_1 = \frac{1}{2}; \quad R_2 = \frac{3}{2}; \quad \bar{R} = 1; \quad \frac{R_2}{R_1} = 3;$$

$$R_0 = \frac{0.43429448}{0.47712125} = 0.91023926\, h;$$

$$v_0 = 0.08976074\, h;$$

$$S_1 = \frac{0.820478}{v_0} = \frac{9.1406\,M}{b\,h^2};\quad \frac{S_1}{S} = 1.5234.$$

$$S_2 = \frac{0.393174}{v_0} = \frac{4.3803\,M}{b\,h^2};\quad \frac{S_2}{S} = 0.7300.$$

The logarithm of 3 in the expression for R_0 was taken from Macmillan's "Logarithmic and Trigonometric Tables," p. 133.

TABLE XXIX.—DISPLACEMENT OF NEUTRAL SURFACE AND RELATIVE STRESSES IN EXTREME FIBERS OF CURVED BEAM OF RECTANGULAR SECTION

Ratio of depth to inner radius $\frac{h}{R_1}$	Distance of neutral axis		Ratio of unit stress in extreme fibers to stress in straight beam	
	From center of curvature $\frac{R_0}{h}$	From center of gravity $\frac{v_0}{h}$	$\frac{S_1}{S}$	$\frac{S_2}{S}$
0.50	2.466303	0.033697	1.1532	0.8799
1.00	1.442695	0.057305	1.2875	0.8103
1.50	1.091357	0.075310	1.4098	0.7639
2.00	0.910239	0.089761	1.5234	0.7300
3.00	0.721347	0.111986	1.7324	0.6814
4.00	0.621335	0.128665	1.9240	0.6514
5.00	0.558111	0.141889	2.1020	0.6282

Table XXIX gives the ratio of the stress in the extreme fibers of a curved beam to the corresponding stress in a straight beam. These ratios are plotted as ordinates on Fig. 284. The curve for the unit stress at the concave surface does not differ greatly from the broken straight line for which the equation is

$$S_1 = S\left(1 + 0.25\frac{h}{R_1}\right);\quad S_1 = \frac{6\,M}{b\,h^2}\left(1 + 0.25\frac{h}{R_1}\right). \tag{4}$$

Problems

1. Verify Table XXIX for $\frac{h}{R_1} = 1.5$.

2. A curved cast-steel beam is 5 in. wide, has an inner radius of 2 in. and an outer radius of 8 in. By means of the table, find the unit stress at the inner and outer fibers under a bending moment of 20,000 ft.-lb. Find the unit stress at 1 in. from the outer surface by means of Eq. (3).
Ans. 13,859 lb./in.^{2}; 5,451 lb./in.^{2}; 4,545 lb./in.^{2}

3. A cast-steel beam is 6 in. wide, has an inner radius of 2 in. and an outer radius of 7 in. Find the stress in the extreme fibers when M = 9,000

ft.-lb. Solve approximately by Eq. (4). Compare with the true curve of Fig. 284.

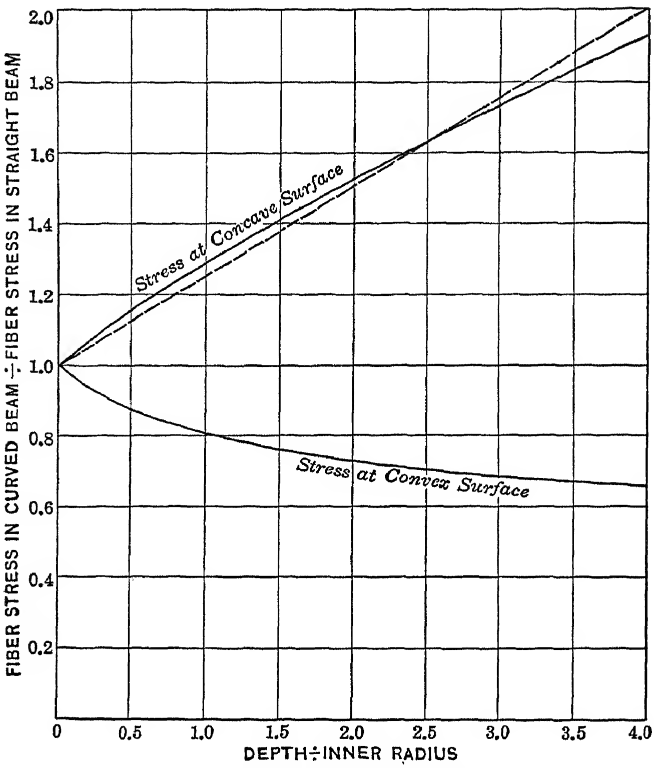


FIG. 284.—Stress at outer fibers of curved beam of rectangular section.

Example III

Find the stress in the extreme fibers of the section of Fig. 285. Compare with the stress in a straight beam.

$$A = 20 \text{ in.}^2; \quad \bar{R} = 6.2 \text{ in.}$$

$$\int \frac{dA}{r} = 6\Big[\log_e r\Big]_4^6 + 2\Big[\log_e r\Big]_6^{10} = 3.45444; \quad R_0 = \frac{20}{3.45444} = 5.78965 \text{ in.};$$

$$v_0 = 0.41035; \quad S_1 = \frac{1.45582 - 1}{8.2070} M = 0.054515\, M; \quad S_2 = ?$$

Problems

4. A hollow beam is 1 in. square outside and 0.8 in. square inside. The beam is curved to an inner radius of 0.5 in. Find the ratio of the stress in the outer fibers to the stress in a straight beam.

Ans. $\dfrac{S_1}{S} = 1.3458; \dfrac{S_2}{S} = 0.8131.$

5. Solve Problem 4 if the inside is 0.6 in. square.

$$Ans.\ \frac{S_1}{S} = 1.4372;\ \frac{S_2}{S} = 0.7811.$$

6. Solve Problem 4 if the inside is 0.4 in. square

$$Ans.\ \frac{S_1}{S} = 1.4918;\ \frac{S_2}{S} = 0.9547.$$

7. Compare the answers of the last three problems with a solid rectangular section for which $\frac{h}{R_1}$ is the same.

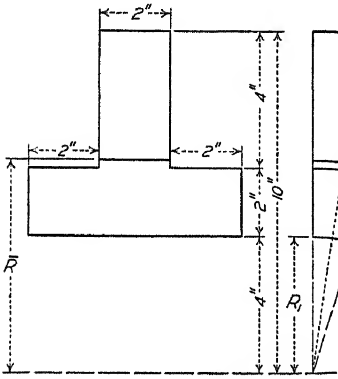


FIG. 285.—Curved beam of T section.

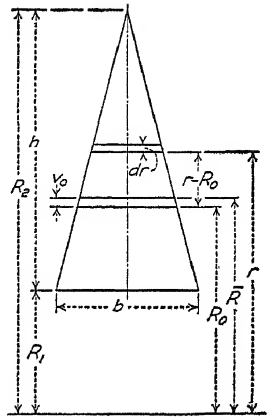


FIG. 286.—Curved beam of triangular section.

217. Triangular Beam.—Figure 286 shows a triangular beam with its base at a distance R_1 from the center of curvature.

$$dA = \frac{b}{h}(R_2 - r)dr; \quad s = k\left(1 - \frac{R_0}{r}\right);$$

$$\int_{R_1}^{R_2} s\,dA = \frac{b\,k}{h}\int_{R_1}^{R_2}\left(1 - \frac{R_0}{r}\right)(R_2 - r)dr = \frac{b\,k}{h}\int_{R_1}^{R_2}\left(R_2 - r + R_0 - \frac{R_0R_2}{r}\right)dr; \quad (1)$$

$$0 = \left[R_2r - \frac{r^2}{2} + R_0r - R_0R_2\log_e r\right]_{R_1}^{R_2} = R_2h - \frac{R_2^2 - R_1^2}{2} + R_0h - R_0R_2\log\frac{R_2}{R_1}; \quad (2)$$

$$R_0 = -\frac{h^2}{2\left(h - R_2\log\frac{R_2}{R_1}\right)} = \frac{h}{2\left(\frac{R_2}{h}\log\frac{R_2}{R_1} - 1\right)}. \quad (3)$$

Example

Locate the neutral surface in a curved beam of triangular section for which $\frac{h}{R_1} = 5$. Find the ratio of the stress in the outer fibers to the stress in a straight beam of the same section.

$$R_1 = 0.2\,h; \quad R_2 = 1.2\,h; \quad \frac{R_2}{R_1} = 6.$$

$$1.2 \times 1.79175947 - 1 = 1.150111364.$$

$$R_0 = 0.5\,h \div 1.150111364 = 0.4347405\,h$$

$$\bar{R} = 0.5333333\,h$$

$$v_0 = 0.0985928\,h$$

$$S_2 = \frac{\left(1 - \frac{0.4347405}{1.2}\right) M}{\frac{b\,h}{2} \times 0.0985928\,h} = \frac{12.9364\,M}{b\,h^2};$$

$$\frac{S_2}{S} = \frac{12.9364\,M}{b\,h^2} \times \frac{b\,h^2}{24\,M} = 0.5390.$$

$$S_1 = \frac{\left(\frac{0.4347405}{0.2} - 1\right)}{\frac{b\,h}{2} \times 0.0985928\,h} = \frac{23.8089\,M}{b\,h^2};$$

$$\frac{S_1}{S} = \frac{23.8089\,M}{b\,h^2} \times \frac{b\,h^2}{12\,M} = 1.9841.$$

TABLE XXX.—DISPLACEMENT OF NEUTRAL SURFACE AND RELATIVE STRESSES IN EXTREME FIBERS OF CURVED BEAM OF TRIANGULAR SECTION

Ratio of depth to inner radius $\frac{h}{R_1}$	Distance of neutral axis from		Stress in extreme fibers in terms of $\frac{M}{b\,h^2}$		Ratio of stress to stress in straight beam	
	Center of curvature $\frac{R_0}{h}$	Center of gravity $\frac{v_0}{h}$	S_1	S_2	$\frac{S_1}{\frac{12\,M}{b\,h_2}}$	$\frac{S_2}{\frac{24\,M}{b\,h_2}}$
0.5	2.310586	0.022747	13.6540	20.2053	1.1378	0.8419
1.0	1.294350	0.038983	15.1014	18.1015	1.2584	0.7542
1.5	0.948495	0.051505	16.4156	16.7325	1.3680	0.6972
2.0	0.771702	0.061631	17.6341	15.7561	1.4695	0.6565
3.0	0.589350	0.077317	19.8675	14.4337	1.6556	0.6014
4.0	0.494170	0.089163	21.9079	13.5631	1.8257	0.5651
5.0	0.434740	0.098593	23.8089	12.9364	1.9841	0.5390

Problems

1. Make the calculations of Table XXX for $\frac{h}{R_1} = 2$, using five-place natural logarithms to determine R_0.
2. Make the calculations of Table XXX for $\frac{h}{R_1} = 4$, using five-place common logarithms.
3. A curved beam of triangular section is 4 in. high and 3 in. wide. The inner radius from the center of curvature to the base of the triangle is 12 in. Find the unit stress in the extreme fibers for a bending moment of 1,500 ft.-lb.
4. Plot curves for the ratios of Tables XXIX and XXX on the same sheet. What can you learn from these curves?

218. Curved Beams of Converging Trapezoidal Section.—Figure 287 shows a curved beam of trapezoidal section with the

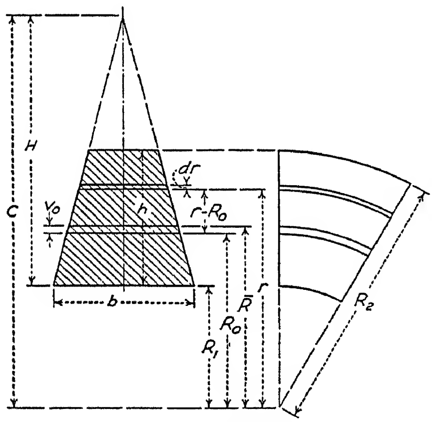


FIG. 287.—Curved beam of trapezoidal section, sides converging.

outer breadth smaller than the inner breadth. The height from the inner surface to the outer surface is h. The height from the inner surface to the intersection of the sides is H; and the radius from the center of curvature to this intersection is C.

$$\text{Total stress} = k\int\left(1 - \frac{R_0}{r}\right)dA = k\left(\int_{R_1}^{R_2} dA - R_0\int_{R_1}^{R_2}\frac{dA}{r}\right).$$

$$dA = \frac{b}{H}(C - r)dr;$$

$$0 = \int_{R_1}^{R_2} dA - \frac{R_0 b}{H}\int_{R_1}^{R_2}\left(\frac{C}{r} - 1\right)dr$$

$$= A - \frac{R_0 b}{H}\left[C \log_e \frac{R_2}{R_1} - h\right]_{R_1}^{R_2}.$$

$$R_0 = \frac{A\,H}{b\,h\left(\frac{C}{h}\log_e\frac{R_2}{R_1} - 1\right)}.$$

TABLE XXXI.—DISPLACEMENT OF NEUTRAL SURFACE AND RELATIVE STRESSES IN EXTREME FIBERS OF CURVED BEAM OF TRAPEZOIDAL SECTION WITH OUTER BASE ONE-HALF OF INNER BASE

$$\text{Area} = \frac{3\,b\,h}{4};\quad \bar{y} = \frac{4\,h}{9};\quad I_c = \frac{13\,b\,h^3}{216};\quad Z_1 = \frac{13\,b\,h^2}{96};$$
$$Z_2 = \frac{13\,b\,h^2}{120}$$

Ratio of depth to inner radius $\frac{h}{R_1}$	Distance of neutral axis from: Center of curvature $\frac{R_0}{h}$	Distance of neutral axis from: Center of gravity $\frac{v_0}{h}$	Stress in extreme fibers in terms of $\frac{M}{b\,h^2}$: S_1	Stress in extreme fibers in terms of $\frac{M}{b\,h^2}$: S_2	Ratio of stress to stress in straight beam: $S_1 \div \frac{96\,M}{13\,b\,h^2}$	Ratio of stress to stress in straight beam: $S_2 \div \frac{120\,M}{13\,b\,h^2}$
0.5	2.412117	0.032327	8.4989	8.0825	1.1509	0.8756
1.0	1.389607	0.054837	9.4731	7.4207	1.2828	0.8039
1.5	1.039183	0.071928	10.3580	6.9786	1.4026	0.7560
2.0	0.858845	0.085599	11.1791	6.6580	1.5138	0.7213
3.0	0.671235	0.106543	12.6860	6.2144	1.7179	0.6732
4.0	0.572249	0.122195	14.0649	5.9162	1.9047	0.6409
5.0	0.509879	0.134565	15.3521	5.6984	2.0789	0.6173

Example

A curved beam of trapezoidal section has an inner base of 2 in., an outer base of 1 in., a height of 3 in., and an inner radius of 1 in. Find the location of the neutral surface, the stress in the extreme fibers in terms of the bending moment, and the ratio of these stresses to the stresses in a straight beam of the same section.

$R_1 = 1$; $R_2 = 4$; $C = 7$; $H = 6$. Using common logarithms,

$$R_0 = \frac{4.5 \times 6 \times 0.43429448}{2 \times 3(7/3 \times 0.6020600 - 0.43429448)} = \frac{1.95432516}{0.97051219}.$$

$$\bar{R} = 2.333333 \qquad I_c = 1\tfrac{3}{4};\quad Z_1 = 1\tfrac{3}{4} \div \tfrac{4}{3} = \tfrac{39}{16};$$
$$R_0 = 2.013705$$
$$\overline{}$$
$$v_0 = 0.319628 \qquad Z_2 = 1\tfrac{3}{4} \div \tfrac{5}{3} = \tfrac{39}{20}.$$

$$1 - \frac{2.013705}{4} = 0.496574,$$

$$S_2 = \frac{0.496574\,M}{4.5 \times 0.319628} = 0.345244\,M;$$

$$\frac{S_2}{S} = 0.345244\,M \div \frac{20\,M}{39} = 0.6732.$$

$$\frac{2.013705}{1} - 1 = 1.013705; \quad S_1 = \frac{1.013705\ M}{1.438326} = 0.704781\ M.$$

$$\frac{S_1}{S} = 0.704781\ M \div \frac{16\ M}{39} = 1.7179.$$

Problems

1. A curved beam of trapezoidal section has an inner radius of 3 in., inner base of 2 in., outer radius of 6 in., and outer base of 1 in. Find the unit stress at the inner and outer surfaces for a moment of 2,000 ft.-lb. by means of Table XXXI without using the stress in a straight beam.
Ans. $S_1 = 12{,}630$ lb./in.^{2}; $S_2 = 9.890$ lb./in.^{2}
2. The inner radius of a trapezoidal curved beam is equal to the height, and the inner base is four times the outer base. Find the expression for the stresses in the extreme fibers and the ratios of these stresses to the stresses in a straight beam.
Ans. $S_1 = \dfrac{11.1668\ M}{b\ h^2}$; $S_2 = \dfrac{10.37351\ M}{b\ h^2}$; $\dfrac{S_1}{S} = 1.2795$; $\dfrac{S_2}{S} = 0.7925.$

219. Curved Beam of Diverging Trapezoidal Section.—Figure 288 shows a trapezoidal section which is wider at the convex than at the concave surface. The distance C from the center of curvature to the vertex of the triangle is less than R_1. $dA = \dfrac{b}{H}(r - C)dr.$

$$R_0 = \frac{A\ H}{b\ h\left(1 - \dfrac{C}{h}\log\dfrac{R_2}{R_1}\right)}. \qquad (1)$$

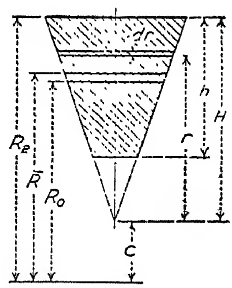


FIG. 288.—Curved beam of trapezoidal section, sides diverging.

Example I

A curved beam of trapezoidal section has inner radius of 3 in., outer radius of 6 in., inner base of 1 in., and outer base of 4 in. Solve for the stresses in the extreme fibers and the ratios of these to the stresses in a straight beam.

$R_1 = 3, \quad R_2 = 6, \quad \bar{R} = 4.8, \quad A = 7.5, \quad b = 4,$
$h = 3, \quad H = 4, \quad C = 2.$

$$R_0 = \frac{7.5 \times 4}{4 \times 3(1 - \frac{2}{3}\log_e 2)} = \frac{2.5}{0.53790188} = 4.647679.$$

$$v_0 = 4.8 - 4.647679 = 0.152321; \quad S_1 = 0.48079\ M;$$

$$\frac{S_1}{S} = 0.48079 \times \frac{11}{4} = 1.3222; \quad S_2 = 0.19730\ M;$$

$$\frac{S_2}{S} = 0.19730 \times \frac{33}{8} = 0.8139.$$

Example II

Solve Example I if the breadth at the inner surface is changed to 2 in. and all other data are unchanged. The inclined surfaces intersect on the center of curvature. Since the width increases directly as r, the neutral surface is midway between the inner and outer surfaces.

$$R_0 = 4.5 \text{ in.}; \quad v_0 = \frac{1}{6} \text{ in.}; \quad S_1 = \frac{M}{3};$$

$$\frac{S_1}{S} = \frac{M}{3} \times \frac{39}{10\,M} = 1.3; \quad S_2 = \frac{M}{6}; \quad \frac{S_2}{S} = \frac{M}{6} \times \frac{39}{8\,M} = 0.8125.$$

Problems

1. Solve Example I if the breadth at the inner surface is 2.4 in. and all the other dimensions remain unchanged.

Ans. $R_1 = 4.45575$ in.; $v_0 = 0.16925$ in.; $\frac{S_1}{S} = 1.2957$.

2. A beam of trapezoidal section has an outer radius of 6 in., an inner radius of 2 in., an outer breadth of 5 in., and an inner breadth of 3 in. Find the ratios of the stresses in the extreme fibers to the stresses in a straight beam of the same section.

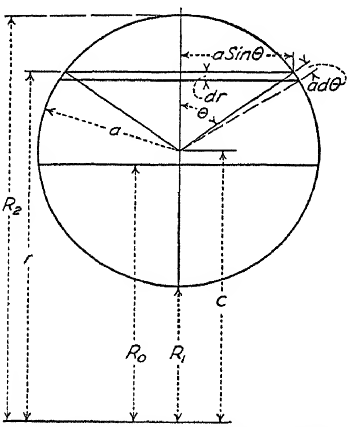


FIG. 289.—Curved beam of circular section.

220. Curved Beams of Circular Section.—Figure 289 shows a circular section of diameter D with its center at a distance c from the center of curvature. The expressions for finding R_0 are

$dA = -2\,a \sin \theta \, dr$; $r = c + a \cos \theta$; $dr = -a \sin \theta \, d\theta$; $dA = 2\,a \sin^2 \theta \, d\theta$, in which a is the radius of the circle and θ is the angle which the radius to the end of the element makes with the vertical upward.

$$\frac{dA}{r} = \frac{2\,a^2 \sin^2 \theta \, d\theta}{c + a \cos \theta}. \tag{1}$$

$$\frac{dA}{r} = \left(-2\,a \cos \theta + 2\,c - \frac{2(c^2 - a^2)}{c + a \cos \theta}\right) d\theta. \tag{2}$$

$$\int \frac{dA}{r} = \left[-2\,a \sin \theta + 2\,c\,\theta + 2\sqrt{c^2 - a^2}\left(\sin^{-1}\frac{a + c \cos \theta)}{c + a \cos \theta}\right)\right]_0^{\pi} \tag{3}$$

$$= 2c\pi + 2\sqrt{c^2 - a^2}\left(\sin^{-1}\frac{a-c}{c-a} - \sin^{-1}\frac{a+c}{c+a}\right) \tag{4}$$

$$= 2c\pi - 2\pi\sqrt{c^2 - a^2} = 2\pi(c - \sqrt{c^2 - a^2}). \tag{5}$$

$$R_0 = \frac{\pi a^2}{2\pi(c - \sqrt{c^2 - a^2})} = \frac{c + \sqrt{c^2 - a^2}}{2}. \tag{6}$$

$$v_0 = c - R_0 = \frac{c - \sqrt{c^2 - a^2}}{2}. \tag{7}$$

Example I

A curved beam of circular section is 4 in. in diameter and the radius of curvature at the inner surface is 2 in. Find the displacement of the neutral surface, the unit stress at the inner surface and at the outer surface, and the ratio of these stresses to the stress in the extreme fibers of a straight beam.

$$R_0 = \frac{4 + \sqrt{16 - 4}}{2} = 2 + \sqrt{3} = 3.73205; \quad v_0 = 2 - \sqrt{3} = 0.26795.$$

$$S_1 = \frac{\left(\frac{R_0}{2} - 1\right)M}{4\pi(2 - \sqrt{3})} = \frac{\frac{\sqrt{3}}{2}M}{4\pi(2 - \sqrt{3})} = \frac{\sqrt{3}(2 + \sqrt{3})M}{8\pi} = \frac{2(\sqrt{3} + 3)M}{8\pi}$$

$$= \frac{0.80801\,M}{\pi}.$$

For a straight beam of circular section 4 in. in diameter,

$$S = \frac{M}{2\pi}; \quad \frac{S_1}{S} = 1.6160.$$

Problems

1. Solve Example I for S_2 and $\frac{S_2}{S}$.
2. Solve Example I if the inner radius is 1 in. instead of 2 in. with all the other data given remaining unchanged. Compare with Table XXXII.
3. Solve Example I if the inner radius is 4 in. and all the other data remain unchanged. Compare with Table XXXII.

$$\text{Since } c = \frac{R_1 + R_2}{2} \text{ and } a = \frac{R_2 - R_1}{2}, \tag{8}$$

$$R_0 = \frac{R_1 + 2\sqrt{R_1R_2} + R_2}{4} = \frac{(\sqrt{R_1} + \sqrt{R_2})^2}{4}; \tag{9}$$

$$v_0 = \frac{R_1 - 2\sqrt{R_1R_2} + R_2}{4} = \frac{(\sqrt{R_1} - \sqrt{R_2})^2}{4}. \tag{10}$$

$$S_1 = \frac{\frac{R_0 - R_1}{R_1}M}{\pi a^2 v_0} = \frac{(-3R_1 + 2\sqrt{R_1R_2} + R_2)M}{\pi a^2 R_1(R_1 - 2\sqrt{R_1R_2} + R_2)}; \tag{11}$$

$$S_2 = \frac{(3R_2 - 2\sqrt{R_1R_2} - R_1)M}{\pi a^2 R_2(R_1 - 2\sqrt{R_1R_2} + R_2)}. \tag{12}$$

Since $S = \frac{4\,M}{\pi\,a^3}$, for a straight beam,

$$\frac{S_1}{S} = \frac{(-3\,R_1 + 2\sqrt{R_1R_2} + R_2)\,a}{4\,R_1\,(R_1 - 2\sqrt{R_1R_2} + R_2)}; \tag{13}$$

$$\frac{S_2}{S} = \frac{(3R_2 - 2\sqrt{R_1R_2} - R_1)\,a}{4\,R_2\,(R_1 - 2\sqrt{R_1R_2} + R_2)}. \tag{14}$$

Equations (13) and (14) make it possible to solve for ratio of the stress in the extreme fibers to the stress in a straight beam by direct substitution.

Example II

Find the ratio of the unit stress in the extreme fibers to the maximum stress in a straight beam when the inner radius is two-thirds the diameter.

$$R_1 = \frac{2\,D}{3}, \quad R_2 = \frac{5\,D}{3}, \quad a = \frac{D}{2}.$$

To simplify the calculations let $R_1 = 2$, $R_2 = 5$, and $a = \frac{3}{2}$.

$$\frac{S_1}{S} = \frac{(-6 + 2\sqrt{10} + 5)\frac{3}{2}}{4 \times 2(2 - 2\sqrt{10} + 5)} = \frac{3(-1 + 2\sqrt{10})}{16(7 - 2\sqrt{10})}$$

$$= \frac{3(-1 + 2\sqrt{10})(7 + 2\sqrt{10})}{16(49 - 40)};$$

$$\frac{S_1}{S} = \frac{-7 + 12\sqrt{10} + 40}{48} = \frac{70.9473}{48} = 1.47807.$$

$$\frac{S_2}{S} = \frac{51 + 12\sqrt{10}}{120} = 0.74123.$$

Figure 290 is plotted from Table XXXII. The relative stress at the concave surface does not differ largely from that represented by the straight line

$$S_1 = S\left(1 + 0.3\,\frac{D}{R_1}\right) = \frac{4\,M}{\pi\,a^3}\left(1 + 0.3\,\frac{D}{R_1}\right) \tag{15}$$

The calculations by Equations (11) and (12) are so easy that it is hardly worth while to use this approximate relation.

Problems

4. Find the unit stress in the extreme fibers of a curved beam which is 2.5 in. in diameter and has an inner radius of 2 in. when the moment is 15,708 in.-lb. *Ans.* $S_1 = 14{,}400$ lb./in.^{2}; $S_2 = 7{,}822$ lb./in.^{2}
5. A curved beam, 4 in. in diameter, has an inner radius of 1 in. Find the moment which will give unit stress of 1,200 lb. per sq. in. at the inner fibers. What will be the stress at the outer fibers?

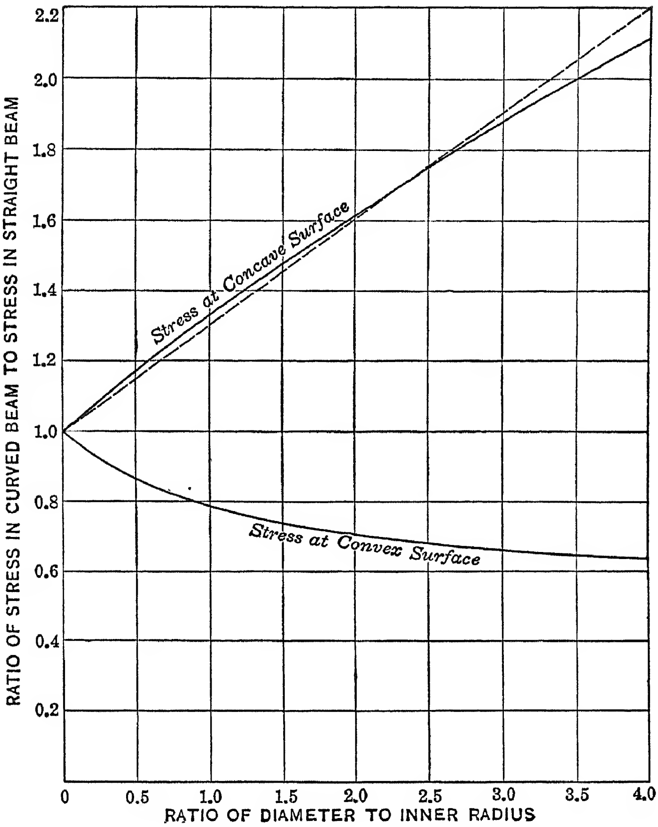


FIG. 290.—Unit stress in outer fibers of curved beam of circular section.

TABLE XXXII.—RELATIVE STRESSES IN EXTREME FIBERS OF CURVED BEAMS OF CIRCULAR SECTION

Ratio of diameter to inner radius	Distance of neutral axis from		Ratio of stress to stress in a straight beam	
$\frac{D}{R_1}$	Center of curvature R_0	Center of gravity v_0	$S_1 \div \frac{4M}{\pi a^3}$	$S_2 \div \frac{4M}{\pi a^3}$
0.5	2.474745	0.025355	1.17487	0.86658
1.0	1.457106	0.042893	1.33211	0.79105
1.5	1.110380	0.056287	1.47807	0.74123
2.0	0.933013	0.066987	1.61603	0.70534
3.0	0.750000	0.083333	1.87500	0.65625
4.0	0.654508	0.095492	2.11803	0.62361
5.0	0.594950	0.105050	2.34974	0.59996

221. Curved Beams of Semicircular Section.—Figure 291 shows a semicircular section with the diameter which bounds it on the convex surface of the beam. The limits of θ in Equation (3) of the preceding article are $\frac{\pi}{2}$ and π.

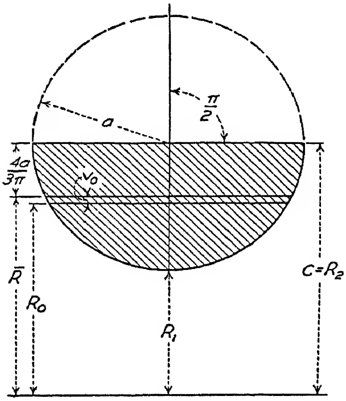


FIG. 291.—Semicircular curved beam, diameter on convex surface.

$$\left[\sin^{-1}\frac{a + c\cos\theta}{c + a\cos\theta}\right]_{\frac{\pi}{2}}^{\pi} = \sin^{-1}\frac{a - c}{c - a} - \sin^{-1}\frac{a}{c}; \quad (1)$$

$$\int_{\frac{\pi}{2}}^{\pi}\frac{dA}{r} = 2a + c\pi - 2\sqrt{c^2 - a^2}\left(\frac{\pi}{2} + \sin^{-1}\frac{a}{c}\right). \quad (2)$$

Example I

Find the displacement of the neutral surface, the stress in the extreme fibers, and the ratio of these stresses to the stress in a straight beam for a curved beam of semicircular section with the radius of curvature of the inner surface equal to radius of the section.

$$a = 1, \quad c = 2, \quad A = 0.5\,\pi\,a^2, \quad \sin^{-1}0.5 = \frac{\pi}{6}.$$

$$R_0 = \frac{0.5\,\pi\,a^2}{2 + 2\,\pi - \sqrt{3}\,\frac{4\,\pi}{3}} = \frac{0.5\,\pi\,a}{(0.63662 + 2 - 2.30940)\pi}; \quad (3)$$

$$R_0 = \frac{0.5\,a}{0.32722} = 1.52802\,a.$$

$$\bar{R} = \left(2 - \frac{4}{3\,\pi}\right)a = 1.57559\,a; \quad v_0 = 0.04757\,a; \quad S_1 = \frac{7.0664\,M}{a^3}.$$

At the convex surface of the *section* of a straight beam,

$$S = \frac{M}{0.19069\,a^3}; \quad \frac{S_1}{S} = 7.0664 \times 0.19069 = 1.3475;$$

$$S_2 = \frac{3.1582\,M}{a^3}; \quad \frac{S_2}{S} = 3.1582 \times 0.25861 = 0.8167.$$

Compare these ratios with previous tables for the same $\frac{h}{R_1}$.

Problems

1. Solve preceding example for $a = 2\,R_1$.

$$v_0 = 0.04725\,a; \quad S_1 = \frac{8.59704\,M}{a^3}; \quad \frac{S_1}{S} = 1.6923; \quad S_2 = \frac{2.8504\,M}{a^3}; \quad \frac{S_2}{S} = 0.7372.$$

2. Solve preceding example for $a = 4\,R_1$.

Figure 292 shows a semicircular section with the diameter which bounds it on the concave surface of the curved beam. The limits of θ in Equation (3) of the preceding article are 0 and $\frac{\pi}{2}$.

$$\left[\sin^{-1}\frac{a + c\cos\theta}{c + a\cos\theta}\right]_0^{\frac{\pi}{2}} = \sin^{-1}\frac{a}{c} - \sin^{-1}\frac{a + c}{c + a};$$

$$\int_0^{\frac{\pi}{2}}\frac{dA}{r} = -2\,a + c\,\pi - 2\sqrt{c^2 - a^2}\left(\frac{\pi}{2} - \sin^{-1}\frac{a}{c}\right). \qquad (4)$$

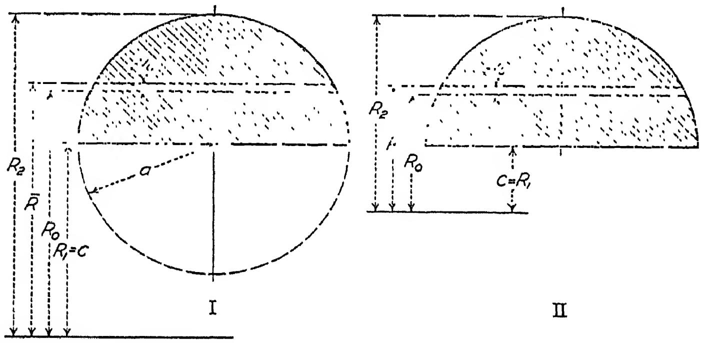


FIG. 292.—Semicircular curved beam, diameter on concave surface.

Example II

A curved beam of semicircular section has the diameter of each section on the concave surface of the beam at a distance 2 a from the center of curvature. Find the ratio of the stresses in the extreme fibers to the stresses in a straight beam.

$$c = R_1 = 2\,a, \quad A = 0.5\,\pi\,a^2, \quad \sin^{-1} 0.5 = \frac{\pi}{6}.$$

$$R_0 = \frac{0.5\,\pi\,a}{-2 + 2\,\pi - \sqrt{3}\,\frac{2\,\pi}{3}} = \frac{0.5\,\pi\,a}{(-0.63662 + 2 - 1.15470)\pi};$$

$$R_0 = \frac{0.5\,a}{0.20868} = 2.39601\,a; \quad v_0 = 2.42441\,a - R_0 = 0.02840\,a.$$

$$S_1 = \frac{0.198005\,M}{0.5\,\pi\,a^2 \times 0.02840\,a} = \frac{4.4600\,M}{a^3}; \quad \frac{S_1}{S} = 1.1536;$$

$$S_2 = \frac{0.201663\,M}{0.0446107\,a^3} = \frac{4.5205\,M}{a^3}; \quad \frac{S_2}{S} = 0.8610.$$

Problems

3. A curved beam similar to Fig. 292 has an inner radius of length a. Solve for the stresses in the extreme fibers.

Ans. $v_0 = 0.04844\,a; \frac{S_1}{S} = 1.2779; \frac{S_2}{S} = 0.7832.$

4. Solve Problem 3 if the inner radius is $\frac{5\,a}{4}$.

$$\textit{Ans.}\ R_0 = 1.63325\,a;\quad v_0 = 0.04116\,a;\quad S_1 = \frac{4.7422\,M}{a^3};\quad \frac{S_1}{S} = 1.2264;$$
$$S_2 = \frac{4.5205\,M}{a^3};\quad \frac{S_2}{S} = 0.8087.$$

When a is greater than c (Fig. 292, II) the expression $\sqrt{c^2 - a^2}$ in Equation (3) of Art. 220 becomes imaginary. For a greater than c,

$$\frac{d\theta}{c + a\cos\theta} = \frac{1}{\sqrt{a^2 - c^2}} \log_e \frac{a + c\cos\theta + \sqrt{a^2 - c^2}\sin\theta}{c + a\cos\theta}.^* \tag{5}$$

$$\int \frac{dA}{r} = -2\,a\sin\theta + 2\,c\,\theta + 2\sqrt{a^2 - c^2}\,\log_e \frac{a + c\cos\theta + \sqrt{a^2 - c^2}\sin\theta}{c + a\cos\theta}. \tag{6}$$

For the semicircle of Fig. 292, the limits are 0 and $\frac{\pi}{2}$.

$$\int_0^{\frac{\pi}{2}} \frac{dA}{r} = -2\,a + c\,\pi + 2\sqrt{a^2 - c^2} \times \left(\log_e \frac{a + \sqrt{a^2 - c^2}}{c} - \log_e \frac{a + c}{c + a}\right); \tag{7}$$

$$\int_0^{\frac{\pi}{2}} \frac{dA}{r} = -2\,a + c\,\pi + 2\sqrt{a^2 - c^2}\,\log_e \frac{a + \sqrt{a^2 - c^2}}{c}. \tag{8}$$

Example III

Find the stresses in the extreme fibers of the beam of Fig. 292 for $c = 0.5\,a$.

$$\log_e (2 + \sqrt{3}) = \log_e 3.73205 = 1.316944;$$
$$\sqrt{3} \times 1.316944 = 2.280812;\quad 2.280812 + 1.570796 - 2 = 1.851606;$$
$$\frac{1.570796\,a^2}{1.851606\,a} = R_0 = 0.84835\,a;\quad v_0 = 0.92441\,a - 0.84835\,a = 0.07606\,a.$$
$$\frac{0.84835\,a}{0.5\,a} - 1 = 0.69676;\quad S_1 = \frac{0.69676\,M}{0.119477\,a^3} = \frac{5.8318\,M}{a^3};$$
$$\frac{S_1}{S} = 5.8318 \times 0.25861 = 1.50816.$$

222. Hooks.—A hook is equivalent to a curved beam which is subjected to eccentric tension. As in a short block which is eccentrically loaded, the total pull P may be replaced by a pull P through the center of gravity of the section and a bending

* PEIRCE, B. O., "Short Table of Integrals," abridged ed., p. 22.

moment $P e$, in which e is the distance of the center of gravity of the section from the line of the load. The direct tensile stress is $\frac{P}{A}$. If the hook is straight at the section under consideration, the tensile stress which is due to bending is $\frac{P e v}{I}$ and the total stress at the innermost fibers is

$$S_t = \frac{P}{A}\left(1 + \frac{e c}{r^2}\right). \tag{1}$$

At the outermost fibers, if the hook is straight at the section under consideration,

$$S = \frac{P}{A}\left(1 - \frac{e c}{r^2}\right). \tag{2}$$

Since the eccentricity of the load on a hook is always so great that $\frac{e c}{r^2}$ is greater than unity, S is a compressive stress.

While Equations (1) and (2) afford an *approximate* method of finding the maximum stresses in a hook or curved bar which is subjected to tension or compression, there is an error on the side of danger, unless the curvature at the section under consideration is relatively small. For accurate results, it is necessary to regard the hook as a curved beam in calculating the bending stress. At the innermost fibers of a hook,

$$S_t = \frac{P}{A} + S_1, \tag{3}$$

in which S_1 is the unit stress at the concave surface calculated for a curved beam with the moment $P e$. At the outermost fibers,

$$S_c = S_2 - \frac{P}{A}, \tag{4}$$

in which S_2 is the bending stress at the convex surface calculated for a curved beam with moment $P e$.

223. Curved Bars of Rectangular Section.—While hooks are not made with rectangular sections, curved bars frequently have this form.

Example

A curved bar of rectangular section is 2 in. wide. At the section farthest from the applied load the inner radius is 3 in. and the outer radius is 6 in.

The load is 3,000 lb. tension and passes through the center of curvature. Find the maximum unit tensile and compressive stress at the most remote section.

$$R_1 = 3 \text{ in.};\quad R_2 = 6 \text{ in.};\quad h = 3 \text{ in.}; \frac{h}{R_1} = 1.$$

$\frac{P}{A} = \frac{3{,}000}{6} = 500$ lb. per sq. in. tension. $M = 3{,}000 \times 4.5 = 13{,}500$ in.-lb. moment. For a straight beam, $S = \frac{13{,}500}{3} = 4{,}500$ lb. per sq. in. Since $\frac{h}{R_1} = 1$, the ratios $\frac{S_1}{S}$ and $\frac{S_2}{S}$ may be taken from Table XXIX without interpolating.

$$S_1 = 4{,}500 \times 1.2876 = 5{,}794 \text{ lb. per sq. in. tension.}$$

Total tension at concave surface $= 5{,}794 + 500 = 6{,}294$ lb. per sq. in.
$S_2 = 4{,}500 \times 0.8104 = 3{,}647$ lb. per sq. in. compression.
Total compression at convex surface $= 3{,}647 - 500 = 3{,}147$ lb. per sq. in.

Problems

1. A curved beam 2 in. square has an inner radius of 3 in. and an outer radius of 5 in. It is subjected to a tension of 2,000 lb. which passes through the center of curvature. Find the tensile stress at the inner surface. Calculate S_1 from the equations of Art. 216, and make an approximate solution by interpolating Table XXIX.

   Ans. $S_t = 7{,}698$ lb./in.^{2}

2. A curved bar of rectangular section is 3 in. wide. The inner radius is 6 in. and the outer radius is 10 in. The load is 6,000 lb. and the line of the load is 3 in. from the concave surface of the bar. Find the maximum tensile stress.

224. Hooks of Circular Section.—The problem of finding the maximum unit tensile stress in a hook of circular section is solved by means of Equation (11) of Art. 220. If tension is regarded as positive, the complete expression for the unit stress at the concave surface is

$$S_t = \frac{Pe(R_2 + 2\sqrt{R_1R_2} - 3R_1)}{\pi a^2 R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} + \frac{P}{\pi a^2}; \tag{1}$$

$$S_t = \frac{P}{\pi a^2}\left(\frac{e(R_2 + 2\sqrt{R_1R_2} - 3R_1)}{R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} + 1\right). \tag{2}$$

At the convex surface, from Equation (14) of Art. 220

$$S_c = \frac{P}{\pi a^2}\left(\frac{e(3R_2 - 2\sqrt{R_1R_2} - R_1)}{R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} - 1\right). \tag{3}$$

Example

A hook of circular section is 2 in. in diameter. The inner radius of curvature is 3 in. and the outer radius is 5 in. The load is 2,000 lb. with the line of its resultant 1 in. inside the concave surface. Find the unit stress in the extreme fibers.

$$R_1 = 3 \text{ in.};\quad R_2 = 5 \text{ in.};\quad a = 1 \text{ in.};\quad e = 2 \text{ in.}$$

$$\frac{P}{\pi a^2} = 636.6 \text{ lb. per sq. in.}$$

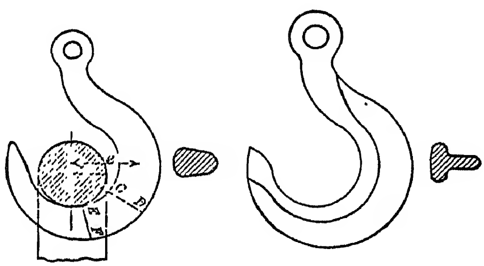
$$S_t = 636.6\left(\frac{2(5 + 2\sqrt{15} - 9)}{3(3 - 2\sqrt{15} + 5)} + 1\right).$$

$$S_t = 636.6\left(\frac{2 \times 3.746}{3 \times 0.254} + 1\right) = 636.6 \times 10.83 = 6{,}894 \text{ lb. per sq. in.}$$

$$S_c = 636.6\left(\frac{2 \times 4.254}{5 \times 0.254} - 1\right) = 636.6 \times 5.699 = 3{,}628 \text{ lb. per sq. in.}$$

Problem

A hook of circular section is 3 in. in diameter. The inner radius of curvature is 4 in. and the distance from the center of the section to the load line is 3 in. If the maximum allowable unit stress is 10,000 lb. per sq. in., what is the safe load?



I II

FIG. 293.—Hooks.

225. Hooks of Trapezoidal Section.—Hooks are frequently made of trapezoidal section with the larger base toward the center of curvature. In the actual hooks the corners are rounded as in Fig. 293. Such a hook may be calculated as if it were the full trapezoidal section and the bending stress in the actual hook may then be computed by multiplying the stress obtained from the full trapezoid by the ratio of the moment of inertia of the full trapezoid to the moment of inertia of the actual section.

Example

A hook of trapezoidal section is 2 in. wide at the concave surface, 1 in. wide at the convex surface, and 4 in. deep between these surfaces at the

section most remote from the line of the load. The inner radius is 4 in. and the line of the load is 2 in. from the concave surface. Find the unit stress in the extreme fibers when the load is 8,000 lb.

Since the inner breadth is twice the outer breadth, the ratios of Table XXXI apply for $\frac{h}{R_1} = 1$.

The calculated stress at the 2-in. base as a straight beam is $S = M\frac{3}{13}$. Since the center of gravity of the section is $\frac{16}{9}$ in. from the base, the moment is $8{,}000 \times \frac{34}{9}$. The ratio from the table is 1.2828.

$$S_1 = \frac{3}{13} \times 8{,}000 \times \frac{34}{9} \times 1.2828 = 8{,}947 \text{ lb. per sq. in.}$$

$$S_1 + \frac{P}{A} = 8{,}947 + 1{,}333 = 10{,}280 \text{ lb. per sq. in.}$$

which is the unit stress at the inner surface.

Problems

1. Calculate the stress in the outer fibers for the hook of the preceding example.
2. Solve the preceding example if the inner width is 3 in. and all other data are unchanged.

CHAPTER XX

PROPERTIES OF AREAS

226. Center of Gravity of Some Areas.—The location of the center of gravity of an area is determined by the expressions

$$\bar{x} = \frac{\int x\,dA}{A}; \quad \bar{y} = \frac{\int y\,dA}{A}. \tag{1}$$

The center of gravity of an area is frequently called the *centroid*.

Example

Locate the center of gravity of the area which is bounded by the X axis, the ordinate $x = x_1$, and the curve $y = x^n$.

$$\text{Area} = \int_0^{x_1} x^n dx = \frac{\left[x^{n+1}\right]_0^{x_1}}{n+1} = \frac{x_1^{n+1}}{n+1}. \tag{2}$$

$$\text{Moment of area} = \int_0^{x_1} x^{n+1}\,dx = \frac{\left[x^{n+2}\right]_0^{x_1}}{n+2} = \frac{x_1^{n+2}}{n+2}. \tag{3}$$

$$\bar{x} = \frac{n+1}{n+2}x_1. \tag{4}$$

If $y = x^n$ and $y_1 = x_1^n$,

$$\bar{y} = \frac{\int_0^{x_1} x^{2n}dx}{2A} = \frac{x_1^{2n+1}}{2n+1} \times \frac{2(n+1)}{x_1^{n+1}} = \frac{(n+1)x_1^n}{2(2n+1)} = \frac{n+1}{2(2n+1)}y_1. \tag{5}$$

Table XXXIV gives the center of gravity of some areas which are largely used in the application of area moments.

227. Moment of Inertia of Areas.—The moment of inertia of an area is defined mathematically by the expression $\int r^2 dA$, in which dA is an element of area and r is the distance of every part of this element from the axis of reference.

Polar moment of inertia is taken with reference to an axis which is perpendicular to the plane of the area. In rectangular coördinates, $r^2 = x^2 + y^2$ and $dA = dx\,dy$. In polar coördinates, $dA = r\,d\theta\,dr$.

The polar moment of inertia of a circle of radius a is $J = \frac{\pi a^4}{2}$; the square of the radius of gyration is $k^2 = \frac{a^2}{2}$. For a hollow circle of outside radius a and inside radius b,

$$J = \frac{\pi(a^4 - b^4)}{2}; \quad k^2 = \frac{a^2 + b^2}{2}. \tag{1}$$

The moment of inertia of a plane area with respect to an axis in its plane is called the *moment of inertia* of the area without any descriptive term. The moment of inertia with respect to the X axis is

$$I_x = \int y^2 dA.$$

The moment of inertia with respect to the Y axis is

$$I_y = \int x^2 dA.$$

Moment of inertia of any area may be transferred from any axis through the center of gravity to a parallel axis by means of the equation

$$I = I_c + A\,d^2, \tag{2}$$

in which I_c is the moment of inertia with respect to the axis through the center of gravity and d is the distance from this axis to the parallel axis.

Table XXXIII gives the moment of inertia of a few areas which are convenient in the calculation of the properties of sections which are used in structures.

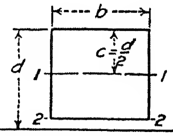
Problems

1. Derive the expression for the moment of inertia of a rectangle with respect to an axis through the center of gravity parallel to the base (see any textbook of mechanics for the method). By transfer of axis find the moment of inertia with respect to the base.
2. Find the moment of inertia of a 6-in. by 8-in. rectangle with respect to an axis which is outside the rectangle at a distance of 4 in. from one 6-in. edge. Solve by transfer formula. Solve also by assuming that the rectangle is extended to the axis of inertia.

Ans. $\frac{6 \times 8^3}{12} + 48 \times 64 = 3{,}328$ in.^{4}; $\frac{6 \times 12^3}{3} - \frac{6 \times 4^3}{3} = 3{,}328$ in.^{4}

3. By integration derive the expression for the moment of inertia of a triangle with respect to an axis through the vertex parallel to the base. Then transfer *to* the parallel line through the center of gravity. Finally transfer *from* the center of gravity to the base.

TABLE XXXIII.—MOMENTS OF INERTIA AND SECTION MODULI

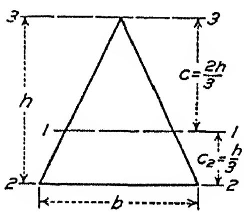


Rectangle: $A = b\,d.$

Axis 1–1: $I = \dfrac{b\,d^3}{12} \qquad k = \dfrac{d}{\sqrt{12}} = 0.288675\,d.$

$c = \dfrac{d}{2} \qquad Z = \dfrac{b\,d^2}{6}.$

Axis 2–2: $I = \dfrac{b\,d^3}{3} \qquad k = \dfrac{d}{\sqrt{3}} = 0.577350\,d.$



Any triangle: $A = \dfrac{b\,h}{2}.$

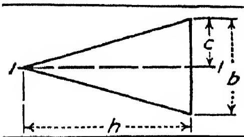
Axis 1–1: $I = \dfrac{b\,h^3}{36} \qquad k = \dfrac{h}{\sqrt{18}} = 0.235702\,h.$

$c = \dfrac{2\,h}{3}; \qquad Z = \dfrac{b\,h^2}{24}.$

$c_2 = \dfrac{h}{3}; \qquad Z_2 = \dfrac{b\,h^2}{12}.$

Axis 2–2: $I = \dfrac{b\,h^3}{12} \qquad k = \dfrac{h}{\sqrt{6}} = 0.408248\,h.$

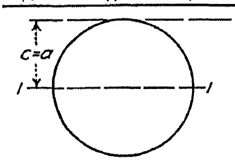
Axis 3–3: $I = \dfrac{b\,h^3}{4} \qquad k = \dfrac{h}{\sqrt{2}} = 0.707107\,h.$



Isosceles triangle: $A = \dfrac{b\,h}{2}.$

Axis 1–1: $I = \dfrac{h\,b^3}{48} \qquad k = \dfrac{b}{\sqrt{24}} = 0.204124\,b.$

$c = \dfrac{b}{2}; \qquad Z = \dfrac{h\,b^2}{24}.$

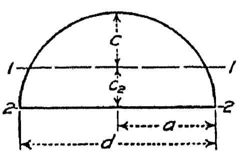


Circle: $A = \pi\,a^2 = 0.785398\,d^2$

Axis 1–1: $I = \dfrac{\pi\,a^4}{4} = \dfrac{\pi\,d^4}{64} = 0.0490874\,d^4$

$k = \dfrac{a}{2}; \quad c = a = \dfrac{d}{2}.$

$Z = \dfrac{\pi\,a^3}{4} = \dfrac{\pi\,d^3}{32} = 0.0981748\,d^3.$



Semicircle: $A = \dfrac{\pi\,a^2}{2} = 1.570796\,a^2.$

Axis 1–1: $I = \left(\dfrac{\pi}{8} - \dfrac{8}{9\,\pi}\right)a^4 = 0.109756\,a^4.$

$k = \dfrac{\sqrt{9\,\pi^2 - 64}}{6\,\pi}\,a = 0.264336\,a.$

$c = a - \dfrac{4\,a}{3\,\pi} = 0.5755868\,a; \qquad Z = 0.190687\,a^3.$

$c_2 = \dfrac{4\,a}{3\,\pi} = 0.4244132\,a; \qquad Z_2 = 0.258609\,a^3.$

The isosceles triangle consists of two triangles of base h and altitude $\dfrac{b}{2}.$ From the expression for the moment of inertia with respect to the base

$$I = 2 \times \frac{h\left(\dfrac{b}{2}\right)^3}{12} = \frac{h\,b^3}{48}.$$

4. By integration, find the moment of inertia of a circle with respect to a diameter. Find the moment of inertia of a semicircle with respect to any diameter.

Ans. I of semicircle $= \dfrac{\pi a^4}{8}$.

5. By transfer find the moment of inertia of a semicircle with respect to a line which is parallel to the bounding diameter and passes through the center of gravity.

$$Ans.\ I_c = \frac{\pi a^4}{8} - \frac{\pi a^2}{2} \times \frac{16 a^2}{9 \pi^2} = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) a^4.$$

Example I

A trapezoid has a lower base of 16 in., an upper base of 4 in., and a height of 12 in. Find the moment of inertia with respect to the lower base; then find the moment of inertia with respect to an axis which is parallel to the base and passes through the center of gravity.

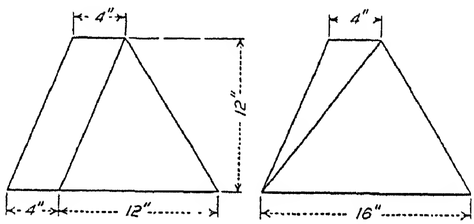


FIG. 294.—Subdivision of trapezoid for computation.

Figure 294, I, shows the trapezoid divided into a parallelogram and a triangle. Figure 294, II, is made up of two triangles. $\bar{y} = 4.8$ in. The moment of inertia with respect to the base is

$$\text{Parallelogram: } \frac{4 \times 12 \times 12 \times 12}{3} = 2{,}304 \quad \text{in.}^4$$

$$\text{Triangle: } \frac{12 \times 12 \times 12 \times 12}{12} = 1{,}728$$

$$I = 4{,}032 \quad \text{in.}^4$$

$$I_c = 4{,}032 - 120 \times 4.8^2 = 4{,}032 - 2{,}764.8 = 1{,}267.2 \text{ in.}^4$$

Problems

6. Find the center of gravity and the moment of inertia with respect to the base for the trapezoid of Example I by means of the triangles of Fig. 294, II.

7. Solve Example I by taking the moment of inertia with respect to the upper base and then transferring to the parallel line through the center of gravity.

8. Find the section modulus of the trapezoid of Fig. 294 with respect to upper and lower extreme fibers. *Ans.* $Z = 176$ in.^{3}; $Z_2 = 264$ in.^{3}

Example II

Find the moment of inertia and radius of gyration of 8-in. by 7-in. by 1-in. angle section with respect to an axis through the center of gravity parallel to the 7-in. leg (Fig. 295).

Dividing the angle into an 8-in. by 1-in. rectangle and a 1-in. by 6-in rectangle and taking moments about the back of the 7-in. leg,

$$
\begin{aligned}
8 \times 4 &= 32 \\
6 \times 0.5 &= 3 \\
\hline
14\,\bar{y} &= 35 \\
\bar{y} &= 2.5 \text{ in.}
\end{aligned}
$$

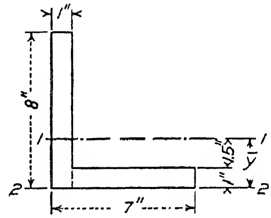


FIG. 295.—Angle section.

The moment of inertia with respect to the axis 1-1 may be found by taking the moment of inertia of each rectangle with respect to a parallel axis through its center of gravity and then transferring to axis 1-1.

$$I_c = \frac{1 \times 8^3}{12} + 8 \times \frac{9}{4} + \frac{6 \times 1^3}{12} + 6 \times 2^2 = 42\frac{2}{3} + 18 + \frac{1}{2} + 24 = 85\frac{1}{6}.$$

Another method is to find the moment of inertia about some axis which is a common base for both rectangles and then transfer *to* the center of gravity. Using axis 2-2 at the back of the 7-in. leg,

$$I = \frac{1 \times 8^3}{3} + \frac{6 \times 1^3}{3} = 172\,\frac{2}{3}; \quad I_c = 172\,\frac{2}{3} - 14 \times \frac{25}{4} = 85\frac{1}{6}.$$

Problems

9. Divide the angle section of Fig. 295 into two 7-in. rectangles; find the sum of the moments of inertia with respect to the right edge of the 7-in. leg; and transfer 1.5 in. to axis 1-1.
10. Find the moment of inertia of the section of Fig. 295 with respect to the axis through the center of gravity parallel to the 8-in. leg. Solve by two methods.
11. Compare the moments of inertia, area, and radii of gyration of the section of Fig. 295 with the values of a similar section which is given in the handbook.
12. Find the location of the center of gravity for a 6-in. by 5-in. by 1-in. angle section. Compute the moment of inertia for axes through the center of gravity parallel to each leg. Calculate the radius of gyration for each of these and compare all the results with the corresponding values for a 3-in. by 2½-in. by ½-in. angle section which is given in the A.I.S.C. handbook.

Figure 296 shows a standard channel section. The moment of inertia with respect to the axis 1–1 is the moment of inertia

of the rectangle of breadth b and height d minus the moment of inertia of the trapezoid which is included between the flanges. This trapezoid is a portion of an isosceles triangle which has a base c. Since the slope of the inner surfaces of the flanges in most standard channels and I-beams is 1:6, the altitude of this

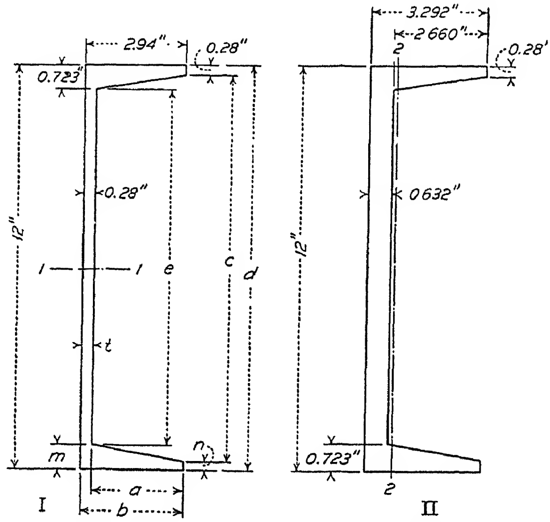


FIG. 296.—Standard channel section.

triangle is $3\,c$. The altitude of that portion of the triangle from the right face of the web to the vertex is $3\,e$.

$$e = c - \frac{a}{3}, \tag{3}$$

in which a is the net width of the flange.

$$I = \frac{b\,d^3}{12} - \frac{3\,c \times c^3 - 3\,e \times e^3}{48} = \frac{b\,d^3}{12} - \frac{c^4 - e^4}{16}. \tag{4}$$

In a similar way the moment of inertia of a standard I-beam is

$$I = \frac{b\,d^3}{12} - \frac{c^4 - e^4}{8}. \tag{5}$$

Example III

Find the moment of inertia of a 12-in. 20.7-lb. standard channel section with respect to the axis of symmetry.

$$c = 12 - 2 \times 0.28 = 11.44; \quad m = 0.28 + \frac{2.66}{6} = 0.72\,\frac{1}{3};$$

$$e = 12 - 2 \times 0.72\tfrac{1}{3} = 10.55\,\frac{1}{3}.$$

The "Pocket Companion" on page 34 gives $m = 0.723$ which makes $e = 10.554$. If this value is used in Eq. (4), the moment of inertia is found to be 128.3 instead of 128.1, which is given in the handbook on page 36. Raising e to the fourth power multiplies the error of this term by 4. Since the final value is taken from the difference of two relatively large quantities, an error of 1 part of 15,000 becomes 1 part in 640 in final result.

$$I = \frac{2.94 \times 12^3}{12} - \frac{11.44^4 - \left(\frac{31.66}{3}\right)^4}{16};$$

$$I = 423.36 - \frac{17{,}127.9 - 12{,}403.9}{16} = 423.36 - 295.25;$$

$$I = 128.1$$

The dimensions of the standard beams and channels which are shown on the drawings of the handbooks apply to the lightest section of that depth. The heavier sections are made by increasing the thickness of the web. The gross width of the flange is increased by the same amount, while the net dimensions of flange are unchanged.

Problems

13. Look up in the tables of the handbook the additional thickness of the 30-lb. 12-in. channel. From the answer of Example III and the moment of inertia of the added rectangle, calculate the moment of inertia of this channel with respect to the axis of symmetry.

14. Solve Problem 13 for the 12-in. 40-lb. channel.

15. Find the moment of inertia of a 20-in. 81.4-lb. standard I-beam with respect to the axis of symmetry which is perpendicular to the web.

Ans. $c = 20 - 1.30 = 18.7;\ e = 18.7 - \frac{3.2}{3} = \frac{52.9}{3};$

$$I = \frac{14{,}000}{3} - \frac{122{,}283.1 - 96{,}680.6}{8} = 4{,}666.7 - 3{,}200.4;$$

$$I = 1{,}466.3.$$

Example IV

Find the moment of inertia of a 12-in. 35-lb. standard channel with respect to the axis through the center of gravity which is parallel to the web (Fig. 296, II).

The moment of inertia is calculated first with respect to the right edge of the web. This is a common base for the three rectangles and two triangles which make up the section.

$$I = \frac{12 \times 0.632^3}{3} + \frac{0.56 \times 2.66^3}{3} + \frac{0.887 \times 2.66^3}{12} = 5.9126 \text{ in.}^4$$

$$10.252\,\bar{x} = 0.6294; \quad \bar{x} = 0.0693 \text{ in.}$$

$$I_c = 5.9126 - 10.26 \times 0.0693^2 = 5.88 \text{ in.}^4$$

Problems

16. Solve Example IV for a 12-in. 20.7-lb. standard channel. Make use of part of the work of the example.

17. Find the moment of inertia of a 20-in. 95-lb. standard I-beam with respect to the axis parallel to the web through its center of gravity. Find the moment of inertia of the one large rectangle and the two smaller rectangles about the required axis. Find the moment of inertia of one of the four equal triangles with respect to a parallel axis through its center of gravity and then transfer.

18. Find the moment of inertia of the section of Fig. 297 with respect to the axis of symmetry and also with respect to the axis through the center of gravity parallel to the plate. Get as many of the data from the handbook as possible. This column is latticed at the top of the figure. The latticing is not considered in the calculations.

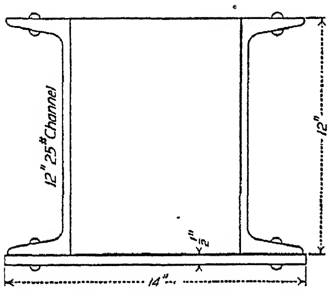


FIG. 297.—Fabricated section.

The sum of the moments of inertia of a plane area with respect to a pair of axes at right angles to each other is equal to the polar moment of inertia of area with respect to the axis which is perpendicular to this plane and passes through the intersection of the two axes which lie in the plane.

If one of these axes in the plane is the X axis,

$$I_x = \int y^2 dA. \tag{6}$$

If the other axis at right angles is the Y axis,

$$I_y = \int x^2\, dA. \tag{7}$$

For the polar moment of inertia, $r^2 = x^2 + y^2$;

$$J = \int (x^2 + y^2) d\,A = \int x^2\, dA + \int y^2 dA. \tag{8}$$

For angle sections with unequal legs, the handbooks give the moments of inertia with respect to axes parallel to each leg and also give the minimum radius of gyration. The maximum moment of inertia may be found from the relation

$$I_x + I_y = J = I_{\min} + I_{\max}. \tag{9}$$

Problems

19. Find the maximum moment of inertia for a 4-in. by 3-in. by ½-in. angle section for an axis through the center of gravity. Get I_x and I_y from the handbook. From the minimum radius of gyration, which is given, calculate the minimum moment of inertia, then solve for I_{max} by Eq. (9).
20. Solve Problem 19 for a 6-in. by 3½-in. by ⅜-in. Zee.

228. Change of Direction of Axes for Moment of Inertia.—By means of Equation (2) of Art. 227, moment of inertia may be transferred from one axis to a parallel axis, provided one of these axes passes through the center of gravity. It is frequently necessary to find the moment of inertia with respect to an axis which is inclined to the principal lines of the figure in such a way that the solution by direct integration is difficult. If the moment of inertia of an area is known for any two axes in the plane at right angles to each other, the moment of inertia for any other axis at a known angle with those axes may be calculated.

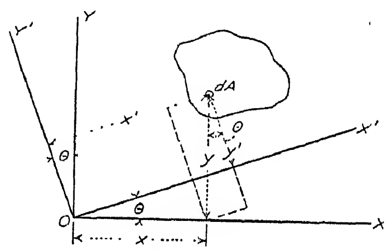


FIG. 298.—Change of direction of axes.

Figure 298 represents any area in the $X\,Y$ plane. The moment of inertia of this area with respect to the X axis may be designated by I_x and the moment of inertia with respect to the Y axis may be designated by I_y.

$$I_x = \int y^2 dA;$$
$$I_y = \int x^2 dA.$$

The line $O\,X'$ is a new axis which makes an angle θ with the X axis and $O\,Y'$ is an axis at right angles to $O\,X'$. The coördinates of an element of area dA with respect to these new axes are (x', y').

The moment of inertia of the area with respect to $O\,X'$ is

$$I = \int y'^2 dA. \tag{1}$$

From the geometry of the figure

$$y' = y \cos\theta - x \sin\theta. \tag{2}$$

$$I = \int (y^2 \cos^2\theta - 2\,x\,y \cos\theta \sin\theta + x^2 \sin^2\theta)dA, \tag{3}$$

$$I = I_x \cos^2\theta + I_y \sin^2\theta - 2\cos\theta\sin\theta \int x\,y\,dA, \tag{4}$$

$$I = I_x \cos^2\theta + I_y \sin^2\theta - \sin 2\theta \int x\,y\,dA, \tag{5}$$

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - \sin 2\theta \int x\,y\,dA. \tag{6}$$

229. Product of Inertia.—The expression $\int x\,y\,dA$, which occurs in Equations (5) and (6) of the preceding article, is called the *product of inertia*. The product of inertia has no physical meaning but it is a convenient tool in finding the moment of inertia of a plane area with respect to any axis.

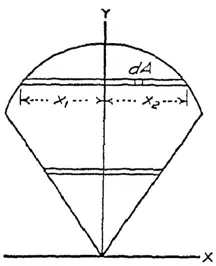


FIG. 299.—Symmetrical section.

If an area is symmetrical with respect to either one of a pair of rectangular axes, its product of inertia with respect to that pair of axes is zero. Figure 299 represents an area which is symmetrical with respect to the Y axis. If this is integrated first with respect to x,

$$H = \frac{1}{2}\int \Big[x^2\Big]_{x_1}^{x_2} y\,dy = \frac{1}{2}\int (x_2^2 - x_1^2)y\,dy.$$

If the area is symmetrical with respect to the Y axis, the lower limit x_1 is numerically equal and opposite in sign to the upper limit x_2, and the squares are the same in magnitude and sign; consequently the term in the brackets vanishes and

$$H = 0.$$

When the product of inertia is known with respect to a pair of rectangular axes through the center of gravity of an area, it may be calculated for a second pair of parallel axes in the plane of the area by formula which is similar to the transfer equation for moment of inertia.

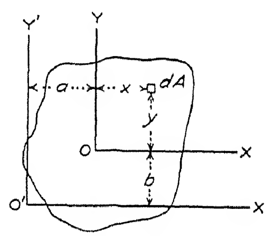


FIG. 300.—Transfer of product of inertia.

Let OX, OY (Fig. 300) be the original pair of axes through the center of gravity, and let (x, y) be the coördinates of an element dA with reference to these axes. Let $O'X'$, $O'Y'$ be a new pair of parallel axes. Let (a, b) be the coördinates of the center of gravity of the area with respect to the new axes.

If H is the product of inertia with respect to the new axes,

$$H = \int (a + x)(b + y)dA. \tag{1}$$

$$H = a\,b\int dA + b\int x\,dA + a\int y\,dA + \int x\,y\,dA. \tag{2}$$

$$H = a\,bA + 0 + 0 + H_c. \tag{3}$$

where H_c is the product of inertia with respect to the axes through the center of gravity.

If the center of gravity falls in the first or third quadrant with respect to the axes for which the product of inertia is desired, the

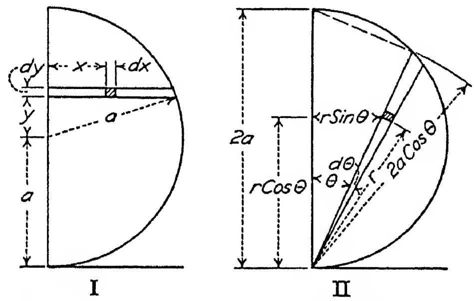


FIG. 301.—Product of inertia of semicircle.

product $a\,b\,A$ is positive, and H will be positive unless H_c is negative and numerically greater than $a\,b\,A$. If the center of gravity falls in the second quadrant, $a\,b\,A$ is negative since a is negative; if it falls in the fourth quadrant, $a\,b\,A$ is negative because b is negative.

Problems

1. By integration find the product of inertia of a rectangle of base b and altitude d with respect to the lower and left edges as axes.

$$Ans.\ H = \frac{b^2d^2}{4}.$$

2. Solve Problem 1 with respect to the lower and right edges as axes.

$$Ans.\ H = -\frac{b^2d^2}{4}.$$

Example

Find the product of inertia of the semicircle of Fig. 301 with respect to the diameter and the tangent at the bottom. By rectangular coördinates with the origin at the center of the circle,

$$H = \int\int (a + y)x\,dx\,dy = \int \frac{a + y}{2}\Big[x^2\Big]_0^{a^2 - y^2} dx; \tag{4}$$

$$H = \frac{1}{2}\int (a + y)(a^2 - y^2)dy = \frac{1}{2}\left[a^3y + \frac{a^2y^2}{2} - \frac{a\,y^3}{3} - \frac{y^4}{4}\right]_{-a}^{a};$$

$$H = \frac{a^4}{2}\left(1 + 1 + \frac{1}{2} - \frac{1}{2} - \frac{1}{3} - \frac{1}{3} - \frac{1}{4} + \frac{1}{4}\right) = \frac{2\,a^4}{3}.$$

By polar coördinates with the origin at the tangent point,

$$H = \int\int \sin\theta\cos\theta\, r^3\, dr\, d\theta = \int \sin\theta\cos\theta\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta;$$

$$H = 4\,a^4\int \cos^5\theta\sin\theta\, d\theta = -\frac{2\,a^4}{3}\left[\cos^6\theta\right]_0^{\frac{\pi}{2}} = \frac{2\,a^4}{3}.$$

By transfer,

$$a = \frac{4\,a}{3\,\pi};\; b = a;\; A = \frac{\pi\,a^2}{2}.$$

$$H = 0 + \frac{4\,a \times a \times \pi\,a^2}{2 \times 3 \times \pi} = \frac{2\,a^4}{3}.$$

Problems

3. By integration find the product of inertia for the right-angled triangle of Fig. 302 with respect to the base and the left edge as axes. Integrate first with respect to y.

Ans. The first integration gives $H = \dfrac{h^2}{2\,b^2}\displaystyle\int_0^b (b - x)^2x\, dx;\quad H = \dfrac{h^2b^2}{24}.$

4. By transfer find the product of inertia of the triangle of Fig. 302 with respect to the parallel axes through the center of gravity.

Ans. $H_c = -\dfrac{b^2h^2}{72}.$

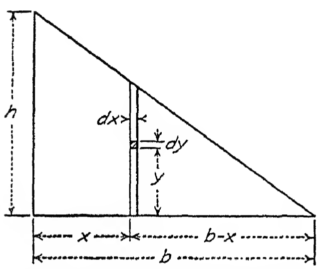


FIG. 302.—Product of inertia of right-angled triangle.

5. Find the product of inertia of a 6-in. by 5-in. by 1-in. angle section with respect to axes through the center of gravity parallel to the legs. The section may be divided into two rectangles. The product of inertia of each of these with respect to the axes through their center of gravity is zero. Transferring to the axes 1–1, 2–2 and adding

$$H_c = 4 \times (-1.5) \times 1.5 + 0 + 6 \times 1 \times (-1) + 0 = -9 - 6 = -15 \text{ in.}^4$$

The problem may be solved more readily in another way. Since 3–3 is an axis of symmetry for the horizontal leg, the product of inertia of this rectangle for 3–3 and any other axis (as 4–4) is zero. Since axis 4–4 is a line of symmetry for the vertical leg, the product of inertia of this rectangle for axes 4–4 and 3–3 is zero. The product of inertia for the entire section for axes 3–3 and 4–4 is the sum of products of inertia of the separate rectangles, therefore, $H = 0$. When the product of inertia is transferred from 3–3, 4–4 to axes 1–1, 2–2, the equation is

$$0 = H_c + 10 \times 1.5 \times 1;\quad H_c = -15 \text{ in.}^4$$

230. Calculation of Moment of Inertia.—For the calculation of moment of inertia, Equation (6) of Art. 228 becomes

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H \sin 2\theta. \tag{1}$$

Equation (5) becomes

$$I = I_x \cos^2\theta + I_y \sin^2\theta - 2\,H \sin\theta \cos\theta. \tag{2}$$

Equation (1) is generally used. However, when $\sin\theta$ and $\cos\theta$ are fractions which may be conveniently squared, Equation (2) is preferable.

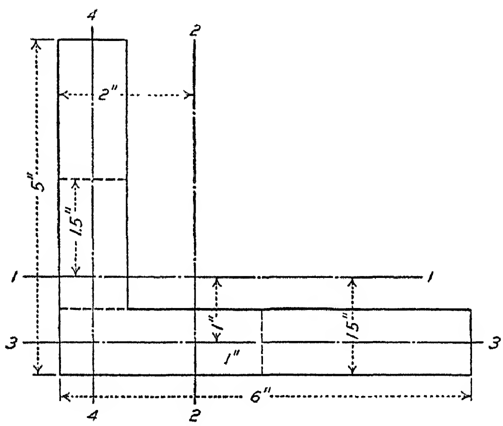


FIG. 303.—Product of inertia of angle section.

Problems

1. Find the moment of inertia of a 4-in. by 3-in. rectangle (Fig. 304) with respect to an axis through the lower left corner making an angle of 20° upward from the 4-in. edge.

$$I_x = 36 \text{ in.}^4, \quad I_y = 64 \text{ in.}^4, \quad H = 36 \text{ in.}^4$$

$$I = \frac{36 + 64}{2} + \frac{36 - 64}{2} \cos 40° - 36 \sin 40°;$$

$$I = 50 - 14 \times 0.7660 - 36 \times 0.6428 = 16.14 \text{ in.}^4$$

2. Solve Problem 1 if the axis is 20° below the direction of the 4-in. edge. *Ans.* $I = 62.42$ in.^{4}

3. Solve Problems 1 and 2 if the axes are $\tan^{-1} 3/4$ and $\tan^{-1}(-4/3)$. Use Eq. (2) without trigonometric tables. Check one of these by means of the moment of inertia of the two equal triangles with respect to the diagonal as the common base.

4. Find the moment of inertia of the 4-in. by 4-in. by 1/2-in. angle section of Fig. 305 with respect to the axis 3–3 by means of Eq. (1). Take I_x from the handbook.

5. Find the moment of inertia of the section of Fig. 305 with respect to the line which bisects the angle. Check by subtracting the moment of inertia of a 3-in. square with respect to its diagonal from the moment of inertia of a 4-in. square.
6. Check the answers of Problems 4 and 5 by means of Eq. (9) of Art. 227.

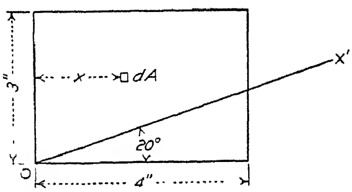


FIG. 304.—Change of direction of axis.

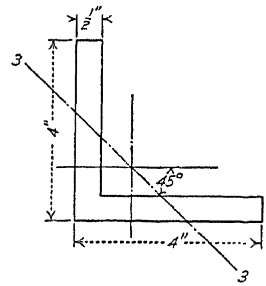


FIG. 305.—A principal axis of angle section

231. Change of Direction of Axes for Product of Inertia.—To derive the expression for the product of inertia for the axes OX', OY' of Fig. 298 the fundamental integral is

$$H' = \int x'y'dA. \tag{1}$$

$$H' = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dA, \tag{2}$$

$$H' = (\cos^2\theta - \sin^2\theta)\int x\,y\,dA + \cos\theta\sin\theta\int(y^2 - x^2)dA, \tag{3}$$

$$H' = H\cos 2\theta + \frac{I_x - I_y}{2}\sin 2\theta. \tag{4}$$

When the expression to the right of the equality sign in Equation (4) is made equal to zero,

$$\tan 2\theta = \frac{2H}{I_y - I_x} \tag{5}$$

Equation (5) gives the angle at which the product of inertia is zero.

Problems

1. In the 4-in. by 3-in. rectangle of Fig. 304 what will be the angle between $O\,X'$ and the 4-in. edge if the product of inertia with respect to $O\,X'$ and the axis through O normal to it is zero? *Ans.* $\theta = 34° \, 32'$.
2. Find the direction of the pair of axes through the center of gravity of the 6-in. by 5-in. by 1-in. angle section of Fig. 303 for which the product of inertia is zero.

232. Direction of Axis for Maximum Moment of Inertia.—From Equation (6) of Art. 228 the moment of inertia with respect to an axis at an angle θ with the X axis is

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H \sin 2\theta. \tag{1}$$

The direction of the axis for maximum or minimum moment of inertia is found by differentiating Equation (1) with respect to θ and equating the derivative to zero.

$$\frac{dI}{d\theta} = (I_y - I_x) \sin 2\theta - 2H \cos 2\theta, \tag{2}$$

from which the condition of maximum or minimum is

$$\tan 2\theta = \frac{2H}{I_y - I_x}. \tag{3}$$

A comparison of Equation (3) with Equation (5) of the preceding article shows that the condition for maximum and minimum moment of inertia is identical with the condition for zero product of inertia. There are two solutions for Equation (3), which give values of 2θ differing by 180 degrees and values of θ differing by 90 degrees. One of these gives the direction of the axis for which the moment of inertia is a maximum and the other gives the direction of the axis for which the moment of inertia is a minimum.

Since the product of inertia for an axis of symmetry is zero, the moment of inertia with respect to an axis of symmetry is greater or less than the moment of inertia for any other axis through any given point in its line.

The line which bisects the angle between the legs of an angle section of equal legs is a line of symmetry and the moment of inertia for this axis is greater than for any other axis through the center of gravity, while the moment of inertia for the axis at right angles to this line of symmetry (the axis 3–3 of Fig. 305) is smaller than that for any other axis through the center of gravity.

The maximum and minimum moments of inertia for axes through the center of gravity are called the *principal moments of inertia* and the axes are the *principal* axes of the area. After the moment of inertia and the product of inertia have been found with respect to any convenient pair of axes which are perpendicular to each other in the plane of the area, the direction

of the principal axes is found by Equation (3) or from the condition that an axis of symmetry is a principal axis. The angle thus found is then substituted in Equation (1) or in Equation (2) of Art. 230 to find the required principal moments of inertia.

Example

Find the principal moments of inertia for the 6-in. by 5-in. by 1-in. angle section for axes through the center of gravity.

$$\tan 2\theta = \frac{-30}{\frac{200}{6} - \frac{125}{6}} = -2.4; \quad 2\theta = 112°\ 37'.2 \text{ or } 292°\ 37'.2.$$

From the first angle, $\cos 2\theta = -5/13$ and $\sin 2\theta = 12/13$;

$$I_m = \frac{325}{12} - \frac{75}{12}\left(-\frac{5}{13}\right) - (-15)\frac{12}{13} = \frac{130}{3} = I_{\text{max}}.$$

After the calculation has been completed, the result is found to be the maximum, since it is larger than I_x or I_y

$$I_{\text{min}} = \frac{325}{12} - \frac{75}{12} \times \frac{5}{13} - (-15)\left(-\frac{12}{13}\right) = \frac{325}{12} - \frac{195}{12} = \frac{65}{6}.$$

When the sine of θ and the cosine of θ may be squared conveniently, Eq. (2) of Art. 230 may be used. From $\tan 2\theta = -2.4$, $\tan \theta$ is found to be $-2/3$, for which θ is $-33°\ 41'$, or $\tan \theta = 3/2$, for which $\theta = 56°\ 19'$.

Problems

1. Solve the foregoing example for maximum I from $\tan \theta = 3/2$. Calculate the sine and cosine without the use of tables.

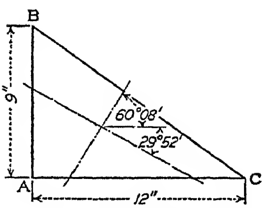


FIG. 306.—Principal axes of a right-angled triangle.

2. Solve the example for I_{min} by means of the single angle.
3. Find the moment of inertia for the angle section of the foregoing example at 45° and at 60°. Compare with I_{max}.
4. Find the maximum and minimum moments of inertia of a 4-in. by 3-in. rectangle with respect to axes through the lower left corner.
   Ans. $I_{\text{min}} = 11.374$ in.^{4} at $34°\ 22'.5$; $I_{\text{max}} = 88.626$ in.^{4}
5. Find the principal moments of inertia of the right-angled triangle of Fig. 306.
   Ans. $I_{\text{max}} = 525.05$ in.^{4} with respect to an axis at 60° 08′ with the horizontal toward the right;
   $I_{\text{min}} = 337.500 - 47.616 - 139.932 = 149.95$ in.^{4}
6. A cantilever, 10 ft. long, has a solid triangular section equivalent to Fig. 306 which is rotated 29° 52′ to bring the principal axis of minimum moment of inertia into the horizontal. Find the stress at each corner when a load of 300 lb. is placed on the free end.

The perpendicular distance from the corner A to the principal axis is $c = 3 \cos 29° 52' + 4 \sin 29° 52' = 3 \times 0.86719 + 4 \times 0.49798$;

$c = 4.5935$; $s = \dfrac{36{,}000 \times 4.5935}{149.95} = 1{,}103$ lb. per sq. in. compression.

At B, $s = \dfrac{36{,}000 \times 3.2112}{149.95} = 771$ lb. per sq. in. tension.

At C, $s = \dfrac{36{,}000 \times 1.3823}{149.95} = 332$ lb. per sq. in. tension.

233. Calculation by Means of the Tangent.—Equation (1) or Equation (2) of Art. 230 may be used to calculate the moment of inertia at any angle and to find the minimum moments of inertia after the angle has been determined by Equation (5) of Art. 231. Another form, which is applicable to transfer to or from maximum or minimum moments of inertia only, makes use of tangent of the single angle, $\phi = \dfrac{1}{2} \tan^{-1} \dfrac{2\,H}{I_y - I_x}$.

By using ϕ and 2ϕ instead of θ and 2θ in the previous equations in order to express the limitations of the equations which follow,

$$\tan 2\phi = \frac{\sin 2\phi}{\cos 2\phi} = \frac{2H}{I_y - I_x}; \tag{1}$$

$$I_y \sin \phi \cos \phi - I_x \sin \phi \cos \phi - H\,(\cos^2\phi - \sin^2\phi) = 0; \tag{2}$$

$$I_y \sin^2 \phi - I_x \sin^2 \phi - H \sin \phi \cos \phi + H\,\frac{\sin^3 \phi}{\cos \phi} = 0. \tag{3}$$

From Equation (2) of Art. 230,

$$I_m = I_y \sin^2\phi + I_x \cos^2\phi - 2\,H \sin \phi \cos \phi. \tag{4}$$

Subtracting Equation (3) from Equation (4),

$$I_m = I_x - H \sin \phi\,\frac{\sin^2\phi + \cos^2\phi}{\cos \phi} = I_x - H \tan \phi. \tag{5}$$

If the position of the second axis at 90° from the first is represented by ϕ',

$$\tan 2\phi' = \frac{\sin 2\phi'}{\cos 2\phi'} = \frac{2\,H}{I_y - I_x} = -\frac{2 \sin \phi \cos \phi}{\sin^2\phi - \cos^2\phi}.$$

Since $\sin \phi' = \sin\left(\dfrac{\pi}{2} + \phi\right) = \cos \phi$, and $\cos \phi' = -\sin \phi$,

$$-I_y \sin \phi \cos \phi + I_x \sin \phi \cos \phi - H \sin^2 \phi + H \cos^2 \phi = 0. \tag{6}$$

Multiplying by $\dfrac{\sin \phi}{\cos \phi}$,

$$I_x \sin^2\phi - I_y \sin^2\phi - H \frac{\sin^3\phi}{\cos \phi} + H \sin \phi \cos \phi = 0. \qquad (7)$$

$$I'_m = I_x \cos^2\phi' + I_y \sin^2\phi' - 2\,H \sin \phi' \cos \phi'; \qquad (8)$$

$$I'_m = I_x \sin^2\phi + I_y \cos^2\phi + 2\,H \sin \phi \cos \phi. \qquad (9)$$

Subtracting Equation (7) from Equation (9),

$$I'_m = I_y (\sin^2\phi + \cos^2\phi) + H \sin \phi \cos \phi + H \frac{\sin^3\phi}{\cos \phi}; \qquad (10)$$

$$I'_m = I_y + H \tan \phi. \qquad (11)$$

Example

Find the principal moments of inertia of the 6-in. by 5-in. by 1-in. angle section by Eqs. (5) and (11).

$$\tan \phi = -\tfrac{2}{3},\quad H = -15,\quad H \tan \phi = 10,\quad I_{\min} = \tfrac{125}{6} - 10 = \tfrac{65}{6};$$
$$I_{\max} = \tfrac{100}{3} + 10 = \tfrac{130}{3}.$$
$$H \tan \phi' = -15\tfrac{3}{2} = -\tfrac{45}{2},\quad I_{\min} = \tfrac{200}{6} - \tfrac{45}{2} = \tfrac{65}{6}.$$
$$I_{\max} = \tfrac{125}{6} + \tfrac{45}{2} = \tfrac{130}{3}.$$

Given ϕ or ϕ', there are two values of the tangent, which are numerically the tangent and cotangent of either angle. The smaller product $H \tan \phi$ or

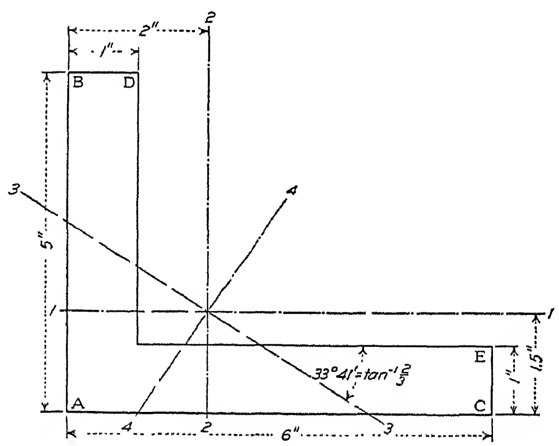


FIG. 307.—Principal axes of angle section with unequal legs.

$H \tan \phi'$ is added to the larger I_x or I_y and subtracted from the smaller I_x or I_y to get $I_{\max}$ and $I_{\min}$; or the larger product is added to the smaller I_x or I_y and subtracted from the larger. No attention need be paid to the sign of H or of the tangents.

Problems

1. Calculate the principal moments of inertia for the right-angled triangle of Fig. 306 by Eqs. (5) and (11) as described in foregoing statement.
2. Find the maximum and minimum moments of inertia for the 4-in. by 3-in. rectangle of Fig. 304 for axes through the lower left corner.

234. Stress and Deflection with Principal Axes Inclined.—When the bending moment is not in the plane of a principal axis of inertia, it is resolved into components perpendicular to the principal axes, and the effects of these components are added to get the total stress and deflection. This method was used in Arts. 84 and 164 without proof.

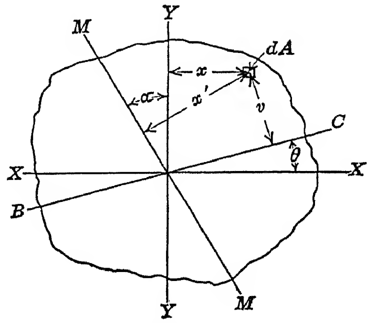


FIG. 308.—Moment at angle with principal axis.

Figure 308 represents any section of a beam for which $X\,X$ and $Y\,Y$ are the principal axes of inertia. The line $M\,M$, at an angle α with $Y\,Y$, is in the plane of the bending moment, and the line $B\,C$ at an angle θ with $X\,X$, is the neutral axis. The element dA, whose coördinates with respect to the *principal axes are* x and y, and whose distance from the *neutral axis* is v, is subjected to a stress which varies as v.

$$v = y \cos \theta - x \sin \theta. \tag{1}$$

$$s = k\,v = k(y \cos \theta - x \sin \theta) \tag{2}$$

Since the external moment is in the plane $M\,M$, the resisting moment must lie in the same plane; therefore, the sum of the moments of all the stress on the entire area about the line $M\,M$ must be zero. The perpendicular distance from dA to the line $M\,M$ is

$$x' = y \sin \alpha + x \cos \alpha. \tag{3}$$

$$\int s\,x'\,dA = 0 = k \int v\,x'\,dA, \tag{4}$$

$$\int y^2 \cos \theta \sin \alpha \, dA + \int x\,y \cos \theta \cos \alpha \, dA - \int x\,y \sin \theta \sin \alpha \, dA - \int x^2 \sin \theta \cos \alpha \, dA = 0; \tag{5}$$

$$I_x \cos \theta \sin \alpha - I_y \sin \theta \cos \alpha = 0, \tag{6}$$

in which I_x is the moment of inertia with respect to the axis $X\,X$, and I_y is the moment of inertia with respect to $Y\,Y$. The second

and third terms of Equation (5) include the product of inertia with respect to the principal axes, which is zero.

$$\tan \theta = \frac{I_x}{I_y} \tan \alpha.$$

Example I

A 6-in. by 8-in. beam is subjected to a load perpendicular to its length making an angle of 30° with the planes of the 8-in. faces. Find the angle between the neutral axis and the planes of the 6-in. faces.

When the line through the center parallel to the 6-in. faces is taken as the X axis,

$$I_x = \frac{6 \times 8^3}{12} = 256,$$

$$I_y = \frac{8 \times 6^3}{12} = 144.$$

$$\tan \theta = \frac{256}{144} \times 0.5774 = 1.0264,$$

$$\theta = 45° \, 45'.$$

The neutral axis makes an angle of 15° 45′ with the line normal to the bending moment.

From Fig. 308 the component of the bending moment perpendicular to the neutral axis is $M \cos(\theta - \alpha)$. The moment of inertia with respect to the neutral axis is $I_x \cos^2 \theta + I_y \sin^2 \theta$, and $v = y \cos \theta - x \sin \theta$.

$$s = \frac{M \cos(\theta - \alpha) v}{I_x \cos^2 \theta + I_y \sin^2 \theta}; \tag{7}$$

$$s = \frac{M(\cos \theta \cos \alpha + \sin \theta \sin \alpha)(y \cos \theta - x \sin \theta)}{I_x \cos^2 \theta + I_y \sin^2 \theta}; \tag{8}$$

$$s = \frac{My(\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha) - Mx(\cos \theta \sin \theta \cos \alpha + \sin^2 \theta \sin \alpha)}{I_x \cos^2 \theta + I_y \sin^2 \theta}. \tag{9}$$

$$My(\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha) = My \cos \alpha (\cos^2 \theta + \cos \theta \sin \theta \tan \alpha) = My \cos \alpha \left(\cos^2 \theta + \frac{I_y}{I_x} \sin^2 \theta \right); \tag{10}$$

$$\frac{My(\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha)}{I_x \cos^2 \theta + I_y \sin^2 \theta} = \frac{My \cos \alpha}{I_x}. \tag{11}$$

In a similar way the second part of Equation (3) may be shown to be

$$\frac{Mx \sin \alpha}{I_y},$$

and

$$s = \frac{My \cos \alpha}{I_x} - \frac{Mx \sin \alpha}{I_y} \tag{12}$$

To find the fiber stress at any point in a beam when the bending moment is inclined to the principal axes of inertia, resolve the bending moment (or the applied forces) perpendicular to the two axes and compute the stress for each component separately. The actual unit stress is the sum of the results of these two, taken with the proper sign.

Example II

The 6-in. by 5-in. by 1-in. angle section of Fig. 307 is used as a cantilever 5 ft. long to carry a load of 500 lb. on the free end. The 6-in. leg is horizontal. Find the unit stress at corner A.

Since I_x and I_y represent the principal moments of inertia in Eq. (12), it is convenient to draw a sketch with the 6-in. leg at 33° 41′, which makes the axis of mimimum moment of inertia lie in the usual horizontal X axis.

$$I_x = \frac{65}{6}, \quad I_y = \frac{130}{3}, \quad \tan \alpha = \frac{2}{3}, \quad \sin \alpha = \frac{2}{\sqrt{13}}, \quad \cos \alpha = \frac{3}{\sqrt{13}}.$$

$$\frac{M \cos \alpha}{I_x} = \frac{30{,}000 \times 0.83205}{65/6} = 2{,}304; \quad \frac{M \sin \alpha}{I_y} = \frac{30{,}000 \times 0.55470}{130/3} = 384.$$

$$\begin{array}{rlrl} y = 1.5 \times 0.43205 = & 1.24807 & x = 2.0 \times 0.83205 = & 1.66410 \\ 2.0 \times 0.55470 = & 1.10940 & 1.5 \times 0.55470 = & -0.83205 \\ \hline & 2.35747 & & 0.83205 \end{array}$$

$$\begin{aligned} 2{,}304 \times 2.35747 &= 5{,}431.6 \text{ lb. per sq. in. compression.} \\ 384 \times 0.83205 &= 319.5 \text{ lb. per sq. in. tension.} \\ S_a &= 5{,}112 \text{ lb. per sq. in. compression.} \end{aligned}$$

Problems

1. Solve the foregoing example for the stress at B.

Ans. $y = \dfrac{3.5 \times 3}{\sqrt{13}} - \dfrac{2 \times 2}{\sqrt{13}} = \dfrac{6.5}{\sqrt{13}} = 3.60555 \times 0.5 = 1.8028;$

$x = \dfrac{7 + 6}{\sqrt{13}} = 3.6056.$ $S_b = 4154 + 1384 = 5{,}538$ lb./in.^{2} tension.

2. Solve Example II for the unit stress at C and D.

Example III

A short post has the right-angled triangular section of Fig. 306. The compression of 10,800 lb. parallel to the length is applied along a line which is 2 in. from the 12-in. face and 2 in. from the 9-in. face.

From the example of the preceding article,

$$I_{\min} = I_x = 149.95 \text{ at } 29° \, 52' \text{ with the 12-in. face;}$$
$$I_{\max} = I_y = 525.05.$$

The moment arm of the load with respect to the axis of I_x is $1 \times 0.86719 + 2 \times 0.49798 = 1.86315$. Stress from this moment $= \dfrac{10{,}800 \times 1.86315\,y}{149.95} = 134.19\,y$. The moment arm of the load with respect to the axis of I_y is $2 \times 0.86719 - 1 \times 0.49798 = 1.2364$ in. Stress from this moment is $\dfrac{10{,}800 \times 1.2364\,x}{525.05} = 25.43\,x$.

For the corner A, $y = 4 \times 0.49798 + 3 \times 0.86719 = 4.5935$.

$$134.19\,y = 616.4 \text{ lb. per sq. in. compression.}$$
$$x = 4 \times 0.86719 - 3 \times 0.49798 = 1.9748.$$
$$25.43\,x = 50.2 \text{ lb. per sq. in. compression.}$$

Total stress at $a = 200 + 616.4 + 50.2 = 867$ lb. per sq. in. compression.

Problems

3. Find the unit stress at B of Fig. 306 for Example III.
4. Find the unit stress at C of Fig. 306 for Example III.

235. Moment of Inertia of Regular Polygons.—*The moment of inertia of a regular polygon with respect to any axis through the center of gravity is a constant.*

A regular polygon has as many axes of symmetry as it has sides. For every axis of symmetry there is a pair of axes for which the product of inertia is zero. When I_x and I_y are taken with respect to a pair of axes one of which is an axis of symmetry, Equation (1) of Art. 230 becomes

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta. \tag{1}$$

Cosine of 2θ decreases continuously from 1 to -1 as the angle increases from 0 to π and the angle θ increases from 0 to $\frac{\pi}{2}$. If $I_x - I_y$ is positive, I_x is the maximum and I_y is the minimum. If $I_x - I_y$ is negative, I_x is the minimum and I_y is the maximum. Since the cosine decreases continuously, there can be no maximum or minimum between I_x and I_y. However, every regular polygon has one or more axes of symmetry in this quadrant for which the moment of inertia must be a maximum. Consequently, $I_x - I_y$ 0. And the moment of inertia is the same for all axes.

Problems

1. Find the moment of inertia of an equilateral triangle of side b with respect to an axis through the center of gravity parallel to one side. Also find the moment of inertia with respect to an axis of symmetry.

$$Ans.\ I_x = \frac{b^4\left(\frac{\sqrt{3}}{2}\right)^3}{36} = \frac{b^4\sqrt{3}}{96};\ I_y = \frac{2\frac{b\sqrt{3}}{2}\left(\frac{b}{2}\right)^3}{12} = \frac{b^4\sqrt{3}}{96}.$$

2. Find the moment of inertia of a square with respect to an axis through the center of gravity parallel to a side and also with respect to a diagonal.

Since the moment of inertia of a regular polygon is the same for every axis through the center of gravity, it is equal to one-half the polar moment of inertia.

$$I = \frac{J}{2}. \tag{2}$$

To find the polar moment of inertia, the regular polygon of n sides is divided into n isosceles triangles, one of which is shown

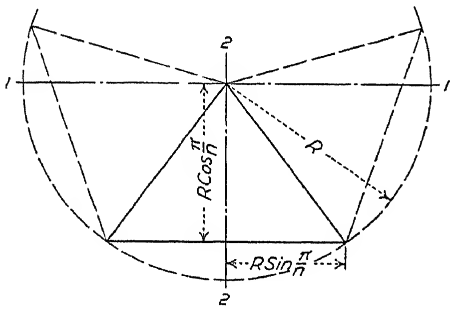


FIG. 309.—Regular polygon.

in Fig. 309. This triangle has equal sides of length R, which is the radius of the circumscribed circle, and a base of $2\,R \sin \frac{\pi}{n}$. The altitude is $R \cos \frac{\pi}{n}$. The polar moment of inertia of the single triangle with respect to the vertex is

$$J = \frac{b\,h^3}{4} + \frac{b^3\,h}{48} = R^4\left(\frac{\cos^2\frac{\pi}{n}}{2} + \frac{\sin^2\frac{\pi}{n}}{6}\right)\sin\frac{\pi}{n}\cos\frac{\pi}{n}. \tag{3}$$

For the entire polygon of n triangles,

$$I = \frac{n\,R^4}{8}\left(\cos^2\frac{\pi}{n} + \frac{1}{3}\sin^2\frac{\pi}{n}\right)\sin\frac{2\pi}{n}. \tag{4}$$

The minimum section modulus is $\frac{I}{R}$;

$$Z_{\min} = \frac{n\,R^3}{8}\left(\cos^2\frac{\pi}{n} + \frac{1}{3}\sin^2\frac{\pi}{n}\right)\sin\frac{2\pi}{n}. \tag{5}$$

The maximum section modulus is $\dfrac{I}{h}$;

$$Z_{\max} = \frac{n\,R^3}{4}\left(\cos^2\frac{\pi}{n} + \frac{1}{3}\sin^2\frac{\pi}{n}\right)\sin\frac{\pi}{n}. \tag{6}$$

Problems

3. Find the moment of inertia of a square by Eq. (4).

$$n = 4, \quad \cos\frac{\pi}{n} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{n}, \quad R = \frac{b}{\sqrt{2}}.$$

$$I = \frac{4\,R^4}{8}\left(\frac{1}{2} + \frac{1}{6}\right)1 = \frac{R^4}{3} = \frac{b^4}{12}.$$

4. Find the moment of inertia of a regular hexagon.

$$R = b, \quad \cos\frac{\pi}{n} = \frac{\sqrt{3}}{2}, \quad \sin\frac{\pi}{n} = \frac{1}{2}, \quad \sin\frac{2\pi}{n} = \frac{\sqrt{3}}{2}.$$

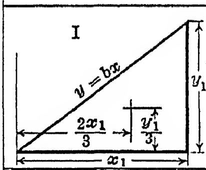
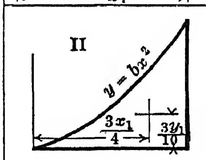
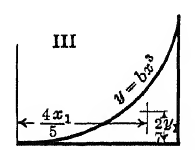
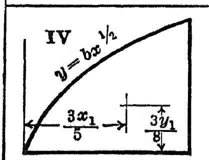
$$I = \frac{6\,R^4}{8}\left(\frac{3}{4} + \frac{1}{12}\right)\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{16}R^4 = 0.5412\,R^4 = 0.5412\,b^4.$$

5. Find the moment of inertia of a regular pentagon.

Ans. $I = 0.4764\,R^4$; $Z_{\min} = 0.4764\,R^3$; $Z_{\max} = 0.5777\,R^3$.

6. Find the moment of inertia and the two principal section moduli for a regular octagon. Check by direct calculations without using Eq. (4).

TABLE XXXIV.—CENTER OF GRAVITY OF SOME PLANE AREAS

	EQUATION	AREA	M_x	$\bar{x}$	$\bar{y}$
	$y = bx$	$\frac{x_1 y_1}{2}$	$\frac{bx_1^3}{3} = \frac{x_1^2 y_1}{3}$	$\frac{2x_1}{3}$	$\frac{y_1}{3}$
	$y = bx^2$	$\frac{x_1 y_1}{3}$	$\frac{bx_1^4}{4} = \frac{x_1^2 y_1}{4}$	$\frac{3x_1}{4}$	$\frac{3y_1}{10}$
	$y = bx^3$	$\frac{x_1 y_1}{4}$	$\frac{bx_1^5}{5} = \frac{x_1^2 y_1}{5}$	$\frac{4x_1}{5}$	$\frac{4y_1}{14}$
	$y = bx^{\frac{1}{2}}$	$\frac{2x_1 y_1}{3}$	$\frac{2bx_1^{\frac{5}{2}}}{5} = \frac{2x_1^2 y_1}{5}$	$\frac{3x_1}{5}$	$\frac{3y_1}{8}$

APPENDIX

TABLE A.—TENSION TEST OF STEEL, S.A.E. 1020

Carbon, 0.20; manganese, 0.52; phosphorus, 0.018; sulfur, 0.032 per cent; mean diameter, 0.752 inch; area, 0.444 square inch; gage length, 8 inches. Tested on 50,000-pound Olsen. Cross-head speed, $\frac{1}{60}$ inch per minute. August 24, 1931

Total load, lb.	Unit stress, lb. per sq. in.	Elongation: In 8-in., inches	Elongation: Unit, inches per inch	Total load, lb.	Unit stress, lb. per sq. in.	Elongation: Total, inches	Elongation: Unit, inches per inch	
0	0	0	0	20,750	46,730	0.35	0.0437	
2,220	5,000	0.00118	0.000147	22,040	49,640	.40	.050*	
4,440	10,000	242	302	23,100	52,030	.50	.0625	
6,660	15,000	398	497	23,940	53,920	.60	.0750	
8,880	20,000	520	650	24,700	55,630	.70	.0875	
11,100	25,000	0.00660	0.000825	25,260	56,890	0.80	0.100	
13,320	30,000	802	0.001002	25,620	57,700	.90	.1125	
15,540	35,000	928	1160	26,030	58,630	1.00	.125	
15,984	36,000	966	1207	26,210	59,030	1.10	.1375	
16,428	37,000	996	1245	26,460	59,590	1.20	.150	
16,680	37,570	0.01010	0.001262	26,540	59,770	1.30	0.1625	
16,690	37,590	1102	1377	26,690	60,110	1.40	.175	
16,660	37,520	1546	1932	26,750	60,250	1.50	.1875	
16,580	37,340	1866	2332	26,770	60,290	1.60	.200	
16,630	37,450	2286	2857	26,770	60,290	1.70	.2125	
16,700	37,610	0.02740	0.003425	26,840	60,450	1.80	0.225	
16,630	37,450	3534	4417	26,840	60,450	1.90	.2375	Diam.
16,740	37,700	5182	6477	26,810	60,380	2.00	.250	neck
16,820	37,880	6226	7782	26,740	60,220	2.10	.2625	0.667
16,860	37,970	7362	9202	26,720	60,100	2.2	.275	.658
16,750	37,730	0.08562	0.010702	26,650	60,020	2.3	0.2875	
16,760	37,750	9302	11627	26,600	59,910	2.4	.300	.655
16,910	38,090	0.10482	13102	26,500	59,680	2.5	.3125	.647
16,790	37,810	11882	14852	26,410	59,480	2.6	.325	.637
16,890	38,040	12902	16127	26,040	58,650	2.7	.3375	.615
16,650	37,500	0.14872	0.018590	24,600	55,400	2.8	0.350	.570
16,430	37,000	16042	20052	22,050	49,660	2.9	.3625	.502
16,390	36,910	16602	20752	Ran entirely at one-sixtieth inch per minute				
16,790	37,810	17096	21370	21,300	47,970	2.91	.3637	.483
17,280	38,920	17642	22052	20,700	46,620	2.92	.3650	.476
				19,370	43,630			
17,480	39,370	0.18042	0.022552	Broke		2.92	.3650	.435
17,650	39,750	[illegible]	[illegible]					
17,930	40,380	[illegible]	[illegible]					
18,370	41,370	[illegible]	[illegible]					
18,660	42,030	21950	27437					
19,190	43,220	0.23990	0.029987					
19,510	43,940	25486	31857					
19,750	44,480	26698	33372					

* Machine ran at 1 in. per minute for 0.08 inch. Then at $\frac{1}{60}$ in. per minute to balance.

Interval	Top	2	3	4	5	6	7	8
Length after rupture, inches...	1.29	1.33	1.72 (neck)	1.36	1.33	1.33	1.30	1.26

TABLE B.—TENSION TEST OF STEEL. S.A.E. 1045

Carbon, 0.44; manganese, 0.70; phosphorus, 0.034; sulfur, 0.023 per cent; mean diameter, 0.749 inch, area 0.441 square inch; gage length, 8 inches. Tested on 50,000-pound Olsen. Cross-head speed, $\frac{1}{60}$ inch per minute. August 31, 1931

Total load, lb.	Unit stress, lb. per sq. in.	Elongation	
		In 8-in., inches	Unit, inches per inch
0	0	0	0
441	1,000	0.00020	0.000025
882	2,000	42	52
1,323	3,000	64	80
1,764	4,000	94	0.000117
2,205	5,000	0.00120	0.000150
4,410	10,000	258	322
6,615	15,000	406	507
8,820	20,000	538	672
11,025	25,000	676	845
13,230	30,000	0.00808	0.001010
13,671	31,000	836	1045
14,112	32,000	862	1075
14,553	33,000	886	1107
14,994	34,000	916	1145
15,435	35,000	0.00924	0.001155
15,876	36,000	974	1217
16,317	37,000	0.01000	1250
16,758	38,000	1034	1292
17,100	38,780	1056	1320
17,640	40,000	0.01080	0.001350
17,840	40,450	1098	1372
18,300	41,500	1126	1407
18,860	42,770	1160	1450
19,620	44,490	1204	1505
20,850	47,280	[illegible]	[illegible]
21,790	49,410	1356	1695
22,260	50,480	1382	1727
23,150	52,500	1442	1802
24,070	54,580	1500	1875
24,390	55,310	0.01520	0.001900
24,710	56,030	1548	1935
24,490	55,540	1630	2037
24,620	55,830	1952	2440
24,480	55,510	2394	2992
Loose scale at top			
24,510	55,790	0.03740	0.004675
24,600	55,780	4820	6025
24,460	55,460	5434	6792
24,520	55,600	6280	7850
24,400	55,330	7520	9400
24,840	56,330	0.07774	0.009717
Scale all over			
25,200	57,140	8094	0.010117
25,580	58,000	8446	0.010557
26,530	60,160	9774	12217
26,950	61,120	0.10362	12952
27,270	61,840	0.10846	0.013557
27,680	62,770	.11214	14017
28,200	63,950	.12186	15232
28,640	64,950	.12924	16155
29,160	66,120	.13610	17012
29,700	67,350	0.14570	0.018212
30,230	68,550	.15522	19402
30,780	69,800	.16522	20662
31,310	70,980	.17504	21880
32,180	72,970	.19066	23832
32,820	74,420	0.20320	0.025400
33,330	75,580	.21388	26735
33,960	77,010	.22906	28632
34,430	78,070	.24030	30037
36,100	81,860	.28398	35497

Elongation taken with dividers. Total elongation 0.28 inch.

Total load*	Unit stress	Elongation		Diameter, inches
		Total	Unit	
[illegible]	87,870	0.4	0.050	
40,420	91,660	0.5	0.0625	0.724
41,550	94,220	0.6	0.075	
42,150	97,580	0.7	0.0875	0.715
42,100	95,470	0.8	0.100	
42,650	96,710	0.9	0.1125	0.706
42,800	97,050	1.0	0.125	
42,720†	96,870	1.1	0.1375	0.696
42,930	97,350	1.21	0.1512	
42,860	97,190	1.3	0.1625	
42,600	96,600	1.41	0.1762	0.681
42,280	95,870	1.51	0.1887	0.665
41,150	93,310	1.6	0.200	0.637
38,000	86,170	....	......	0.575
37,200	84,350	Broke		
After failure		1.68	0.210	0.558

* Cross-head speed 1 inch per minute for about 0.08 inch; then at $\frac{1}{60}$ inch per minute till balanced.

† Cross head ran too far at high speed. Stress increasing when reading was taken.

TABLE C.—TENSION TEST OF HIGH-CARBON STEEL. S.A.E. 1095
Gage length, 8 inches; mean diameter, 0.7466 inch; area, 0.4378 square inch. Tested on 100,000-pound Olsen. Cross-head speed, 0.05 inch per minute. September 14, 1931

Unit stress, lb. per sq. in.	Unit elongation, inches per inch
0	0
1,000	0.000031
2,000	65
3,000	97
4,000	0.000127
5,000	0.000157
10,000	317
15,000	502
20,000	672
25,000	850
30,000	0.001022
35,000	1195
40,000	1372
41,000	1402
42,000	1430
43,000	0.001442
44,000	1502
45,000	1535
50,000	1725
55,000	1877
60,000	0.002047
65,000	2237
70,000	2482
75,000	3182
80,000	4554
85,000	0.006147
90,000	7937
95,000	9930
100,000	0.012275
102,790	13547
105,070	0.014840
107,350	16075
109,640	17370
111,920	18575
114,210	20127
116,490	0.021495
118,780	23067
121,060	24617
123,340	26397
124,490	27337
125,630	0.028312
126,770	29337
127,910	30367
129,050	31562
130,200	32587
131,340	0.033687
136,480	0.034997

Unit stress, lb. per sq. in.	Unit elongation, inches per inch
Elongation with dividers	
134,080	0.03625
135,910	3875
136,480	4125
138,300	425
139,220	4375
139,790	0.04625
140,250	475
141,160	4875
142,070	5125
142,760	525
143,450	0.05625
144,020	5937
144,590	6187
145,040	65
145,390	6875
145,730	0.07187
145,840	7625
145,500	8215
145,040	85
144,930	Broke

Length

Stress	Upper four divisions, in.	Lower four divisions, in.
136,480	4.14	4.14
145,500	4.33	4.32
145,040	4.34	4.34
After fracture..	4.26	4.32

Diameter readings in two planes

Division	Stress 145,500		After fracture	
Top	0.707	0.710	0.711	0.713
2	0.717	0.717	0.729	0.724
3	0.718	0.719	0.726	0.726
4	0.717	0.717	0.723	0.724
5	0.715	0.715	0.724	0.719
6	0.704	0.705	0.704	0.700
7	0.714	0.716	0.722	0.717
8	0.715	0.717	0.726	0.721
Average.	0.7152		0.7192	

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